McGill University
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December 2006

MATH 556

MATHEMATICAL STATISTICS I

SOLUTIONS

1. (a) From first principles (univariate transformation theorem also acceptable): for $y > 0$

$$F_Y(y) = P[Y \le y] = P\left[\frac{1}{X} \le y\right] = P\left[X \ge \frac{1}{y}\right] = 1 - F_X\left(y^{-1}\right)$$

and therefore

$$f_Y(y) = \frac{1}{y^2} f_X\left(y^{-1}\right) = \frac{1}{y^2}\frac{1}{\Gamma(\alpha)}\left(\frac{1}{y}\right)^{\alpha-1}\exp\left\{-\frac{1}{y}\right\} = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{y}\right)^{\alpha+1}\exp\left\{-\frac{1}{y}\right\} \qquad y > 0$$

and zero otherwise.

6 MARKS

By direct calculation

$$E_{f_Y}[Y] = \int_0^\infty \frac{1}{x}\frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}\,dx = \frac{1}{\Gamma(\alpha)}\int_0^\infty x^{(\alpha-1)-1}e^{-x}\,dx = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{1}{\alpha-1}$$

provided $\alpha > 1$; if $\alpha \le 1$, then the expectation does not exist.

3 MARKS

(b) From first principles: range of V is $\mathbb{R}^+$, and thus for $v > 0$

$$F_V(v) = P[V \le v] = P\left[U^2 \le v\right] = P\left[-\sqrt{v} \le U \le \sqrt{v}\right] = F_U\left(\sqrt{v}\right) - F_U\left(-\sqrt{v}\right)$$

and therefore

$$f_V(v) = \frac{1}{2\sqrt{v}}\left[f_U\left(\sqrt{v}\right) + f_U\left(-\sqrt{v}\right)\right]$$

Here $f_U(u) = \exp\{-u - \exp\{-u\}\}$, so

$$f_V(v) = \frac{1}{2\sqrt{v}}\left[\exp\left\{-\sqrt{v} - \exp\left\{-\sqrt{v}\right\}\right\} + \exp\left\{\sqrt{v} - \exp\left\{\sqrt{v}\right\}\right\}\right] \qquad v > 0$$

and zero otherwise.

6 MARKS

(c) We have

$$P[X < Y] = \iint_A f_{X,Y}(x,y)\,dxdy = \iint_A f_X(x)\,f_Y(y)dxdy$$

by independence, where

$$A \equiv \{(x,y) : 0 < x < y < \infty\}$$

3 MARKS

Hence

$$P[X < Y] = \int_0^\infty \left\{\int_0^y f_X(x)\,dx\right\} f_Y(y)dy = \int_0^\infty F_X(y)\,f_Y(y)dy.$$

Changing variables in the integral $y \to t = F_Y(y) \therefore y = F_Y^{-1}(t)$, we have

$$P[X < Y] = \int_0^1 F_X\left(F_Y^{-1}(t)\right) f_Y\left(F_Y^{-1}(t)\right)\frac{dy}{dt}dt.$$

and

$$\frac{dy}{dt} = \left[\frac{dt}{dy}\right]^{-1} = \left[f_Y(y)\right]^{-1} = \left[f_Y\left(F_Y^{-1}(t)\right)\right]^{-1}$$

and the result follows.

7 MARKS

2. (a) Using the multivariate transformation theorem

(a) We have that $\mathbb{X}^{(2)} \equiv \mathbb{R} \times \mathbb{R}$, and

$$g_1(t_1, t_2) = \frac{t_1}{t_1 + t_2} \qquad g_2(t_1, t_2) = t_1 + t_2$$

(b) Inverse transformations:

$$\left.\begin{array}{l} Y_1 = \dfrac{Z_1}{Z_1 + Z_2} \\ Y_2 = Z_1 + Z_2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} Z_1 = Y_1 Y_2 \\ Z_2 = (1 - Y_1)\, Y_2 \end{array}\right.$$

and thus

$$g_1^{-1}(t_1, t_2) = t_1 t_2 \qquad g_2^{-1}(t_1, t_2) = (1 - t_1)\, t_2$$

(c) Range: straightforwardly we have that $0 < Y_1 < 1, Y_2 > 0$, so $\mathbb{Y}^{(2)} = (0, 1) \times \mathbb{R}^+$

(d) The Jacobian for points $(y_1, y_2) \in \mathbb{Y}^{(2)}$ is

$$D_{y_1,y_2} = \begin{bmatrix} \dfrac{\partial z_1}{\partial y_1} & \dfrac{\partial z_1}{\partial y_2} \\ \dfrac{\partial z_2}{\partial y_1} & \dfrac{\partial z_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ -y_2 & 1 - y_1 \end{bmatrix} \Rightarrow |J(y_1, y_2)| = |\det D_{y_1,y_2}| = y_2$$

