MATH 556 - EXERCISES 1

Not for assessment.

1. The absolutely continuous cdf F_X has density

$$f_X(x) = c\exp\{-\lambda|x-\theta|\} \qquad x \in \mathbb{R}$$

for parameters $\lambda > 0$ and $\theta \in \mathbb{R}$.

Compute

(a) The constant c;
(b) The cdf F_X;
(c) The quantile function Q_X;
(d) The *expectation* $\mathbb{E}_X[X]$, defined by

$$\mathbb{E}_X[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

(e) The *variance* $\text{Var}_X[X]$, defined by

$$\text{Var}_X[X] = \mathbb{E}_X[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

where $\mu = \mathbb{E}_X[X]$.

2. Find the quantile function, $Q_X(p)$ for $0 < p < 1$, for the following cases:

(a) X is distributed as $Weibull(3,2)$ (given on the Distributions Formula Sheet).

(b) X has the discrete distribution with

$$f_X(x) = c\mathbb{1}_{\{1,2,\ldots,10\}}(x) \qquad x \in \mathbb{R}$$

for some c to be determined.

(c) X has the distribution with

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{2} & 0 \leq x < 1 \\ \dfrac{3}{4} & x = 1 \\ (1 - c\exp(-(x-1))) & x > 1 \end{cases}$$

for some c to be determined.

3. Suppose that X has a standard Normal distribution, $X \sim Normal(0,1)$ with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} \qquad x \in \mathbb{R}$$

For a transformation $Y = g(X)$ of X for some real-valued function $g(\cdot)$, we may compute the cdf of Y using the identity

$$F_Y(y) = P_Y[Y \leq y] = P_X[g(X) \leq y] \qquad y \in \mathbb{R}.$$

Compute and sketch (or plot) the pdfs of the following transformed random variables

(a) $Y_1 = X^2$
(b) $Y_2 = |X|$
(c) $Y_3 = 2X - X^2$
(d) $Y_4 = F_X(X)$, where $F_X(.)$ is the cdf of X.
Hint: you do not need to compute the explicit form of $F_X(x)$ in this case, simply use the fact that it is a strictly increasing function.

4. Show that if $X \sim Pareto(\theta, \alpha)$ (parameterized as on the distributions handout), then

$$X \stackrel{d}{=} g(Z)$$

where $Z \sim Exponential(1)$, and $g(.)$ is some transformation to be found.

5. Suppose that X is a continuous random variable with support $\mathbb{X} = \mathbb{R}$, and with cdf $F_X(x)$. Suppose that Y is a transformed variable given by

$$Y = \{F_X(X)\}^k$$

for positive integer k. Find the expectation of Y.
H*int: note that $0 \leq Y \leq 1$ with probability 1; use the result*

$$\mathbb{E}_Y[Y] = \int_0^1 y f_Y(y)\, dy \equiv \int_{-\infty}^{\infty} \{F_X(x)\}^k f_X(x)\, dx.$$

6. A collection of d random variables $X = (X_1, \ldots, X_d)$ are termed *independent* if

$$f_{X_1,\ldots,X_d}(x_1,\ldots,x_d) = \prod_{j=1}^{d} f_{X_j}(x_j) \qquad \text{for all} \quad (x_1,\ldots,x_d) \in \mathbb{R}^d.$$

Consider the joint pdf defined on the unit cube $(0,1)^3$.

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = c(1 - \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3))$$

and zero otherwise, for some constant c.

(a) Are (X_1, X_2) independent ?
(b) Are (X_1, X_2, X_3) independent ?

Justify your answers by considering the relevant marginal distributions.