Name: ______________________________ ID: ______________________________

McGill University

Faculty of Science

Final Examination

MATH 556: Mathematical Statistics I

Examiner: Professor J. Nešlehová

Associate Examiner: Professor D. A. Stephens

Date: Friday, December 9, 2010

Time: 9:00 A.M. – 12:00 P.M.

Instructions

- **This is a closed book exam.**
- **The exam comprises one title page, three pages of questions and two pages of formulas.**
- **Answer all six questions in the examination booklets provided.**
- **Calculators and translation dictionaries are permitted.**
- **A formula sheet is provided.**

Good Luck!

Problem 1

Suppose that U is a uniform random variable on the interval $(0,1)$.

(a) Find the probability density function of the random variable

$$X = \mu - \beta \ln\{-\ln(U)\},$$

where $\mu \in \mathbb{R}$ and $\beta > 0$ are fixed parameters. **(5 marks)**

(b) Prove that the moment generating function of X is of the form

$$M_X(t) = e^{\mu t}\,\Gamma(1 - \beta t).$$

For which values of t does it exist? **(4 marks)**

(c) Let Y be an arbitrary random variable with moment generating function M_Y. Show that

$$\mathrm{E}(Y) = S_Y'(t), \quad \mathrm{Var}(Y) = S_Y''(t),$$

where $S_Y(t) = \ln\{M_Y(t)\}$. **(4 marks)**

(d) Compute the expectation and variance of X from part (a). Use the fact that $\Gamma'(1) = -\gamma$ and $\Gamma''(1) = \pi^2/6 + \gamma^2$, where $\gamma \approx 0.57722$ is the Euler–Mascheroni constant. **(4 marks)**

Problem 2

Let (X, Y) be a random pair of independent, standard normal random variables. Further, let R and Θ denote the polar coordinates of (X, Y).

(a) Show that R has the so-called Rayleigh distribution with density

$$f_R(r) = re^{-r^2/2}, \quad 0 < r < \infty.$$

(5 marks)

(b) Show that R and Θ are independent. **(5 marks)**

(c) Derive the distribution of X/Y. What can you say about its moment generating function? **(6 marks)**

Problem 3

The inverse Gaussian distribution with parameters $\chi > 0$ and $\psi > 0$ has probability density function

$$f(x|\chi,\psi) = \frac{\exp\sqrt{\chi\psi}}{\sqrt{2\pi x^3}}\sqrt{\chi}\exp\left\{-\frac{1}{2}(\chi x^{-1} + \psi x)\right\}, \quad x > 0.$$

(a) Show that the family $f(x|\chi,\psi)$ is an exponential family. Determine the natural parametrization and the natural parameter space. **(4 marks)**

(b) Derive expressions for the expectation and variance of random variables $t_1(X), \dots, t_k(X)$ for a k-parameter exponential family with canonical parameters $\eta_1, \dots, \eta_k$ and functions $t_1(x), \dots, t_k(x)$ in the usual representation. **(4 marks)**

(c) Suppose that X is an inverse Gaussian random variable with parameters $\chi > 0$ and $\psi > 0$. Compute $\mathrm{E}(1/X)$ and $\mathrm{Var}(1/X)$. **(4 marks)**

(d) Prove that if X is an inverse Gaussian random variable with parameters $\chi > 0$ and $\psi > 0$,

$$\mathrm{cov}(X, 1/X) = -\frac{1}{\sqrt{\chi\psi}}.$$

(4 marks)

(e) List three pitfalls of the linear correlation coefficient. **(3 marks)**

Problem 4

(a) Let X be a random variable such that $X|M = m$ is $\mathcal{N}(m, \sigma^2)$ where $M \sim \mathcal{N}(\mu, \tau^2)$. Determine the distribution of X. **(4 marks)**

(b) Prove that for any three variables X, Y and Z with finite variances,

$$\mathrm{Cov}(X,Y) = \mathrm{E}(\mathrm{Cov}(X,Y|Z)) + \mathrm{Cov}(\mathrm{E}(X|Z), \mathrm{E}(Y|Z))$$

(4 marks)

(c) Let X_1 and X_2 be random variables such that $X_i|M = m$ is $\mathcal{N}(m, \sigma^2)$ for $i = 1, 2$ where $M \sim \mathcal{N}(\mu, \tau^2)$. Suppose further that X_1 and X_2 are independent given $M = m$. Compute the correlation between X_1 and X_2. **(4 marks)**

(d) Are X_1 and X_2 from part (c) (marginally) independent? Justify your answer. **(4 marks)**

Problem 5

Let $X_1, \ldots, X_n$ be a random sample from the Poisson distribution with parameter $\lambda > 0$. Consider

$$Y_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i(X_i - 1)}$$

and recall, without proof, the MGF and the mean and variance of the Poisson distribution.

(a) Determine the exact distribution of $\overline{X}_n = (X_1 + \cdots + X_n)/n$. **(4 marks)**

(b) Show that $\text{Var}\{X_1(X_1 - 1)\} = 4\lambda^3 + 2\lambda^2$. **(4 marks)**

(c) Prove that $\text{E}(Y_n) \le \lambda$. You can use any inequality proved in class. **(4 marks)**

(d) Prove that $Y_n \to \lambda$ in probability. **(4 marks)**

(e) Show how the distribution of Y_n can be approximated in terms of the normal distribution when n is large. **(4 marks)**

Problem 6

Consider i.i.d. random variables $X_1, X_2, \ldots$ with density f and distribution function F.

(a) Prove that the distribution function of $Y_n = \min(X_1, \ldots, X_n)$ equals $1 - \{1 - F(x)\}^n$ for all $x \in \mathbb{R}$. **(3 marks)**

(b) Suppose that there exists $a \in \mathbb{R}$ such that the density f satisfies $f(x) = 0$ for $x \in (0, a)$ and $f(x) > 0$ for $x \in [a, \infty)$ (in particular, $f(a) > 0$). Prove that $n(Y_n - a)$ converges in distribution to an exponential random variable with parameter $f(a)$. **(5 marks)**

(c) Under the conditions of part (b), prove that $Y_n \to a$ in probability. **(4 marks)**

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(r,p)$	$\{0,1,2,...\}$	$r \in \mathbb{Z}^+, p \in (0,1)$	$\binom{r+x-1}{x}p^r(1-p)^x$		$\dfrac{r(1-p)}{p}$	$\dfrac{r(1-p)}{p^2}$	$\left(\dfrac{p}{1-e^t(1-p)}\right)^r$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \quad \alpha > 0$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu + \sigma\mathrm{E}_{f_X}[X] \qquad \mathrm{Var}_{f_Y}[Y] = \sigma^2\mathrm{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\frac{1}{\beta-\alpha}$	$\frac{x-\alpha}{\beta-\alpha}$	$\frac{(\alpha+\beta)}{2}$	$\frac{(\beta-\alpha)^2}{12}$	$\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
χ^2_ν	$\mathbb{R}^+$	$\nu\in\mathbb{N}$	$\frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}}x^{(\nu/2)-1}e^{-x/2}$		ν	2ν	$(1-2t)^{-\nu/2}$
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\frac{\theta}{\theta+x}\right)^\alpha$	$\frac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	