MATH 556 - EXERCISES 2 : SOLUTIONS

1. (a) By direct calculation, using a derivation as in lecture notes,

$$\varphi_X(t) = \mathbb{E}_X[e^{itX}] = \int_{-\infty}^{\infty} e^{itx}\frac{1}{2}|x|\exp\{-|x|\}\,dx = \int_0^{\infty}\cos(tx)xe^{-x}\,dx$$

as the pdf is an even function around zero. Integrating by parts,

$$\begin{aligned}
\varphi_X(t) &= \left[\frac{1}{t}\sin(tx)xe^{-x}\right]_0^{\infty} - \frac{1}{t}\int_0^{\infty}\sin(tx)[-xe^{-x}+e^{-x}]\,dx \\
&= \frac{1}{t}\int_0^{\infty}\sin(tx)xe^{-x}\,dx - \frac{1}{t}\int_0^{\infty}\sin(tx)e^{-x}\,dx \\
&= \frac{1}{t}\left[-\frac{1}{t}\cos(tx)xe^{-x}\right]_0^{\infty} + \frac{1}{t^2}\int_0^{\infty}\cos(tx)[-xe^{-x}+e^{-x}]\,dx \\
&\qquad -\frac{1}{t}\left[-\frac{1}{t}\cos(tx)e^{-x}\right]_0^{\infty} + \frac{1}{t^2}\int_0^{\infty}\cos(tx)e^{-x}\,dx \\
&= -\frac{1}{t^2}\int_0^{\infty}\cos(tx)xe^{-x}\,dx - \frac{1}{t^2} + \frac{2}{t^2}\int_0^{\infty}\cos(tx)e^{-x}\,dx
\end{aligned}$$

where the first term is equal to $-\varphi_X(t)/t^2$. Now

$$\begin{aligned}
\int_0^{\infty}\cos(tx)e^{-x}\,dx &= \left[-\frac{1}{t}\sin(tx)e^{-x}\right]_0^{\infty} - \frac{1}{t}\int_0^{\infty}\sin(tx)e^{-x}\,dx = 0 - \frac{1}{t}\int_0^{\infty}\sin(tx)e^{-x}\,dx \\
&= -\frac{1}{t}\left[-\frac{1}{t}\cos(tx)e^{-x}\right]_0^{\infty} - \frac{1}{t^2}\int_0^{\infty}\cos(tx)e^{-x}\,dx = -\frac{1}{t^2}\int_0^{\infty}\cos(tx)e^{-x}\,dx
\end{aligned}$$

Thus, on rearrangement, verifying the result from lectures, we have

$$\int_0^{\infty}\cos(tx)e^{-x}\,dx = \frac{1}{1+t^2}.$$

Returning to the previous expression, we have that

$$\varphi_X(t) = -\frac{1}{t^2}\varphi_X(t) - \frac{1}{t^2} + \frac{2}{t^2}\frac{1}{1+t^2} \qquad \therefore \qquad \varphi_X(t) = \frac{1-t^2}{(1+t^2)^2} \qquad t\in\mathbb{R}.$$

Note that an alternative proof can be obtained using Gamma integrals: we have that

$$\begin{aligned}
\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{2}|x|e^{-|x|}\,dx &= \frac{1}{2}\int_{-\infty}^{0} e^{itx}(-x)e^{x}\,dx + \frac{1}{2}\int_0^{\infty} e^{itx}xe^{-x}\,dx && \text{(splitting at } x=0\text{)} \\
&= \frac{1}{2}\int_0^{\infty} e^{-itx}xe^{-x}\,dx + \frac{1}{2}\int_0^{\infty} e^{itx}xe^{-x}\,dx && (x \longrightarrow -x \text{ in first term.}) \\
&= \frac{1}{2}\int_0^{\infty} xe^{-x(1+it)}\,dx + \frac{1}{2}\int_0^{\infty} xe^{-x(1-it)}\,dx
\end{aligned}$$

Now, by appealing to the Gamma pdf integral, we can deduce that

$$\int_0^{\infty} xe^{-x(1-it)}\,dx = \frac{\Gamma(2)}{(1-it)^2} \qquad \text{and} \qquad \int_0^{\infty} xe^{-x(1+it)}\,dx = \frac{\Gamma(2)}{(1+it)^2}$$

so therefore, as $\Gamma(2) = 1$,

$$\varphi_X(t) = \frac{1}{2}\left[\frac{\Gamma(2)}{(1-it)^2} + \frac{\Gamma(2)}{(1+it)^2}\right] = \frac{1-t^2}{(1+t^2)^2}$$

Note that the Gamma pdf can be defined for complex scale parameter. Here the mgf exists in a neighbourhood of zero, so we can mimic the entire computation by looking at the mgf, M_X, and then substituting in it for t at the last line.

