MATH 556 - EXERCISES 5

Not for Assessment.

1. If possible, write the distribution in the Exponential Family form, and find the natural (canonical) parameterization. If the function does not specify an Exponential Family, explain why not.

   (a) The continuous $Uniform(\theta_1, \theta_2)$ distribution:

   $$f_X(x; \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \qquad \theta_1 < x < \theta_2$$

   and zero otherwise, for parameters $\theta_1 < \theta_2$.

   (b) The distribution defined by

   $$f_X(x; \theta) = \frac{-1}{\log(1-\theta)} \frac{\theta^x}{x} \qquad x = 1, 2, 3, \ldots$$

   and zero otherwise, for parameter θ, where $0 < \theta < 1$.

2. For scalar random variable X, consider a one parameter Exponential Family distribution in its natural parameterization,

   $$f_X(x; \eta) = h(x) \exp\{\eta\, T(x) - K(\eta)\}$$

   and natural parameter space $\mathcal{H}$. Suppose that $\mathcal{H}$ is an open interval in $\mathbb{R}$, so that for every $\eta \in \mathcal{H}$, there exists an $\epsilon > 0$ such that

   $$\eta' \in \mathcal{H} \quad \text{if} \quad |\eta - \eta'| < \epsilon$$

   (a) Show that the natural parameter space $\mathcal{H}$ is a convex set, that is, for $0 \leq \lambda \leq 1$,

   $$\eta_1, \eta_2 \in \mathcal{H} \qquad \Longrightarrow \qquad \lambda\eta_1 + (1-\lambda)\eta_2 \in \mathcal{H}$$

   (b) Suppose that $\eta_1, \eta_2 \in \mathcal{H}$. Find the form of the log likelihood ratio, $\ell(x; \eta_1, \eta_2)$, where

   $$\ell(x; \eta_1, \eta_2) = \log \frac{f_X(x; \eta_1)}{f_X(x; \eta_2)}.$$

3. Consider the distribution for continuous random variable X with pdf specified via the two dimensional parameter $\theta = (\psi, \gamma)$ as

   $$f_X(x; \psi, \gamma) = \mathbb{1}_{(0,\infty)}(x) \sqrt{\frac{1}{2\pi\gamma x^3}} \exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right\}$$

   for $\psi, \gamma > 0$ and

   (a) Is this a location-scale family distribution ? Justify your answer.

   (b) Is this an Exponential Family distribution ? Justify your answer.

   (c) For this model, the result concerning the expected score holds, that is

   $$\mathbb{E}_X[\mathbf{S}(X; \theta)] = \mathbf{0} \qquad (2 \times 1)$$

   where

   $$\mathbf{S}(x; \theta) = \begin{pmatrix} S_1(x; \theta) \\ \\ S_2(x; \theta) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial\psi} \log\{f_X(x; \psi, \gamma)\} \\ \dfrac{\partial}{\partial\gamma} \log\{f_X(x; \psi, \gamma)\} \end{pmatrix}$$

   Using this result, find $\mathbb{E}_X[X]$ and $\mathbb{E}_X[1/X]$