MATH 556 - EXERCISES 3 : SOLUTIONS

1. Using the Chebychev Lemma with $h(x) = e^{tx}$ and $c = e^{at}$, for $t > 0$,

$$P_X\left[X \geq a\right] = P_X\left[tX \geq at\right] = P_X\left[\exp\{tX\} \geq \exp\{at\}\right] \leq \frac{\mathbb{E}_X[e^{tX}]}{e^{at}} = \frac{M_X(t)}{e^{at}}$$

provided $t < h$ also. Using similar methods,

$$P_X\left[X \leq a\right] \leq e^{-at} M_X(t) \qquad \text{for } -h < t < 0$$

For the second result, we have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds}\left\{K_X(t)\right\}_{s=t} = \frac{d}{ds}\left\{\log M_X(t)\right\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = \mathbb{E}_X[\,X\,]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t) M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\left\{M_X(t)\right\}^2}$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0) M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\left\{M_X(0)\right\}^2} = \mathbb{E}_X[\,X^2\,] - \left\{\mathbb{E}_X[\,X\,]\right\}^2$$

2. (a) From first principles, for $y > 0$,

$$P[Y \leq y] = P_X[X^2 \leq y] = P_X[-\sqrt{y} \leq X \leq \sqrt{y}] = \Phi(\sqrt{y} - \mu) - \Phi(-\sqrt{y} - \mu)$$

therefore for $y > 0$,

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}}\left(\phi(\sqrt{y} - \mu) + \phi(-\sqrt{y} - \mu)\right) \\ &= \frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{y}}\left(\exp\left\{-\frac{1}{2}(\sqrt{y} - \mu)^2\right\} + \exp\left\{-\frac{1}{2}(-\sqrt{y} - \mu)^2\right\}\right) \\ &= \frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{y}}\exp\left\{-\frac{1}{2}(y + \mu^2)\right\}\left(\exp\left\{\sqrt{y}\mu\right\} + \exp\left\{-\sqrt{y}\mu\right\}\right) \end{aligned}$$

Let $\lambda = \mu^2$. We rewrite the density

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{y}}\exp\left\{-\frac{1}{2}(y + \lambda)\right\}\left(\exp\left\{\sqrt{y\lambda}\right\} + \exp\left\{-\sqrt{y\lambda}\right\}\right)$$

Using the exponential series expansion, we have that

$$\begin{aligned} \exp\left\{\sqrt{y\lambda}\right\} + \exp\left\{-\sqrt{y\lambda}\right\} &= \sum_{j=0}^{\infty}\frac{1}{j!}y^{j/2}\lambda^{j/2} - \sum_{j=0}^{\infty}\frac{1}{j!}(-1)^j y^{j/2}\lambda^{j/2} \\ &= 2\sum_{j=0}^{\infty}\frac{(\lambda y)^j}{(2j)!} \end{aligned}$$

Therefore the density is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} \exp\left\{-\frac{1}{2}(y+\lambda)\right\} \sum_{j=0}^{\infty} \frac{(\lambda y)^j}{(2j)!}$$

Rewriting this, we have for $y > 0$,

$$\begin{aligned}
f_Y(y) &= \exp\left\{-\frac{\lambda}{2}\right\} \sum_{j=0}^{\infty} \frac{(\lambda y)^j}{(2j)!} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} \exp\left\{-\frac{y}{2}\right\} \\
&= \exp\left\{-\frac{\lambda}{2}\right\} \sum_{j=0}^{\infty} \frac{\lambda^j}{(2j)!} \frac{1}{\sqrt{2\pi}} y^{j-1/2} \exp\left\{-\frac{y}{2}\right\} \\
&= \exp\left\{-\frac{\lambda}{2}\right\} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{(2j)!} \frac{2^{j+1/2}}{\sqrt{\pi}} y^{j-1/2} \exp\left\{-\frac{y}{2}\right\} \\
&= \exp\left\{-\frac{\lambda}{2}\right\} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{\Gamma(2j+1)} \frac{2^{j+1/2}}{\sqrt{\pi}} y^{j-1/2} \exp\left\{-\frac{y}{2}\right\}
\end{aligned}$$

Note that

$$\Gamma(2j+1) = (2j)! = (2j)(2j-1)(2j-2)....3.2.1 = 2^{2j} j(j-1/2)(j-1)...(3/2)(2/2)(1/2).$$

Now, $Z_m \sim \chi^2_m \equiv \text{Gamma}(m/2, 1/2)$, we have

$$f_{Z_m}(z) = \frac{(1/2)^{m/2}}{\Gamma(m/2)} z^{m/2-1} \exp\{-z/2\} \qquad z > 0$$

so if $m = 2j + k$,

$$f_{Z_{2j+k}}(z) = \frac{(1/2)^{j+k/2}}{\Gamma(j+k/2)} z^{j+k/2-1} \exp\{-z/2\} \qquad z > 0$$

Therefore we have after some cancellation,

$$f_Y(y) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} f_{Z_{2j+k}}(y) \qquad y > 0$$

with $k = 1$.

