McGill University
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MATH 556

MATHEMATICAL STATISTICS I

SOLUTIONS

1. (a) From first principles (univariate transformation theorem also acceptable): for $y \in \mathbb{R}$

$$F_Y(y) = P[Y \leq y] = \text{Pr}\left[\log\left(\frac{X}{1-X}\right) \leq y\right] = \text{Pr}\left[X \geq \frac{e^y}{1+e^y}\right] = \frac{e^y}{1+e^y}$$

and therefore

$$f_Y(y) = \frac{e^y}{(1+e^y)^2} \qquad y \in \mathbb{R}$$

and zero otherwise. By inspection this expectation is finite, and the pdf is symmetric around zero, so the expectation is equal to zero.

6 MARKS

(b) From first principles (univariate transformation theorem also acceptable): for $z \in (0, 1/4)$

$$F_Z(z) = \text{Pr}[Z \leq z] = \text{Pr}[X(1-X) \leq z] = \text{Pr}[x_1(z) \geq X \geq x_2(z)]$$

where $x_1(z)$ and $x_2(z)$ are the roots of the quadratic

$$x^2 - x + z = 0$$

that is

$$x_1(z) = \frac{1-\sqrt{1-4z}}{2} \qquad x_2(z) = \frac{1+\sqrt{1-4z}}{2}.$$

Hence

$$F_Z(z) = \sqrt{1-4z} \qquad 0 < z < 1/4.$$

and therefore

$$f_Y(y) = \frac{2}{\sqrt{1-4z}} \qquad 0 < z < 1/4$$

and zero otherwise. For the expectation, using the Beta integral

$$E_{f_Z}[Z] = E_{f_X}[X(1-X)] = \int_0^1 x(1-x)\, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

6 MARKS

(c)

$$\text{Pr}\left[X_1 X_2 > \frac{1}{2}\right] = \int_{1/2}^{1}\int_{1/(2x_1)}^{1} dx_2\, dx_1 = \int_{1/2}^{1}(1 - 1/(2x_1))\, dx_1 = \left[x - \frac{1}{2}\log x_1\right]_{1/2}^{1}$$

Hence

$$\text{Pr}\left[X_1 X_2 > \frac{1}{2}\right] = \left(1 - \frac{1}{2}\log 1\right) - \left(\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}\log 2$$

As the distributions of X_1 and $1 - X_1$ are identical, we also have

$$\text{Pr}\left[(1-X_1)(1-X_2) > \frac{1}{2}\right] = \frac{1}{2} - \frac{1}{2}\log 2$$

8 MARKS

2. (a) Using the multivariate transformation theorem

(a) We have that $\mathbb{Z}^{(2)} \equiv \mathbb{R} \times \mathbb{R}$, and

$$g_1(t_1, t_2) = \frac{t_1}{t_2} \qquad\qquad g_2(t_1, t_2) = t_1 + t_2$$

(b) Inverse transformations:

$$\left.\begin{array}{l} X_1 = \dfrac{Z_1}{Z_2} \\ X_2 = Z_1 + Z_2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} Z_1 = \dfrac{X_1 X_2}{1 + X_1} \\ Z_2 = \dfrac{X_2}{1 + X_1} \end{array}\right.$$

and thus

$$g_1^{-1}(t_1, t_2) = t_1 t_2/(1 + t_1) \qquad\qquad g_2^{-1}(t_1, t_2) = t_2/(1 + t_1)$$

(c) Range: straightforwardly we have that $\mathbb{X}^{(2)} \equiv \mathbb{R} \times \mathbb{R}$

(d) The Jacobian for points $(x_1, x_2) \in \mathbb{Y}^{(2)}$ is

$$D_{x_1,x_2} = \left[\begin{array}{cc} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \end{array}\right] = \left[\begin{array}{cc} \dfrac{x_2}{(1+x_1)^2} & \dfrac{x_1}{(1+x_1)} \\ -\dfrac{x_2}{(1+x_1)^2} & \dfrac{1}{(1+x_1)} \end{array}\right] \Rightarrow |J(x_1, x_2)| = \frac{|x_2|}{(1+x_1)^2}$$

(e) For the joint pdf we have for $(x_1, x_2) \in \mathbb{Y}^{(2)}$, bx independence of Z_1 and Z_2

$$f_{X_1,X_2}(x_1, x_2) = f_{Z_1,Z_2}\left(\frac{x_1 x_2}{1+x_1}, \frac{x_2}{1+x_1}\right) \times \frac{|x_2|}{(1+x_1)^2} = \frac{1}{2\pi}\frac{|x_2|}{(1+x_1)^2}\exp\left\{-\frac{1}{2}\left[\frac{x_2^2(1+x_1^2)}{2(1+x_1)^2}\right]\right\}$$

