MATH 556 - EXERCISES 7

Not for Assessment.

1. Let Y_n and Z_n correspond to the *maximum* and *minimum* order statistics derived from an independent sample $X_1, \ldots, X_n$ from population with cdf F_X.

   (a) Suppose $X_1, \ldots, X_n \sim Uniform(0,1)$. Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$.

   (b) Suppose $X_1, \ldots, X_n$ have cdf

   $$F_X(x) = \frac{1}{1+e^{-x}} \qquad x \in \mathbb{R}$$

   Find the cdfs of Y_n and $U_n = Y_n - \log n$ and the limiting distributions of Y_n and U_n as $n \longrightarrow \infty$.

   (c) Suppose $X_1, \ldots, X_n$ have cdf

   $$F_X(x) = 1 - \frac{1}{1+2x} \qquad x > 0$$

   Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$. Find also the cdfs of $U_n = Y_n/n$ and $V_n = nZ_n$, and the limiting distributions of U_n and V_n as $n \longrightarrow \infty$.

2. Using the Central Limit Theorem, construct Normal approximations to probability distribution of a random variable X having

   (a) a Binomial distribution, $X \sim Binomial(n, \theta)$

   (b) a Poisson distribution, $X \sim Poisson(\lambda)$

   (c) a Negative Binomial distribution, $X \sim NegBinomial(n, \theta)$

   (d) a Gamma distribution, $X \sim Gamma(\alpha, \beta)$

3. Suppose $X_1, \ldots, X_n \sim Poisson(\lambda)$ are independent random variables. Let $M_n = \overline{X}_n$. Show that $M_n \stackrel{p}{\longrightarrow} \lambda$ as $n \longrightarrow \infty$. If random variable T_n is defined by $T_n = e^{-M_n}$, show that $T_n \stackrel{p}{\longrightarrow} e^{-\lambda}$, and find an approximation to the probability distribution of T_n as $n \longrightarrow \infty$.

4. For the following sequences of random variables, $\{X_n\}$, decide whether the the sequence converges in *mean-square* (rth mean for $r = 2$) or *in probability* as $n \longrightarrow \infty$.

   (a) $X_n = \begin{cases} 1 & \text{with prob. } 1/n \\ 2 & \text{with prob. } 1 - 1/n \end{cases}$

   (b) $X_n = \begin{cases} n^2 & \text{with prob. } 1/n \\ 1 & \text{with prob. } 1 - 1/n \end{cases}$

   (c) $X_n = \begin{cases} n & \text{with prob. } 1/\log n \\ 0 & \text{with prob. } 1 - 1/\log n \end{cases}$

ADVANCED MATERIAL: ALMOST SURE CONVERGENCE AND THE BOREL-CANTELLI LEMMA.

Let $\{E_n\}$ be a sequence of events in sample space Ω. Then

$$E^{(S)} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$$

is the **limsup** event of the infinite sequence; event $E^{(S)}$ occurs if and only if

- **for all** $n \geq 1$, **there exists** an $m \geq n$ such that E_m occurs, or equivalently
- **infinitely many** of the E_n occur.

Similarly, let

$$E^{(I)} = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m$$

is the **liminf** event of the infinite sequence; event $E^{(I)}$ occurs if and only if

- **there exists** $n \geq 1$, such that **for all** $m \geq n$, E_m occurs, or equivalently
- **only finitely many** of the E_n do not occur.

The Borel-Cantelli Lemma: Let $\{E_n\}$ be a sequence of events in sample space Ω. Then

(i) If

$$\sum_{n=1}^{\infty} P(E_n) < \infty, \qquad \Longrightarrow \qquad P\left(E^{(S)}\right) = 0,$$

that is,

$$P[E_n \text{ occurs infinitely often }] = 0.$$

(ii) If the events $\{E_n\}$ are **independent**

$$\sum_{n=1}^{\infty} P(E_n) = \infty \qquad \Longrightarrow \qquad P\left(E^{(S)}\right) = 1.$$

that is, $P[\ E_n \text{ occurs infinitely often }] = 1$.

