MATH 556 - EXERCISES 2

Not for assessment.

1. Suppose that X is a continuous rv with pdf f_X and characteristic function (cf) φ_X. Find $\varphi_X(t)$ if

   (a)
   $$f_X(x) = \frac{1}{2}|x| \exp\{-|x|\} \qquad x \in \mathbb{R}.$$

   (b)
   $$f_X(x) = \exp\{-x - e^{-x}\} \qquad x \in \mathbb{R}.$$

   (c)
   $$f_X(x) = \frac{1}{\cosh(\pi x)} = \frac{2}{e^{-\pi x} + e^{\pi x}} = \sum_{k=0}^{\infty} (-1)^k \exp\{-(2k+1)\pi|x|\} \qquad x \in \mathbb{R}.$$

   Integrate term by term to find the cf, and leave your answer as an infinite sum if necessary.

2. Using the inversion formulae, find pmf or pdf $f_X(x)$ for the following cfs defined for $t \in \mathbb{R}$.

   (a)
   $$\varphi_X(t) = \mathbb{1}_{(-1,1)}(t)(1 - |t|)$$

   where as previously defined, the function $\mathbb{1}_A(x)$ is the **indicator function** for set A

   $$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} .$$

   (b)
   $$\varphi_X(t) = \frac{2(1 - \cos t)}{t^2}$$

   Hint: recall the result in part (a).

   (c) For some parameter $\theta > 0$
   $$\varphi_X(t) = \cos(\theta t).$$

3. By considering derivatives at $t = 0$, and the implied moments, assess whether the function

   $$\varphi(t) = \frac{1}{1 + t^4}$$

   is a valid cf for a pmf or pdf.

4. Suppose that cf $\varphi_X(t)$ takes the form

   $$\varphi_X(t) = \frac{1}{2}\left(\cos(t) + \cos(\pi t)\right).$$

   (a) Is the distribution of X (absolutely) continuous ? Justify your answer.
   (b) Comment on the finiteness or existence of $\mathbb{E}_X[X^r]$ for $r \geq 1$.

5. Consider the function
   $$\varphi(t) = \frac{1}{(1 + 2t^2 + t^4)}.$$

   Assess whether this function is a valid cf, and if it is valid, describe in as much detail as possible the distribution to which it corresponds.

6. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed Cauchy rvs each with

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad x \in \mathbb{R} \qquad\qquad \varphi_X(t) = \exp\{-|t|\} \qquad t \in \mathbb{R}.$$

Let continuous random variable Z_n be defined by

$$Z_n = \frac{1}{\overline{X}} = \frac{n}{\sum\limits_{j=1}^{n} X_j}.$$

Find an expression for $P_{Z_n}[|Z_n| \leq c]$ for constant $c > 0$.

7. A probability distribution for rv X is termed *infinitely divisible* if, for all positive integers n, there exists a sequence of independent and identically distributed rvs $Z_{n1}, \ldots, Z_{nn}$ such that X and

$$Z_n = \sum_{j=1}^{n} Z_{nj}$$

have the same distribution, that is, the characteristic function of X can be written $\varphi_X(t) = \{\varphi_Z(t)\}^n$ for some characteristic function φ_Z. Show that the $Gamma(\alpha, \beta)$ distribution is infinitely divisible.

8. Suppose that X has a finite mixture distribution with cdf

$$F_X(x) = \sum_{k=1}^{K} \omega_k F_k(x) \qquad x \in \mathbb{R}$$

where K is a positive integer, $F_1, \ldots, F_K$ are distinct cdfs, and $\omega_1, \ldots, \omega_K$ satisfy

$$0 < \omega_k < 1 \quad \text{for all } k \qquad \sum_{k=1}^{K} \omega_k = 1.$$

Find the cf for X in terms of the cfs corresponding to $F_1, \ldots, F_K$.

9. Prove that if f_X is pdf for a continuous random variable, then $|\varphi_X(t)| \longrightarrow 0$ as $|t| \longrightarrow \infty$.
Hint: Use the fact that f_X can be approximated to arbitrary accuracy by a step-function; for each $\epsilon > 0$, there exists a step-function $f_\epsilon(x)$ defined (for some $K = K(\epsilon)$) as

$$f_\epsilon(x) = \sum_{k=1}^{K} c_k \mathbb{1}_{A_k}(x)$$

where $c_k, k = 1, \ldots, K$ are real constants, and $A_1, \ldots, A_K$ form a partition of $\mathbb{R}$, such that

$$\int_{-\infty}^{\infty} |f_X(x) - f_\epsilon(x)|\, dx < \epsilon.$$

10. A sufficient condition for a distribution defined on $\mathbb{R}$ to be (absolutely) continuous is that

$$\int_{-\infty}^{\infty} |\varphi(t)|\, dt < \infty.$$

where $|\varphi(t)|$ is the modulus of the complex-valued quantity $\varphi(t)$. By finding a suitable counterexample, show that this is not a necessary condition for (absolute) continuity. That is, find an (absolutely) continuous distribution with cf $\varphi(t)$ for which

$$\int_{-\infty}^{\infty} |\varphi(t)|\, dt = \infty.$$

Hint: check the distributions formula sheet, and deduce the cfs from the mgfs.