(b) We have for $\varphi_Y(t)$, from the definition

$$\varphi_Y(t) = \mathbb{E}_Y[e^{itY}] = \mathbb{E}_X[e^{itX^2}] = \int_{-\infty}^{\infty} e^{itx^2} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-\mu)^2\right\} \, dx$$

We may complete the square in the integral to obtain

$$\varphi_Y(t) = \exp\left\{\frac{\mu^2 it}{1-2it}\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-2it)}{2}\left(x - \frac{\mu}{(1-2it)}\right)^2\right\} \, dx$$

and thus, integrating the normal kernel,

$$\varphi_Y(t) = \frac{1}{\sqrt{1-2it}} \exp\left\{\frac{\mu^2 it}{1-2it}\right\}$$

(here as the integral is wrt x, we may just treat the quantity i as if it were a real quantity during the manipulation).

(c) We have

$$\mathcal{L}_Y(t) = \varphi_Y(it) = \frac{1}{\sqrt{1+2t}} \exp\left\{-\frac{\mu^2 t}{1+2t}\right\}$$

(d) In this case the mgf exists, and

$$M_Y(t) = \varphi_Y(-it) = \frac{1}{\sqrt{1-2t}} \exp\left\{\frac{\mu^2 t}{1-2t}\right\}$$

so

$$K_Y(t) = -\frac{1}{2}\log(1-2t) + \frac{\mu^2 t}{1-2t}.$$

Thus

$$\begin{aligned} \mathbb{E}_Y[Y] = \frac{d}{dt}\{K_Y(t)\}_{t=0} &= \left\{\frac{1}{1-2t} + \frac{(1-2t)\mu^2 + 2\mu^2 t}{(1-2t)^2}\right\}_{t=0} = \left\{\frac{1}{1-2t} + \frac{\mu^2}{(1-2t)^2}\right\}_{t=0} \\ &= 1 + \mu^2 \end{aligned}$$

and

$$\text{Var}_Y[Y] = \frac{d^2}{dt^2}\{K_Y(t)\}_{t=0} = \left\{\frac{2}{(1-2t)^2} + \frac{4\mu^2}{(1-2t)^3}\right\}_{t=0} = 2 + 4\mu^2$$

(e) Using mgfs, we have for integers ν_i, $i = 1, \dots, n$,

$$M_S(t) = \left(\frac{1}{1-2t}\right)^{\nu/2} \exp\left\{\frac{\lambda t}{1-2t}\right\}$$

where

$$\nu = \sum_{i=1}^{n} \nu_i \qquad \lambda = \sum_{i=1}^{n} \lambda_i$$

To see this, note that if $n = 2$ with $\nu_1 = \nu_2 = 1$, but $\lambda_1 = \mu_1^2$ and $\lambda_2 = \mu_2^2$, we have

$$\begin{aligned} M_S(t) = M_{Y_1}(t) M_{Y_2}(t) &= \frac{1}{\sqrt{1-2t}} \exp\left\{\frac{\lambda_1 t}{1-2t}\right\} \frac{1}{\sqrt{1-2t}} \exp\left\{\frac{\lambda_2 t}{1-2t}\right\} \\ &= \left(\frac{1}{\sqrt{1-2t}}\right)^2 \exp\left\{\frac{(\lambda_1 + \lambda_2) t}{1-2t}\right\} \end{aligned}$$

We can use this as a recursion to generate the mgf for any integer ν_1. Thus S has a noncentral chi-square distribution with ν degrees of freedom, and noncentrality λ.

3. Differentiating under the integral wrt t, we have

$$\frac{d^r}{dt^r}\mathcal{L}_X(t) = \int \left\{\frac{d^r}{dt^r} e^{-tx}\right\} dF_X(x) = \int (-x)^r e^{-tx}\, dF_X(x) = (-1)^r \int x^r e^{-tx}\, dF_X(x)$$

and the result follows multiplying both sides by $(-1)^r$, and recalling that the variable is nonnegative.