Note: This result is useful for assessing almost sure convergence. For a sequence of random variables $\{X_n\}$ and limit random variable X, suppose, for $\epsilon > 0$, that $A_n(\epsilon)$ is the event

$$A_n(\epsilon) \equiv \{\omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}$$

The Borel-Cantelli Lemma says that for arbitrary $\epsilon > 0$,

(i) **if**

$$\sum_{n=1}^{\infty} P(A_n(\epsilon)) = \sum_{n=1}^{\infty} P[\,|X_n - X| \geq \epsilon\,] < \infty$$

then

$$X_n \xrightarrow{a.s.} X$$

(ii) **if**

$$\sum_{n=1}^{\infty} P(A_n(\epsilon)) = \sum_{n=1}^{\infty} P[\,|X_n - X| \geq \epsilon\,] = \infty$$

with the X_n **independent then**

$$X_n \xrightarrow{a.s.} X$$

Proof

(i) Note first that

$$\sum_{n=1}^{\infty} P(E_n) < \infty \quad \Longrightarrow \quad \lim_{n \longrightarrow \infty} \sum_{m=n}^{\infty} P(E_m) = 0.$$

because if the sum on the left-hand side is finite, then the tail-sums on the right-hand side tend to zero as $n \longrightarrow \infty$. But for every $n \geq 1$,

$$E^{(S)} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m \subseteq \bigcup_{m=n}^{\infty} E_m \qquad \therefore \qquad P\left(E^{(S)}\right) \leq P\left(\bigcup_{m=n}^{\infty} E_m\right) \leq \sum_{m=n}^{\infty} P(E_m). \quad (1)$$

Thus, taking limits as $n \longrightarrow \infty$, we have that

$$P\left(E^{(S)}\right) \leq \lim_{n \longrightarrow \infty} \sum_{m=n}^{\infty} P(E_m) = 0.$$

(ii) Consider $N \geq n$, and the union of events

$$E_{n,N} = \bigcup_{m=n}^{N} E_m.$$

$E_{n,N}$ corresponds to the collection of sample outcomes that are in *at least one* of the collections corresponding to events $E_n, ..., E_N$. Therefore, $E'_{n,N}$ is the collection of sample outcomes in Ω that are **not in any** of the collections corresponding to events $E_n, ..., E_N$, and hence

$$E'_{n,N} = \bigcap_{m=n}^{N} E'_m \quad (2)$$

Now,

$$E_{n,N} \subseteq \bigcup_{m=n}^{\infty} E_m \qquad \Longrightarrow \qquad P(E_{n,N}) \leq P\left(\bigcup_{m=n}^{\infty} E_m\right)$$

and hence, by assumption and independence,

$$\begin{aligned} 1 - P\left(\bigcup_{m=n}^{\infty} E_m\right) &\leq 1 - P\left(\bigcup_{m=n}^{N} E_m\right) = 1 - P(E_{n,N}) = P\left(E'_{n,N}\right) = P\left(\bigcap_{m=n}^{N} E'_m\right) = \prod_{m=n}^{N} P\left(E'_m\right) \\ &= \prod_{m=n}^{N} (1 - P(E_m)) \leq \exp\left\{-\sum_{m=n}^{N} P(E_m)\right\}, \end{aligned}$$

as $1 - x \leq \exp\{-x\}$ for $0 < x < 1$. Now, taking the limit of both sides as $N \longrightarrow \infty$, for fixed n,

$$1 - P\left(\bigcup_{m=n}^{\infty} E_m\right) \leq \lim_{N \longrightarrow \infty} \exp\left\{-\sum_{m=n}^{N} P(E_m)\right\} = 0$$

as, by assumption $\sum_{n=1}^{\infty} P(E_n) = \infty$. Thus, for each n, we have that

$$P\left(\bigcup_{m=n}^{\infty} E_m\right) = 1 \qquad \therefore \qquad \lim_{n \longrightarrow \infty} P\left(\bigcup_{m=n}^{\infty} E_m\right) = 1.$$