(b) By direct calculation

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} \exp\{-x - e^{-x}\}\, dx = \int_{-\infty}^{\infty} \exp\{-(1-it)x - e^{-x}\}\, dx = \int_0^{\infty} y^{-it} e^{-y}\, dy = \Gamma(1-it)$$

after setting $y = e^{-x}$, using the Gamma integral technique from above.

(c) By direct calculation, using the series expansion and integrating term by term;

$$\begin{aligned}
\varphi_X(t) = \mathbb{E}_X[e^{itX}] &= \int_{-\infty}^{\infty} e^{itx} \sum_{k=0}^{\infty} (-1)^k \exp\{-(2k+1)\pi|x|\}\, dx \\
&= \sum_{k=0}^{\infty} (-1)^k \int_{-\infty}^{\infty} e^{itx} \exp\{-(2k+1)\pi|x|\}\, dx \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \int_{-\infty}^{\infty} e^{i(t/(2k+1)\pi)y} \exp\{-|y|\}\, dy
\end{aligned}$$

setting $y = (2k+1)\pi x$, after exchanging the order of integration and differentiation, which is legitimate in this context as the sum and integral are convergent. Using the result used for computing the cf for the Double Exponential distribution,

$$\int_{-\infty}^{\infty} e^{ity} \frac{1}{2} \exp\{-|y|\}\, dy = \frac{1}{1+t^2} \quad \therefore \quad \int_{-\infty}^{\infty} e^{i\frac{t}{(2k+1)\pi}y} \exp\{-|y|\}\, dy = \frac{2}{1 + \left\{\dfrac{t}{(2k+1)\pi}\right\}^2}$$

$$\therefore \quad \varphi_X(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \frac{2}{1 + \left\{\dfrac{t}{(2k+1)\pi}\right\}^2} = \sum_{k=0}^{\infty} (-1)^k \frac{2(2k+1)\pi}{(2k+1)^2\pi^2 + t^2} \qquad (1)$$

It can be shown that

$$\varphi_X(t) = \frac{2e^{t/2}}{e^{t/2} + e^{-t/2}} = \frac{1}{\cosh(t/2)} \qquad t \in \mathbb{R}$$

and this result can be verified using the inversion formula. It is then clear that the pdf and its cf are identical in form.

There is a comprehensive list of cfs and integrals in the books

- W. Feller *An Introduction to Probability Theory and Its Applications, Volume 2*, (1971).
- M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables* (1965), Dover Publications.
- I. S. Gradshteyn, I. M. Ryzhik et al., *Table of Integrals, Series, and Products, Sixth Edition* (2000), Academic Press.

2. (a) The cf is integrable, and $|\varphi_X(t)| \longrightarrow 0$ as $|t| \longrightarrow \infty$, so the inversion formula for continuous pdfs can be used:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_X(t)\,dt \quad = \quad \frac{1}{2\pi}\int_{-1}^{1} e^{-itx}(1-|t|)\,dt = \frac{1}{\pi}\int_{0}^{1}(1-t)\cos(tx)\,dt$$

after writing $e^{-itx} = \cos(tx) - i\sin(tx)$, and splitting the integral into two halves over $(-1 < t < 0)$ and $(0 < t < 1)$ in the usual way. Integrating by parts yields

$$f_X(x) = \frac{1}{\pi x^2}(1-\cos x) \qquad x \in \mathbb{R}$$

It is straightforward to verify that this is a valid pdf.