(e) For the joint pdf we have for $(y_1, y_2) \in \mathbb{Y}^{(2)}$, by independence of Z_1 and Z_2

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{Z_1,Z_2}(y_1 y_2, (1 - y_1)\, y_2) \times y_2 \\ &= f_{Z_1}(y_1 y_2) \times f_{Z_2}((1 - y_1)\, y_2) \times y_2 \\ &= \exp\{-y_1 y_2\} \exp\{-(1 - y_1)\, y_2\} \times y_2 = y_2 \exp\{-y_2\} \end{aligned}$$

and zero otherwise. Note that Y_1 and Y_2 are independent, as their joint pdf factorizes into the respective marginal pdfs, that is, $f_{Y_1,Y_2}(y_1, y_2) = \{1\} \times \{y_2 \exp\{-y_2\}\}$ - not necessary for full marks.

15 MARKS

(b) (i) For $0 \leq x \leq n$, using the Beta integral function

$$\begin{aligned} f_X(x) &= \int_0^1 f_{X|Y}(x|y)\, f_Y(y) dy = \int_0^1 \binom{n}{x} y^x (1 - y)^{n-x} dy \\ &= \binom{n}{x} \frac{\Gamma(x+1)\Gamma(n - x + 1)}{\Gamma(n + 2)} = \frac{1}{n+1} \end{aligned}$$

5 MARKS

(ii) For $x > 0$, using the Gamma integral

$$\begin{aligned} f_X(x) &= \int_0^\infty f_{X|Y}(x|y)\, f_Y(y) dy = \int_0^\infty y e^{-xy} \beta e^{-\beta y} dy \\ &= \beta \int_0^\infty y e^{-(x+\beta) y} dy = \beta \frac{\Gamma(2)}{(x + \beta)^2} = \frac{\beta}{(x + \beta)^2} \end{aligned}$$

5 MARKS

3. (a) We have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds}\left\{K_X(t)\right\}_{s=t} = \frac{d}{ds}\left\{\log M_X(t)\right\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = E_{f_X}[\, X \,]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\left\{M_X(t)\right\}^2}$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\left\{M_X(0)\right\}^2} = E_{f_X}[\, X^2 \,] - \{E_{f_X}[\, X \,]\}^2$$

and hence $K_X^{(2)}(0) = Var_{f_X}[\, X \,]$

8 MARKS

(b) By inspection, $c = \lambda/2$, and so

$$C_X(t) = E_{f_X}[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{itx}\lambda e^{-\lambda|x|}\, dx$$

But f_X is symmetric about zero, so

$$C_X(t) = \int_0^{\infty} \cos(tx)\lambda e^{-\lambda x}\, dx = \int_0^{\infty} \cos(sy)e^{-y}\, dy$$

where $s = t/\lambda$. Integrating by parts yields

$$C_X(t) = \frac{1}{1+(t/\lambda)^2} = \frac{\lambda^2}{\lambda^2+t^2}$$

as

$$\begin{aligned} C_X(t) &= \int_0^{\infty} \cos(ty)e^{-y}\, dy = \left[-\cos(ty)e^{-y}\right]_0^{\infty} - \int_0^{\infty} t\sin(ty)e^{-y}\, dy \\ &= 1 - t\left[\sin(ty)e^{-y}\right]_0^{\infty} - t\int_0^{\infty} t\cos(ty)e^{-y}\, dy \\ &= 1 - t^2 C_X(t) \end{aligned}$$

gives

$$C_X(t) = \frac{1}{1+t^2}.$$

8 MARKS

(c) The cf for $Z \sim N(0,1)$ can be written

$$C_Z(t) = \exp\{-t^2/2\} = \{\exp\{-t^2/(2n)\}\}^n = \{C_{X_n}(t)\}^n$$

for $n = 1, 2, \ldots$, where $C_{X_n}(t)$ is the cf of $X_n \sim N(0, \sigma^2/n)$. This holds for arbitrary positive integer n, so Z is infinitely divisible.

4 MARKS

(d) We have

$$C_X(t) = \cos(t) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$$

and hence it follows that X has a discrete distribution with pmf with equal probability on -1 and 1, that is, is symmetric, and hence the skewness is zero.