8 MARKS

(b) To get the marginal for X_1, we integrate out X_2;

$$\begin{aligned} f_{X_1}(x_1) &= \frac{1}{2\pi}\frac{1}{(1+x_1)^2}\int_{-\infty}^{\infty} |x_2| \exp\left\{-\frac{x_2^2(1+x_1^2)}{2(1+x_1)^2}\right\} dx \\ &= \frac{1}{\pi}\frac{1}{(1+x_1)^2}\int_{0}^{\infty} x_2 \exp\left\{-\frac{x_2^2(1+x_1^2)}{2(1+x_1)^2}\right\} dx \\ &= \frac{1}{\pi}\frac{1}{(1+x_1)^2}\left[-\frac{(1+x_1)^2}{(1+x_1^2)}\exp\left\{-\frac{x_2^2(1+x_1^2)}{2(1+x_1)^2}\right\}\right]_0^{\infty} = \frac{1}{\pi}\frac{1}{1+x_1^2} \end{aligned}$$

so $X_1 \sim Cauchy$.

4 MARKS

(c) The covariance between random variables Y_1 and Y_2 is

$$Cov_{f_{Y_1,Y_2}}[Y_1, Y_2] = E_{f_{Y_1,Y_2}}[Y_1 Y_2] - E_{f_{Y_1}}[Y_1]E_{f_{Y_2}}[Y_2] \equiv E_{f_{Z_1}}[Z_1^5] - E_{f_{Z_1}}[Z_1^2]E_{f_{Z_1}}[Z_1^3] = 0$$

as the odd moments of the standard normal are zero.

4 MARKS

(d) Find the mgf of V is

$$M_V(t) = E_{f_V}[e^{tV}] = E_{f_{Z_1,Z_2}}[\exp\{t(\alpha Z_1 + \beta Z_2)\}] = M_{Z_1}(\alpha t)M_{Z_2}(\beta t) = \exp\{(\alpha^2 + \beta^2)t^2/2\}$$

4 MARKS

3. (a) By inspection

$$C_X(t) = E_{f_X}[e^{itX}] = \frac{1}{2\sigma}\int_{-\infty}^{\infty} e^{itx}\lambda e^{-|x/\sigma|}\ dx$$

But f_X is symmetric about zero, so

$$C_X(t) = \frac{1}{\sigma}\int_0^{\infty} \cos(tx)e^{-x/\sigma}\ dx = \int_0^{\infty} \cos(sy)e^{-y}\ dy$$

where $s = \sigma t$, after changing from x to $y = x/\sigma$. Integrating by parts yields

$$C_X(t) = \frac{1}{1+\sigma^2 t^2}$$

as

$$\begin{aligned} C_X(t) &= \int_0^{\infty} \cos(ty)e^{-y}\ dy = \left[-\cos(ty)e^{-y}\right]_0^{\infty} - \int_0^{\infty} t\sin(ty)e^{-y}\ dy \\ &= 1 - t\left[\sin(ty)e^{-y}\right]_0^{\infty} - t\int_0^{\infty} t\cos(ty)e^{-y}\ dy = 1 - t^2 C_X(t) \end{aligned}$$

8 MARKS

(b) (i) $X_1, \ldots, X_n$ are continuous random variables, as $|C_X(t)| \longrightarrow 0$ as $t \longrightarrow \infty$

2 MARKS

(ii) For the distribution to be infinitely divisible, the function

$$\{\exp\{-|t|^\alpha\}\}^{1/n} = \exp\left\{-\left|\frac{t}{n^{1/\alpha}}\right|^\alpha\right\}$$

needs to be a valid cf for a probability distribution, for all n. But clearly this cf is the cf of the scale transformed random variable $Y_1 = n^{1/\alpha}X_1$. So the distribution of the X variables **is** infinitely divisible.