But the sequence of events $\{A_n\}$ defined for $n \geq 1$ by

$$A_n \;=\; \bigcup_{m=n}^{\infty} E_m$$

is monotone non-increasing, and hence, by continuity,

$$P\left(\lim_{n\longrightarrow\infty} A_n\right) \;=\; \lim_{n\longrightarrow\infty} P\left(A_n\right). \tag{3}$$

From (4), we have that the right hand side of equation (5) is equal to 1, and, by definition,

$$\lim_{n\longrightarrow\infty} A_n \;=\; \bigcap_{n=1}^{\infty} A_n \;=\; \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m. \tag{4}$$

Hence, combining (2), (3) and (4) we have finally that

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\right) \;=\; 1 \qquad \Longrightarrow \qquad P\left(E^{(S)}\right) = 1.$$

Interpretation and Implications

The Borel-Cantelli result is concerned with the calculation of the probability of the limsup event $E^{(S)}$ occurring for general infinite sequences of events $\{E_n\}$. From previous discussion, we have seen that $E^{(S)}$ corresponds to the collection of sample outcomes in Ω that are in **infinitely many** of the E_n collections. Alternately, $E^{(S)}$ occurs if and only if **infinitely many** $\{E_n\}$ occur. The Borel-Cantelli result tells us conditions under which $P\left(E^{(S)}\right) = 0$ or 1.

EXAMPLE : Consider the event E defined by

"E occurs" $=$ "run of 100^{100} Heads occurs in an infinite sequence of independent coin tosses"

We wish to calculate $P(E)$, and proceed as follows; consider the infinite sequence of events $\{E_n\}$ defined by

"E_n occurs" $=$ "run of 100^{100} Heads occurs in the nth block of 100^{100} coin tosses"

Then $\{E_n\}$ are independent events, and

$$P\left(E_n\right) \;=\; \frac{1}{2^{100^{100}}} > 0 \;\Longrightarrow\; \sum_{n=1}^{\infty} P(E_n) = \infty,$$

and hence by part (b) of the Borel-Cantelli result,

$$P\left(E^{(S)}\right) \;=\; P\left(\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} E_m\right) \;=\; 1$$

so that the probability that infinitely many of the $\{E_n\}$ occur is 1. But, crucially,

$$E^{(S)} \subseteq E \;\Longrightarrow\; P\left(E\right) \;=\; 1.$$

Therefore the probability that E occurs, that is that a run of 100^{100} Heads occurs in an infinite sequence of independent coin tosses, is 1.

ADVANCED EXERCISES

1.* Consider the sequence of random variables defined for $n = 1, 2, 3, \ldots$ by

$$X_n = \mathbb{1}_{[0,n^{-1})}\left(U_n\right)$$

where $U_1, U_2, \ldots$ are a sequence of independent $Uniform(0, 1)$ random variables, and $\mathbb{1}_A$ is the indicator function for set A

$$\mathbb{1}_A\left(\omega\right) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Does the sequence $\{X_n\}$ converge

(a) almost surely ?

(b) in r^{th} mean for $r = 1$?

Hint: Consider the events $A_n \equiv (X_n \neq 0)$ *for* $n = 1, 2, \ldots$.

2.* Let $Z \sim Uniform(0, 1)$, and define a sequence of random variables $\{X_n\}$ by

$$X_n = n\mathbb{1}_{[1-n^{-1},1)}(Z) \qquad n = 1, 2, \ldots$$

where, for set A

$$\mathbb{1}_A(Z) = \begin{cases} 1 & Z \in A \\ 0 & Z \notin A \end{cases}$$

that is, I_A is the indicator random variable associated with the set A.

Does the sequence $\{X_n\}$ converge in any mode to any limit random variable ? Justify your answer.

3.* Suppose, for $n = 1, 2, \ldots$, $X_n \sim Bernoulli(p_n)$ are a sequence of independent random variables where

$$P[X_n = 1] = p_n = \frac{1}{\sqrt{n}}\ .$$

Does $P[\ X_n = 1 \text{ infinitely often }] = 1$? Justify your answer.