McGill University

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Faculty of Science

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MATH 556

MATHEMATICAL STATISTICS I

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MATHEMATICAL STATISTICS I

Final Examination

Date: 8th December 2006 Time: 2pm-5pm

This paper contains six questions.

Credit will be given for all questions attempted.

Calculators may not be used. A Formula Sheet is provided.

1. (a) Suppose that random variable X has a $Gamma\,(\alpha, 1)$ distribution, for parameter $\alpha > 0$. Find the probability density function (pdf) of random variable Y defined by

$$Y = \frac{1}{X}.$$

Find also the expectation of Y.

(b) Suppose that continuous random variable U has a cumulative distribution function (cdf), F_U, given by

$$F_U\,(u) = \exp\{-\exp\{-u\}\} \qquad u \in \mathbb{R}.$$

Find the pdf of random variable V defined by

$$V = U^2.$$

(c) Suppose that X and Y are positive, independent continuous random variables with cdfs F_X and F_Y and pdfs f_X and f_Y respectively. Show that

$$P\,[X < Y] = \iint_A f_X\,(x)\, f_Y(y)\, dx\, dy$$

for a suitably defined set A.

By considering the ranges of integration carefully, and making a change of variables, deduce that

$$P\,[X < Y] = \int_0^1 F_X\left(F_Y^{-1}\,(t)\right) dt.$$

where F_Y^{-1} is the inverse function for the 1-1 function F_Y.

2. (a) Suppose that Z_1 and Z_2 are independent random variables each having an $Exponential(1)$ distribution. Find the joint pdf of random variables Y_1 and Y_2 defined by

$$Y_1 = \frac{Z_1}{Z_1 + Z_2} \qquad Y_2 = Z_1 + Z_2.$$

(b) The joint pmf/pdf of random variables X and Y can be specified in the following way:

$$f_{X,Y}\,(x, y) = f_{X|Y}\,(x|y)\, f_Y\,(y)\,.$$

Find the marginal pmf of X

(i) if

$$X|Y = y \sim Binomial(n, y)$$

for positive integer n, and continuous random variable Y has a standard uniform distribution, $Y \sim Uniform(0, 1)$;

(ii) if

$$X|Y = y \sim Exponential(y)$$

and $Y \sim Exponential(\beta)$ for parameter $\beta > 0$.

3. (a) For a scalar random variable X, define the cumulant generating function (cgf), K_X, and show that

$$K_X^{(1)}(0) = E_{f_X}[X] \qquad K_X^{(2)}(0) = Var_{f_X}[X]$$

where $K_X^{(r)}(t)$ denotes the rth derivative of K_X with respect to t.

Assume that K_X exists, and quote without proof properties of the moment generating function (mgf), M_X.

(b) Suppose that continuous random variable X has pdf given by

$$f_X(x) = c\exp\{-\lambda|x|\} \qquad x \in \mathbb{R}$$

for parameter $\lambda > 0$, and constant c. Find the characteristic function (cf) for X, $C_X(t)$.

(c) Prove that the standard Normal distribution is infinitely divisible.

(d) Suppose that random variable Y has cf defined by

$$C_Y(t) = \cos(t) \qquad t \in \mathbb{R}.$$

Find the skewness of Y, ς, where

$$\varsigma = \frac{E_{f_Y}[(Y-\mu)^3]}{\sigma^3}$$

where μ and σ^2 are the expectation and variance of Y.

4. (a) Suppose that real constants a, b, p, q satisfy $a, b > 0$ and $p, q > 1$ with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove that

$$\frac{1}{p}\,a^p + \frac{1}{q}\,b^q \geq ab$$

with equality if and only if $a^p = b^q$.

Hence prove Hölder's Inequality: If X and Y are two random variables, and $p, q > 1$ satisfy the above identity,

$$|E_{f_{X,Y}}[XY]| \leq E_{f_{X,Y}}[|XY|] \leq \{E_{f_X}[|X|^p]\}^{1/p}\{E_{f_Y}[|Y|^q]\}^{1/q}$$

(b) Prove that for random variables X and Y having joint pdf $f_{X,Y}$,

$$\{Cov_{f_{X,Y}}[X,Y]\}^2 \leq Var_{f_X}[X]Var_{f_Y}[Y]$$

5. (a) Show that the $Poisson(\lambda)$ and $Binomial(n, \theta)$ distributions are one parameter Exponential Family Distributions, and in each case find the natural (or canonical) parameter.

(b) Consider the following three level hierarchical model

LEVEL 3 :	$r \in \{1, 2, \ldots\}$	Fixed parameter
LEVEL 2 :	$V \sim Gamma(r/2, r/2)$	
LEVEL 1 :	$X \vert V = v \sim Normal(0, v^{-1})$	

Find the kurtosis of X, κ, defined by

$$\kappa = \frac{E_{f_X}[(X - \mu)^4]}{\sigma^4}$$

where μ and σ^2 are the expectation and variance of X. State precisely conditions on r for the kurtosis to be finite.

6. (a) Suppose that continuous random variables $X_1, \ldots, X_n$ are independent and identically distributed with cdf F_X specified by

$$F_X(x) = \frac{x^2}{1 + x^2} \qquad x > 0.$$

and zero otherwise. Let Y_n be the maximum order statistic derived from $X_1, \ldots, X_n$, that is,

$$Y_n = \max\{X_1, \ldots, X_n\}$$

Show that, in the limit as $n \to \infty$,the limiting distribution of Y_n does not exist, but that the limiting distribution of Z_n defined by

$$Z_n = Y_n/\sqrt{n}$$

does exist and is a continuous distribution.

(b) Suppose that random variable Y_n can be written as

$$Y_n = \sum_{i=0}^{n} X_i$$

where, for each $i > 0$

$$X_i = \begin{cases} a & \text{with probability } \frac{1}{2} \\ -a & \text{with probability } \frac{1}{2} \end{cases}$$

and for $i = 0$, $X_0 = x_0$, a known constant.

(i) Find an approximation to the distribution of Y_n for large n.

(ii) Show that, for any n,

$$P[|Y_n - x_0| \geq 2\sigma_n] \leq \frac{1}{4}$$

for some σ_n to be defined.

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma(1+2/\alpha)-\Gamma(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	