4 MARKS

(iii) We have by elementary cf results that

$$C_{T_n}(t) = e^{a_n it}\{C_X(b_n t)\}^n = e^{a_n it}\{\exp\{-n|b_n t|^\alpha\}\} = e^{a_n it}\{\exp\{-n|b_n|^\alpha|t|^\alpha\}\}$$

Thus we must have $a_n = 0$ (as $C_X(t)$ is entirely real) and

$$b_n = n^{-1/\alpha}$$

6 MARKS

4. This question is bookwork:

(a) **Chebychev Lemma:** If X is a random variable, then for non-negative function h, and $c > 0$,

$$\Pr\left[h(X) \geq c\right] \leq \frac{E_{f_X}\left[h(X)\right]}{c}$$

Suppose that X has density function f_X which is positive for $x \in \mathbb{X}$. Let $\mathcal{A} = \{x \in \mathbb{X} : h(x) \geq c\} \subseteq X$. Then, as $h(x) \geq c$ on $\mathcal{A}$,

$$\begin{aligned} E_{f_X}\left[h(X)\right] = \int h(x) f_X(x)\, dx &= \int_{\mathcal{A}} h(x) f_X(x)\, dx + \int_{\mathcal{A}'} h(x) f_X(x)\, dx \\ &\geq \int_{\mathcal{A}} h(x) f_X(x)\, dx \\ &\geq \int_{\mathcal{A}} c f_X(x)\, dx = c\Pr\left[X \in \mathcal{A}\right] = c\Pr\left[h(X) \geq c\right] \end{aligned}$$

and the result follows.

Using the Chebychev Lemma with $h(x) = e^{tx}$ and $c = e^{at}$, for $t > 0$,

$$P\left[X \geq a\right] = P\left[tX \geq at\right] = P\left[\exp\{tX\} \geq \exp\{at\}\right] \leq \frac{E_{f_X}\left[e^{tX}\right]}{e^{at}} = \frac{M_X(t)}{e^{at}}$$

10 MARKS

(b) **MINKOWSKI'S INEQUALITY :** Suppose that X and Y are two random variables, and $1 \leq p < \infty$. Then

$$\left\{E_{f_{X,Y}}\left[|X+Y|^p\right]\right\}^{1/p} \leq \left\{E_{f_X}\left[|X|^p\right]\right\}^{1/p} + \left\{E_{f_Y}\left[|Y|^p\right]\right\}^{1/p}$$

Proof. Write

$$\begin{aligned} E_{f_{X,Y}}\left[|X+Y|^p\right] &= E_{f_{X,Y}}\left[|X+Y||X+Y|^{p-1}\right] \\ &\leq E_{f_{X,Y}}\left[|X||X+Y|^{p-1}\right] + E_{f_{X,Y}}\left[|Y||X+Y|^{p-1}\right] \end{aligned}$$

by the triangle inequality $|x+y| \leq |x| + |y|$. Using Hölder's Inequality on the terms on the right hand side, for q selected to satisfy $1/p + 1/q = 1$,

$$E_{f_{X,Y}}\left[|X+Y|^p\right] \leq \left\{E_{f_X}\left[|X|^p\right]\right\}^{1/p} \left\{E_{f_{X,Y}}\left[|X+Y|^{q(p-1)}\right]\right\}^{1/q} + \left\{E_{f_Y}\left[|Y|^p\right]\right\}^{1/p} \left\{E_{f_{X,Y}}\left[|X+Y|^{q(p-1)}\right]\right\}^{1/q}$$

and dividing through by $\left\{E_{f_{X,Y}}\left[|X+Y|^{q(p-1)}\right]\right\}^{1/q}$ yields

$$\frac{E_{f_{X,Y}}\left[|X+Y|^p\right]}{\left\{E_{f_{X,Y}}\left[|X+Y|^{q(p-1)}\right]\right\}^{1/q}} \leq \left\{E_{f_X}\left[|X|^p\right]\right\}^{1/p} + \left\{E_{f_Y}\left[|Y|^p\right]\right\}^{1/p}$$

and the result follows as $q(p-1) = p$, and $1 - 1/q = 1/p$. ∎

10 MARKS

5. (a) (i) A natural Exponential Family has $k = 1$ and takes the form

$$f_X(x|\eta) = h(x)c^\star(\eta)\exp\{\eta x\}$$

where η is the natural parameter. *5 MARKS*

(ii) Let $S(X;\eta)$ be defined by

$$S(X;\eta) = \frac{d}{d\eta}\log f_X(X;\eta) = \frac{d}{d\eta}\{\log c^\star(\eta)\} + X$$

This is the score function, and we know that $E_{f_X}[S(X;\eta)] = 0$, so therefore

$$0 = \frac{d}{d\eta}\{\log c^\star(\eta)\} + E_{f_X}[X] \qquad \therefore \qquad E_{f_X}[X] = -\frac{d}{d\eta}\{\log c^\star(\eta)\}$$

5 MARKS

(iii) By the univariate transformation theorem

$$f_Y(y|\alpha) = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{y}\right)^{\alpha+1}\exp\left\{-\frac{1}{y}\right\} \qquad x > 0$$

Thus, if $\eta = -(\alpha+1)$, we have for $x \in \mathbb{R}$

$$f_Y(y|\eta) = I_{(0,\infty)}(y)\exp\left\{-\frac{1}{y}\right\}\frac{1}{\Gamma(-1-\eta)}\exp\{\eta\log y\}$$

so this is an Exponential Family distribution with natural parameter $\eta = -(\alpha+1)$.