For the second result, we have, on integrating by parts,

$$\begin{aligned}\mathcal{L}_X(t) &= \int_0^\infty e^{-tx} f_X(x)\,dx \\ &= \left[e^{-tx}F_X(x)\right]_0^\infty + t\int_0^\infty e^{-tx}F_X(x)\,dx = t\int_0^\infty e^{-tx}F_X(x)\,dx.\end{aligned}$$

4. Using the mgf of the Gamma from the formula sheet, the mgf of Y must be

$$\left(\frac{\beta_1}{\beta_1 - t}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 + t}\right)^{\alpha_2}$$

thus the characteristic function

$$\left(\frac{\beta_1}{\beta_1 - it}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 + it}\right)^{\alpha_2}$$

In principle we could invert this cf using the inversion formula for continuous rvs. However, the density is not straightforward. The difference of two Gamma random variables has the *Variance Gamma* distribution.

5. To recap, we have by definition that

$$\varphi_k(t) = \int e^{itx}\,dF_k(x)$$

say, for some cdf F_k. Clearly $\varphi(t)$ is finite as the c_k are summable, and the individual cf integrals are finite. We have, because of this, by exchanging the order of summation and integration,

$$\varphi(t) = \sum_{k=1}^n c_k\varphi_k(t) = \sum_{k=1}^n \left\{c_k \int e^{itx}\,dF_k(x)\right\} = \int e^{itx}\left\{\sum_{k=1}^n c_k\,dF_k(x)\right\} = \int e^{itx}\,d\left\{\sum_{k=1}^n c_kF_k(x)\right\}.$$

The function

$$F_X(x) = \sum_{k=1}^n c_kF_k(x)$$

is a valid cdf; this is easily verified by checking the standard properties, as the c_ks sum to one. Therefore $\varphi(t)$ is the cf corresponding to F_X. The distribution characterized by F_X and $\varphi(t)$ is termed a *finite mixture distribution*.

For the limiting case, consider the limiting case function $F_X(x)$

$$F_X(x) = \sum_{k=1}^\infty c_kF_k(x)$$

for any fixed x. The right hand side can be considered as the probability of a disjoint union of the events

$$(X \le x \cap Z = k) \qquad k = 1, 2, \ldots$$

where Z is a discrete random variable on the positive integers, with probabilities $c_1, c_2, \ldots$ attached to $1, 2, \ldots$. Now by standard limit results for event sequences

$$\bigcup_{k=1}^\infty (X \le x \cap Z = k) \equiv (X \le x)$$

and thus $F_X(x)$ is a well-defined cdf. Hence the limiting case as $n \longrightarrow \infty$ provides no difficulty.

6. By inspection of the formula sheet, and the realization that if the mgf $M(t)$ exists for $|t| < h$, then $\varphi(t) = M(it)$, we may deduce that $\varphi_1(t)$ is the cf of the normal density with mean zero and variance 8. For $\varphi_2(t)$, we note that

$$\limsup_{t \longrightarrow \pm\infty} |\varphi_2(t)| = 1$$

so the distribution is discrete; we must find mass function $f_2(x)$ with support $\mathbb{X}$ such that

$$\sum_{x \in \mathbb{X}} e^{ixt} f_2(x) = (3 + \cos(t) + \cos(2t))/5$$

Now, note that $e^{itx} = \cos(tx) + i\sin(tx)$, so we can deduce that $x = 0, 1, 2$ must be in $\mathbb{X}$. Note also that the cf is entirely real, and

$$\cos(tx) = \frac{e^{itx} + e^{-itx}}{2}$$

and so

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \cos(2t) = \frac{e^{it2} + e^{-it2}}{2}$$

Hence $f_2(x)$ can be deduced to be of the form

$$f_2(x) = \begin{cases} 1/10 & x = -2 \\ 1/10 & x = -1 \\ 6/10 & x = 0 \\ 1/10 & x = 1 \\ 1/10 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Hence, using the previous result, the distribution is an equal mixture of the two components,

$$F(x) = \frac{1}{2}\Phi(x/\sqrt{8}) + \frac{1}{2}F_2(x).$$

7. As $X_2 = X_1 + (X_2 - X_1)$, we have by the independence statements

$$\varphi_{X_2}(t) = \varphi_{X_1}(t)\varphi_{X_2 - X_1}(t) = \varphi_{X_1}(t)\varphi_{X_2}(t)\varphi_{X_1}(-t) = \varphi_{X_2}(t)|\varphi_{X_1}(t)|^2$$

with the final step being an elementary property of complex numbers. Now $\varphi_{X_2}(0) = 1$ and thus by continuity $\varphi_{X_2}(t) \neq 0$ for t at least in a neighbourhood of zero. Thus, equating the two sides, we must have $|\varphi_{X_1}(t)|^2 = 1$ for all t. Thus as t varies, $\varphi_{X_1}(t)$ is always a complex valued quantity that lies on the unit circle in the complex plane. Hence we must have

$$\varphi_{X_1}(t) = e^{itc}$$

for some c, and X_1 is degenerate at c.