(b) Here, $|\varphi_X(t)| \longrightarrow 0$ as $|t| \longrightarrow \infty$, so we are dealing with a continuous distribution. Thus, by the inversion formula,

$$f_X(x) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_X(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\frac{2(1-\cos t)}{t^2}\,dt$$

But note that here $\varphi_X(t)$ is of the same functional form (up to proportionality) as the pdf found for part (a). Thus we must have that

$$f_X(x) \propto (1-|x|) \qquad -1 < x < 1$$

and zero otherwise. In fact

$$f_X(x) = (1-|x|) \qquad -1 < x < 1$$

and zero otherwise.

(c) Here, φ_X is in the form of a pure $\cos$ function, and therefore can be written as the sum of complex exponentials, that is

$$\varphi_X(t) = \cos(\theta t) = \frac{1}{2}\left[(\cos(\theta t) + i\sin(\theta t)) + (\cos(\theta t) - i\sin(\theta t))\right] = \frac{1}{2}e^{it\theta} + \frac{1}{2}e^{-it\theta}$$

and hence it follows that X has a **discrete distribution** with pmf given by

$$f_X(x) = \begin{cases} 1/2 & x = -\theta \\ 1/2 & x = \theta \\ 0 & \text{otherwise} \end{cases}$$

3. Suppose that $\varphi(t)$ is a valid cf for random variable X. Then, on repeated differentiation, we see that

$$\varphi^{(1)}(0) = \varphi^{(2)}(0) = \varphi^{(3)}(0) = 0.$$

Hence, we can deduce from the results given in lectures about moments that

$$\mathbb{E}_X[X] = \mathbb{E}_X[X^2] = \text{Var}_X[X] = 0$$

so that f_X must be a **degenerate** distribution, with $P_X[X = x_0] = 1$ for some x_0. But this conflicts with the properties of φ, namely that $\varphi(t) \longrightarrow 0$ as $t \longrightarrow \pm\infty$, so φ cannot be a valid cf.

4. The function

$$\varphi_X(t) = \frac{1}{2}\left(\cos(t) + \cos(\pi t)\right)$$

exhibits the behaviour that

$$\limsup_{|t|\longrightarrow\infty} |\varphi_X(t)| = 1$$

due to the $\cos$ terms. Therefore X is discrete by the result from lectures. Furthermore, $\varphi_X(t)$ is entirely real, and therefore must correspond to a distribution with mass function that is symmetric about zero. From the cf definition, and the nature of the $\cos$ function, we deduce that the pmf can only have support on values $\{-\pi, -1, 1, \pi\}$, and hence

$$f_X(x) = \frac{1}{4} \qquad x \in \{-\pi, -1, 1, \pi\}$$

and zero otherwise. Note that $\cos(-t) = \cos(t)$ and $\cos(-\pi t) = \cos(\pi t)$.

   (a) No, it is the discrete distribution on the finite support identified above.
   (b) As the distribution has finite support, moments of all orders are finite.

5. We have that

$$\varphi(t) = \frac{1}{(1 + 2t^2 + t^4)} = \frac{1}{(1+t^2)^2} = \{\varphi_X(t)\}^2$$

where, from lectures, $\varphi_X(t)$ is the cf for $X \sim Laplace$ with pdf

$$f_X(x) = \frac{1}{2}e^{-|x|} \qquad x \in \mathbb{R}.$$

Hence $\varphi(t)$ is the cf of a continuous random variable, Y say, which can be decomposed as the sum of two independent *Laplace* or *double exponential* random variables.

6. If $X \sim Cauchy$, then $Y = 1/X$ has pdf given by the general transformation theorem

$$f_Y(y) = f_X(1/y) \times |J(y)| = \frac{1}{\pi}\frac{1}{1+(1/y)^2} \times |-1/y^2| = \frac{1}{\pi}\frac{1}{1+y^2} \qquad y \in \mathbb{R}$$

so in fact $Y \sim Cauchy$ also. Now, the sample mean rv $\overline{X}$ has characteristic function given by

$$\{\varphi_X(t/n)\}^n = \{\exp\{-|t/n|\}\}^n = \exp\{-|t|\}$$

so $\overline{X}$ also has a $Cauchy$ distribution. Combining these results gives that $Z_n = 1/\overline{X} \sim Cauchy$. The cdf of the Cauchy distribution takes the form