4 MARKS

(b) Without loss of generality, consider X_1 and X_2. By iterated expectation

$$\begin{aligned} E_{f_{X_1}}[X_1] &= E_{f_M}\left[E_{f_{X_1|M}}[X_1|M=m]\right] = E_{f_M}[M] = \mu \\ E_{f_{X_1}}[X_1^2] &= E_{f_M}\left[E_{f_{X_1|M}}[X_1^2|M=m]\right] = E_{f_M}\left[M^2+\sigma^2\right] = \mu^2+\tau^2+\sigma^2 \end{aligned}$$

so that

$$Var_{f_{X_1}}[X_1] = E_{f_{X_1}}[X_1^2] - \{E_{f_{X_1}}[X_1]\}^2 = \tau^2+\sigma^2.$$

By symmetry $E_{f_{X_2}}[X_2] = \mu$ and $Var_{f_{X_2}}[X_2] = \tau^2+\sigma^2$. Now,

$$E_{f_{X_1,X_2}}[X_1X_2] = E_{f_M}\left[E_{f_{X_1,X_2|M}}[X_1X_2|M=m]\right] = E_{f_M}\left[E_{f_{X_1|M}}[X_1|M=m]\times E_{f_{X_2|M}}[X_2|M=m]\right]$$

by conditional independence. Therefore

$$E_{f_{X_1,X_2}}[X_1X_2] = E_{f_M}[M\times M] = E_{f_M}\left[M^2\right] = \mu^2+\tau^2$$

Hence

$$Cov_{f_{X_1,X_2}}[X_1,X_2] = E_{f_{X_1,X_2}}[X_1X_2] - E_{f_{X_1}}[X_1]E_{f_{X_2}}[X_2] = \mu^2+\tau^2-\mu^2 = \tau^2$$

But this pairwise result holds for all pairs i, j, so the variance-covariance matrix takes the form

$$\Sigma_{ij} = \begin{cases} \tau^2+\sigma^2 & i = j \\ \tau^2 & i \neq j \end{cases}$$

6 MARKS

6. (a) For $0 < x < \infty$,

$$F_{X_n}(x) = \left(\frac{n\lambda x}{1+n\lambda x}\right)^n = \left(1+\frac{1}{n\lambda x}\right)^{-n} \longrightarrow \exp\{-1/(\lambda x)\} = F_X(x)$$

as $n \longrightarrow \infty$.

6 MARKS

(b) We have, from the extreme order statistics result

$$F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n$$

so that for $z \geq 1$

$$F_{Z_n}(z) = 1 - \frac{1}{z^n} \longrightarrow \begin{cases} 0 & z < 1 \\ 1 & z \geq 1 \end{cases}$$

as $n \longrightarrow \infty$, so the distribution is degenerate at $z = 1$. Now, if $U_n = (Z_n)^{\alpha_n}$, then if $\alpha_n = 1/n$,

$$F_{U_n}(u) = \mathsf{Pr}[U_n \leq u] = \mathsf{Pr}[(Z_n)^{\alpha_n} \leq u] = \mathsf{Pr}[Z_n \leq u^{1/\alpha_n}] = 1 - \frac{1}{u} \qquad u \geq 1.$$

8 MARKS

(c) From the formula sheet

$$E_{f_X}[X] = \frac{1}{\lambda} \qquad\qquad Var_{f_X}[X] = \frac{1}{\lambda^2}$$

and so from the Central Limit Theorem

$$\frac{\sqrt{n}(\overline{X}_n - 1/\lambda)}{1/\lambda} \xrightarrow{d} Z \sim N(0,1)$$

or, for large finite n

$$\overline{X}_n \sim AN(1/\lambda, 1/(n\lambda^2))$$

Now, using the Delta Method with function $g(x) = e^{-1/x}$, and $c = 1/\lambda$, we have

$$\dot{g}(x) = \frac{e^{-1/x}}{x^2} \qquad \therefore \qquad \dot{g}(c) = \lambda^2 e^{-\lambda}$$

and therefore for large finite n

$$T_n \sim AN(e^{-\lambda}, \lambda^2 e^{-2\lambda}/n).$$

6 MARKS