8. First, note that

$$M_X(t) = e^{-t} M_Z(t)$$

where

$$M_Z(t) = \frac{9}{(3 + 2t)^2} = \frac{1}{(1 + (2/3)t)^2}$$

and, by linear transformation results for mgfs, $X \stackrel{d}{=} Z - 1$. Hence

$$\mathbb{E}_X[X^r] = \mathbb{E}_Z[(Z-1)^r] = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \mathbb{E}_Z[Z^j].$$

We have that

$$M_Z^{(j)}(t) = (-2)(-3)...(-2-j+1)\frac{(2/3)^j}{(1+(2/3)t)^{2+j}} = (-1)^j (j+1)! \frac{(2/3)^j}{(1+(2/3)t)^{2+j}}$$

so that

$$\mathbb{E}_Z[Z^j] = M_Z^{(j)}(0) = (-1)^j (j+1)!(2/3)^j$$

which we may substitute in above to get

$$\mathbb{E}_X[X^r] = \mathbb{E}_Z[(Z-1)^r] = (-1)^r \sum_{j=0}^{r} \binom{r}{j} (j+1)!(2/3)^j.$$

9. From the formula sheet we have

$$M_X(t) = (1 - \theta + \theta e^t)^n$$

and thus

$$K_X(t) = n \log(1 - \theta + \theta e^t).$$

Using the linear transformation result,

$$M_{Z_n}(t) = e^{b_n t} M_X(a_n t)$$

where

$$a_n = \frac{1}{\sqrt{n\theta(1-\theta)}} \qquad\qquad b_n = -\frac{n\theta}{\sqrt{n\theta(1-\theta)}} = -n\theta a_n$$

we have

$$K_{Z_n}(t) = -n\theta a_n t + K_X(a_n t) \quad = -n\theta a_n t + n \log(1 - \theta + \theta e^{a_n t})$$

Now if

$$g_n(t) = e^{a_n t} - 1 = a_n t + \frac{1}{2} a_n^2 t^2 + \frac{1}{6} a_n^3 t^3 + \cdots$$

we have up to terms in t^3

$$\begin{aligned} n \log\{(1 + \theta g_n(t))\} &= n\theta g_n(t) - n\theta^2 \{g_n(t)\}^2/2 + n\theta^3 \{g_n(t)\}^3/6 \cdots \\ &= n\theta(a_n t + \frac{1}{2} a_n^2 t^2 + \frac{1}{6} a_n^3 t^3 + \cdots) \\ &\quad -\frac{n\theta^2}{2}(a_n^2 t^2 + a_n^3 t^3 + \cdots) \\ &\quad +\frac{n\theta^3}{3}(a_n^3 t^3 + \cdots) + \cdots \end{aligned}$$

Therefore, from the earlier expression, the term in t cancels, and we are left with

$$\begin{aligned} K_{Z_n}(t) &= \frac{n\theta(1-\theta)}{2} a_n^2 t^2 + n\theta \left(\frac{1}{6} - \frac{\theta}{2} + \frac{\theta^2}{3} \right) a_n^3 t^3 + \cdots \\ &= \frac{t^2}{2} + \frac{1}{n^{1/2}} \frac{\theta(1-3\theta+2\theta^2)}{6(\theta(1-\theta))^{3/2}} t^3 + \cdots \end{aligned}$$

The truncation after the second term leads to an approximation which has order na_n^4; this is a constant times $nn^{-2} = n^{-1}$. Hence we may equivalently write

$$K_{Z_n}(t) = \frac{t^2}{2} + \frac{1}{n^{1/2}} \frac{\theta(1-3\theta+2\theta^2)}{6(\theta(1-\theta))^{3/2}} t^3 + \mathrm{O}(n^{-1})$$

or

$$K_{Z_n}(t) = \frac{t^2}{2} + \frac{1}{n^{1/2}} \frac{\theta(1-3\theta+2\theta^2)}{6(\theta(1-\theta))^{3/2}} t^3 + \mathrm{o}(n^{-1/2})$$

as $n \longrightarrow \infty$.