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi}\frac{1}{1+y^2}dy = \frac{1}{\pi}\arctan(x) + \frac{1}{2} \qquad x \in \mathbb{R}$$

and hence

$$P[|Z_n| \leq c] = F_X(c) - F_X(-c) = \frac{1}{\pi}\arctan(c) - \frac{1}{\pi}\arctan(-c) = \frac{2}{\pi}\arctan(c)$$

7. If $X \sim Gamma(\alpha, \beta)$, then the mgf for X is given by

$$M_X(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha} \qquad -\beta < t < \beta,$$

say. If we take

$$Z_{nj} \sim Gamma(\alpha/n, \beta) \qquad j = 1, \ldots, n$$

as a collection of iid variables, and define Z_n as their sum, then the mgf of Z_n is

$$\left\{ \left(\frac{\beta}{\beta - t} \right)^{\alpha/n} \right\}^n = \left(\frac{\beta}{\beta - t} \right)^{\alpha}$$

and Z_n and X have the same distribution. We compute the corresponding cfs by substitution. Hence X has an infinitely divisible distribution.

8. By direct calculation

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} dF_X(x) = \int_{-\infty}^{\infty} e^{itx} d\left(\sum_{k=1}^{K} \omega_k F_k(x) \right) = \sum_{k=1}^{K} \omega_k \left(\int_{-\infty}^{\infty} e^{itx} dF_k(x) \right)$$

(writing out the sum/integral in full) so therefore

$$\varphi_X(t) = \sum_{k=1}^{K} \omega_k \varphi_k(t).$$

9. Let $\varphi_\epsilon(t)$ denote the cf associated with f_ϵ, that is

$$\varphi_\epsilon(t) = \int_{-\infty}^{\infty} e^{itx} f_\epsilon(t)\, dt.$$

Then

$$\begin{aligned} |\varphi_X(t) - \varphi_\epsilon(t)| &= \left| \int_{-\infty}^{\infty} e^{itx} (f_X(x) - f_\epsilon(x))\, dx \right| \leq \int_{-\infty}^{\infty} \left| e^{itx} (f_X(x) - f_\epsilon(x)) \right|\, dx \\ &\leq \int_{-\infty}^{\infty} |(f_X(x) - f_\epsilon(x))|\, dx < \epsilon \end{aligned}$$

by assumption, and hence $\varphi_X(t)$ and $\varphi_\epsilon(t)$ are (uniformly) arbitrarily close. But

$$\varphi_\epsilon(t) = \int_{-\infty}^{\infty} e^{itx} f_\epsilon(x)\, dx = \int_{-\infty}^{\infty} e^{itx} \left\{ \sum_{k=1}^{K} c_k \mathbb{1}_{A_k}(x) \right\}\, dx = \sum_{k=1}^{K} c_k \frac{e^{iu_k t} - e^{il_k t}}{it}$$

where $A_k = (l_k, u_k]$, say, with $l_1 = -\infty$ and $u_K = \infty$. Thus

$$|\varphi_\epsilon(t)| \leq \frac{2}{t} \sum_{k=1}^{K} c_k \longrightarrow 0 \qquad \text{as} \qquad |t| \longrightarrow \infty.$$

as $|(e^{iu_k t} - e^{il_k t})/i| \leq |e^{iu_k t}/i| + |e^{il_k t}/i| \leq 2$. Hence, as $|\varphi_\epsilon(t)| \longrightarrow 0$, $|\varphi_X(t)| \longrightarrow 0$ also.

10. We have that if $X \sim \mathit{Exponential}(1)$, which is absolutely continuous with cdf

$$F_X(x) = 1 - e^{-x} \qquad x > 0$$

and zero otherwise, has the cf that takes the form

$$\varphi_X(t) = \frac{1}{1 - it} \qquad t \in \mathbb{R}.$$

Here we have that

$$|\varphi_X(t)| = \frac{1}{\sqrt{1+t^2}}$$

and which provides the counterexample, as this function integrates to infinity on $\mathbb{R}$.

A strategy for funding such a counterexample involves (a) finding any suitable real-valued function that integrates to infinity, then (b) proving it is a cf; the second step can be achieved either by looking the table of cfs (or Fourier transforms), applying Bochner's Theorem, or checking the sufficient conditions generated by Polya's Theorem.