Name: ______________________________ ID: ______________________________

McGill University

Faculty of Science

Final Examination

MATH 556: Mathematical Statistics I

Examiner: Professor J. Nešlehová | Date: Thursday, December 6, 2012

Associate Examiner: Professor D.A. Stephens | Time: 2:00 P.M. – 5:00 P.M.

Instructions

- **This is a closed book exam.**
- **The exam comprises one title page, three pages of questions and two pages of formulas.**
- **Answer all six questions in the examination booklets provided.**
- **Calculators and translation dictionaries are permitted.**
- **A formula sheet is provided.**

Good Luck!

Problem 1

The Student t_ν distribution with $\nu > 0$ degrees of freedom has density

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}, \quad x \in \mathbb{R}.$$

(a) Determine the distribution of T^2 if T is Student t_ν. Can you recognize it? **(5 Marks)**

(b) Let X be a random variable with density f_X and Y a strictly positive random variable with density f_Y which is independent of X. Prove that $W = X/\sqrt{Y}$ has density

$$f_W(w) = \int_{-\infty}^{\infty} f_X(wz) f_Y(z^2) 2z^2 \mathrm{d}z.$$

(5 Marks)

(c) Suppose that X is Normal$(0,1)$ and νY is χ^2_ν. Verify that $X/\sqrt{Y}$ is Student t_ν. **(4 Marks)**

(d) Suppose that T is a Student t_ν random variable with $\nu > 2$. Show that $\mathrm{E}(T^2) \geq 1$.

(5 Marks)

(e) Let $\bar{X}_n$ and S_n^2 be, respectively, the sample mean and the sample variance of a random sample $X_1, \ldots, X_n$. State the conditions under which $(\sqrt{n}\bar{X}_n)/\sqrt{S_n^2}$ is Student t_{n-1}.

(4 Marks)

Problem 2

Let X and Y be independent Exponential random variables, $X \sim$ Exponential(λ) and $Y \sim$ Exponential(μ). Imagine that it is impossible to observe X and Y directly. Instead, you observe the random variables Z and W, where

$$Z = \min(X, Y) \quad \text{and} \quad W = \begin{cases} 1 & \text{if} \quad Z = X, \\ 0 & \text{if} \quad Z = Y. \end{cases}$$

(a) Find the joint distribution of Z and W. **(4 Marks)**

(b) Find the marginal distributions of Z and W. **(4 Marks)**

(c) Determine the conditional distribution of Z given $W = i$, $i = 0, 1$ and show that Z and W are independent. **(4 Marks)**

Problem 3

(a) Define the exponential family of distributions, and explain what it means to say that the exponential family is (i) a strict k-parameter exponential family; (ii) in its canonical or natural parametrization. **(2 Marks)**

(b) For the following families of distributions, assess whether the family is an exponential family. Where possible, write down the canonical or natural parameter space.

(i) The Inverse Gamma family with density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x}, \quad x > 0$$

and parameters $\alpha > 0$ and $\beta > 0$.

(ii) The Gumbel family with density

$$f(x) = \frac{1}{\beta} \exp\left\{ -\frac{x-\mu}{\beta} - e^{-(x-\mu)/\beta} \right\}, \quad x \in \mathbb{R}$$

and parameters $\mu \in \mathbb{R}$ and $\beta > 0$. **(8 Marks)**

(c) Compute the covariance between $1/X$ and $\ln(1/X)$, where X is an Inverse Gamma random variable with parameters $\alpha > 0$ and $\beta > 0$. **(5 Marks)**

Problem 4

Suppose that N is a Poisson(λ) random variable, independent of the i.i.d. sequence $X_1, X_2, \ldots$ of Gamma($\alpha, 1$) random variables, $\alpha, \lambda > 0$. Let S_N be given by

$$S_N = \sum_{i=1}^{N} X_i.$$

(a) Compute the expectation and variance of S_N. **(5 Marks)**

(b) Show that the moment generating function of S_N is given by

$$M(t) = \exp\left\{ \lambda \left(\frac{1}{1-t} \right)^\alpha - 1 \right\}.$$

For which values of t does it exist? **(5 Marks)**

(c) Compute the saddlepoint approximation to the density of S_N. **(5 Marks)**

(d) Determine the function g for which $\mathrm{E}\{S_N - g(N)\}^2$ is minimized. **(5 Marks)**

Problem 5

Let E_1, E_2, E_3 be independent Exponential(1) random variables and denote by

$$E_{(1)} \le E_{(2)} \le E_{(3)}$$

the corresponding order statistics.

(a) Prove that the variables

$$S_1 = 3E_{(1)}, \quad S_2 = 2\{E_{(2)} - E_{(1)}\}, \quad S_3 = E_{(3)} - E_{(2)}$$

are independent and Exponential(1). **(5 Marks)**

(b) Compute the marginal densities of $E_{(1)}$, $E_{(2)}$ and $E_{(3)}$. **(5 Marks)**

(c) Prove that for all $i \neq j$, $i, j \in \{1, 2, 3\}$,

$$\text{cov}(E_{(i)}, E_{(j)}) > 0.$$

(Computing the joint distribution of $(E_{(i)}, E_{(j)})$ *is NOT necessary).* **(5 Marks)**

Problem 6

(a) Suppose that $X_1, \ldots, X_n$ are Poisson(λ_X) and $Y_1, \ldots, Y_n$ are Poisson(λ_Y), with all variables mutually independent. Consider the random variable M_n defined by $M_n = \bar{X}_n + \bar{Y}_n$, where

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

are the means of the two samples, respectively. Verify the convergence in probability of M_n to μ, for an appropriately chosen constant μ. **(5 Marks)**

(b) For the random variables in part (a), for large n, find a Normal approximation to the distribution of the random variable Z_n defined by $Z_n = \exp(-\bar{X}_n)$. **(5 Marks)**

(c) Suppose that the random variable X has a Poisson distribution with parameter λ. Consider the standardized random variable, Z_λ, defined by

$$Z_\lambda = \frac{X - \lambda}{\sqrt{\lambda}}.$$

Prove that, as $\lambda \to \infty$, Z_λ converges in distribution to a Normal(0, 1) random variable Z.

(5 Marks)

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(r,p)$	$\{0,1,2,...\}$	$r \in \mathbb{Z}^+, p \in (0,1)$	$\binom{r+x-1}{x}p^r(1-p)^x$		$\dfrac{r(1-p)}{p}$	$\dfrac{r(1-p)}{p^2}$	$\left(\dfrac{p}{1-e^t(1-p)}\right)^r$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \quad \alpha > 0$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma\mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2\mathrm{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard: $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard: $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard: $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Normal(\mu,\sigma^2)$ (standard: $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
χ^2_ν	$\mathbb{R}^+$	$\nu\in\mathbb{N}$	$\dfrac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}}x^{(\nu/2)-1}e^{-x/2}$		ν	2ν	$(1-2t)^{-\nu/2}$
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	