MATH 556 - ASSIGNMENT 1 – SOLUTIONS

1. *Consider the pdf, $f_X(x)$, for continuous random variable X that takes the form*

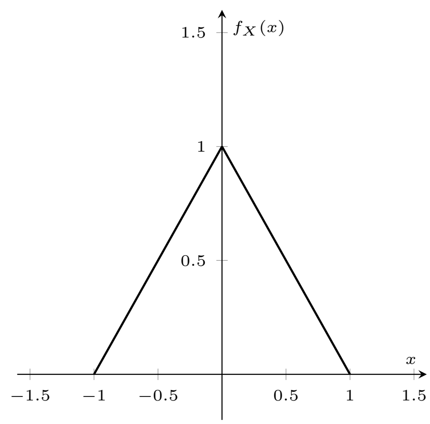
$$f_X(x) = \mathbb{1}_{(-1,0)}(x)(c+x) + \mathbb{1}_{[0,1)}(x)(c-x) \quad x \in \mathbb{R}$$

for some constant c, where $\mathbb{1}_A(x)$ is the indicator function for set A

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases} .$$

(a) *Find the value of c.*

When $x = 0$, the pdf takes the value c, so the distribution is a 'triangular' distribution, bounded by the lines $y = c + x$ and $y = c - x$. As the pdf must integrate to 1, and the area of the triangle is c, we must have $c = 1$. The pdf is depicted below



$f_X(x)$

2 Marks

(b) *Find the form of cdf, $F_X(x)$.*

For the cdf, we compute

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt = \int_{-1}^{x} f_X(t)\, dt$$

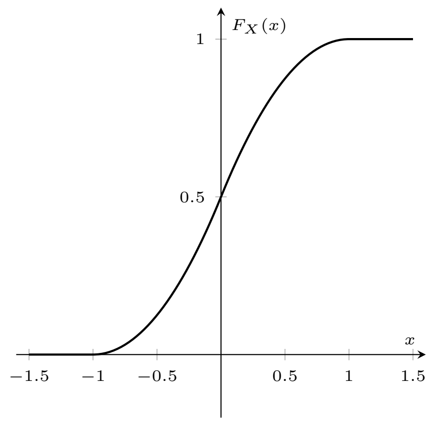
There are two cases to consider: for $-1 < x < 0$,

$$F_X(x) = \int_{-1}^{x} (1+t)\, dt = \frac{1}{2} + x + \frac{x^2}{2}$$

whereas for $0 < x < 1$

$$F_X(x) = \int_{-1}^{x} f_X(t)\, dt = \int_{-1}^{0} f_X(t)\, dt + \int_{0}^{x} f_X(t)\, dt$$

$$= \frac{1}{2} + x - \frac{x^2}{2}$$



$F_X(x)$

3 Marks

(c) *Find the quantile function, $Q_X(p)$ for $0 < p < 1$.*

When $0 < p < 1/2$, we can invert the cdf directly: we solve $F_X(x) = p$ for x, that is

$$p = \frac{1}{2} + x + \frac{x^2}{2}$$

or

$$x^2 + 2x + (1 - 2p) = 0.$$

The solutions to this quadratic are

$$x = \frac{-2 \pm \sqrt{4 - 4(1 - 2p)}}{2} = -1 \pm \sqrt{2p}$$

so we must take the larger root and deduce that for $0 < p < 1/2$,

$$Q_X(p) = -1 + \sqrt{2p}.$$

For $1/2 < p < 1$, we solve

$$p = \frac{1}{2} + x - \frac{x^2}{2}$$

or

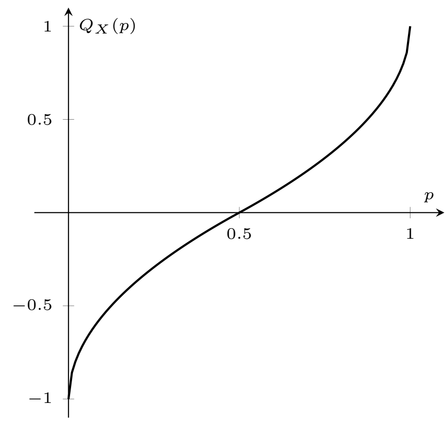
$$x^2 - 2x - (1 - 2p) = 0.$$

The solutions to this quadratic are

$$x = \frac{2 \pm \sqrt{4 + 4(1 - 2p)}}{2} = 1 \pm \sqrt{2(1 - p)}$$

so we must take the smaller root and deduce that for $1/2 < p < 1$,

$$Q_X(p) = 1 - \sqrt{2(1 - p)}.$$



$Q_X(p)$

3 Marks

(d) *Find the expected value of the quantity $|X|$, $\mathbb{E}_X[|X|]$, defined by*

$$\mathbb{E}_X[|X|] = \int_{-\infty}^{\infty} |x| f_X(x)\, dx$$

We have that

$$\begin{aligned}
\mathbb{E}_X[|X|] = \int_{-\infty}^{\infty} |x| f_X(x)\, dx &= \int_{-1}^{1} |x| f_X(x)\, dx \\
&= \int_{-1}^{0} -x(1+x)\, dx + \int_{0}^{1} x(1-x)\, dx \\
&= \left[-\frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{0}^{1} \\
&= \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{1}{3} \\
&= \frac{1}{3}
\end{aligned}$$

2 Marks

Hint: first sketch the pdf.

2. *Suppose that $X = (X_1, X_2)$ where X_1 and X_2 are independent continuous random variables where each has a $Uniform(0,1)$ distribution. Find, by integrating the joint pdf over a suitable set B in $\mathbb{R}^2$, the probability*

$$P_X[X_1 > 2X_2].$$

Hint: first identify the set B.

We have by independence that

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = 1 \qquad 0 < x_1, x_2 < 1$$

and zero otherwise. We must integrate the joint pdf over the set

$$B = \{(x_1, x_2) : 0 < x_1, x_2, 1, x_1 > 2x_2\}.$$

Thus

$$P_X[X_1 > 2X_2] = \int_0^1 \int_0^{x_1/2} dx_2 \, dx_1 = \int_0^1 \frac{x_1}{2} \, dx_1 = \frac{1}{4}$$

4 Marks

3. *One fair red die and one fair blue die are rolled, with the results of the rolls independent events. Let X_1 denote the sum of the scores on the two dice, and let X_2 denote the value equal to the score on the red die minus the score on the blue die.*

   Are X_1 and X_2 independent random variables ? Justify your answer.

   X_1 and X_2 are not independent; we have that

   $$P_X[X_1 > X_2] = 1$$

   so that the support of the joint pmf is not a Cartesian product, and so the two variables cannot be independent.

3 Marks

4. *Suppose that X is a continuous random variable with distribution specified so that*

   $$P_X[X > x] = \begin{cases} 1 & x < 0 \\ \exp\{2(1 - e^x)\} & x \geq 0 \end{cases}$$

   Find the pdf of random variable Y, where

   $$Y = \exp\{X\}.$$

   We have that for $x > 0$

   $$F_X(x) = 1 - \exp\{2(1 - e^x)\}$$

   so therefore for $y > 1$

   $$F_Y(y) = P_Y[Y \leq y] = P_X[e^X < y] = P_X[X < \log y] = F_X(\log y) = 1 - \exp\{2(1 - y)\}.$$

   Hence

   $$f_Y(y) = 2\mathbb{1}_{(1,\infty)}(y) \exp\{2(1 - y)\} \qquad y \in \mathbb{R}$$

3 Marks