5 MARKS

4. This question is bookwork:

(a) Fix $b > 0$. Let
$$g(a; b) = \frac{1}{p}\, a^p + \frac{1}{q}\, b^q - ab.$$
We require that $g(a; b) \geq 0$ for all a. Differentiating wrt a for fixed b yields
$$g^{(1)}(a; b) = a^{p-1} - b$$
so that $g(a; b)$ is minimized (the second derivative is strictly positive at all a) when $a^{p-1} = b$, and at this value of a, the function takes the value
$$\frac{1}{p}\, a^p + \frac{1}{q}\, (a^{p-1})^q - a(a^{p-1}) = \frac{1}{p}\, a^p + \frac{1}{q}\, a^p - a^p = 0$$
as $1/p + 1/q = 1 \Longrightarrow (p-1)q = p$. As the second derivative is strictly positive at all a, the minimum is attained at the **unique** value of a where $a^{p-1} = b$, where, raising both sides to power q yields $a^p = b^q$.

8 MARKS

For the first inequality,
$$E_{f_{X,Y}}[|XY|] = \iint |xy| f_{X,Y}(x, y)\, dxdy \geq \iint xy f_{X,Y}(x, y)\, dxdy = E_{f_{X,Y}}[XY]$$
and
$$E_{f_{X,Y}}[XY] = \iint xy f_{X,Y}(x, y)\, dxdy \geq \iint -|xy| f_{X,Y}(x, y)\, dxdy = -E_{f_{X,Y}}[|XY|]$$
so
$$-E_{f_{X,Y}}[|XY|] \leq E_{f_{X,Y}}[XY] \leq E_{f_{X,Y}}[|XY|] \qquad \therefore \qquad |E_{f_{X,Y}}[XY]| \leq E_{f_{X,Y}}[|XY|].$$
For the second inequality, set
$$a = \frac{|X|}{\{E_{f_X}[|X|^p]\}^{1/p}} \qquad b = \frac{|Y|}{\{E_{f_Y}[|Y|^q]\}^{1/q}}.$$
Then from the previous lemma
$$\frac{1}{p}\, \frac{|X|^p}{E_{f_X}[|X|^p]} + \frac{1}{q}\, \frac{|Y|^q}{E_{f_Y}[|Y|^q]} \geq \frac{|XY|}{\{E_{f_X}[|X|^p]\}^{1/p} \{E_{f_Y}[|Y|^q]\}^{1/q}}$$
and taking expectations yields, on the left hand side,
$$\frac{1}{p}\, \frac{E_{f_X}[|X|^p]}{E_{f_X}[|X|^p]} + \frac{1}{q}\, \frac{E_{f_Y}[|Y|^q]}{E_{f_Y}[|Y|^q]} = \frac{1}{p} + \frac{1}{q} = 1$$
and on the right hand side
$$\frac{E_{f_{X,Y}}[|XY|]}{\{E_{f_X}[|X|^p]\}^{1/p} \{E_{f_Y}[|Y|^q]\}^{1/q}}$$
and the result follows.

12 MARKS

(b) The result follows setting $p = q = 2$ in Hölder's Inequality with random variables $X - \mu_X$ and $Y - \mu_Y$ in the stated version, after squaring both sides.

5 MARKS

5. (a) Going directly to the canonical forms

- $X \sim Poisson(\lambda)$,

$$f_X(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} = h(x)c(\lambda)\exp\{w(\lambda)t(x)\}$$

where

$$h(x) = \frac{I_{\{0,1,\ldots\}}(x)}{x!} \qquad c(\lambda) = e^{-\lambda} \qquad w(\lambda) = \log\lambda \qquad t(x) = x$$

so canonical parameter is $\eta = \log\lambda$.

5 MARKS

- $X \sim Binomial(n,\theta)$.

$$f_X(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} = h(x)c(\theta)\exp\{w(\theta)t(x)\}$$

where

$$h(x) = \binom{n}{x} I_{\{0,1,\ldots,n\}}(x) \qquad c(\theta) = (1-\theta)^n \qquad w(\theta) = \log\left(\frac{\theta}{1-\theta}\right) \qquad t(x) = x$$

so canonical parameter is $\eta = \log\left(\frac{\theta}{1-\theta}\right)$.

5 MARKS

(b) We have that $X = Z/\sqrt{V}$, where $Z \sim N(0,1)$ independent of V. Hence the expectation of X is zero, and using iterated expectation

$$E_{f_X}[X^k] = E_{f_Z}[Z^k]E_{f_V}[V^{-k/2}]$$

Using (say) mgfs, $E_{f_Z}[Z] = E_{f_Z}[Z^3] = 0$, with

$$E_{f_Z}[Z^2] = 1 \qquad E_{f_Z}[Z^4] = 3.$$

Also

$$\begin{aligned} E_{f_V}[V^{-k/2}] &= \int_0^\infty \frac{1}{x^{k/2}}\frac{(r/2)^{r/2}}{\Gamma(r/2)}x^{r/2-1}e^{-rx/2}\,dx \\ &= \frac{(r/2)^{r/2}}{\Gamma(r/2)}\int_0^\infty x^{(r-k)/2-1}e^{-rx/2}\,dx \\ &= \frac{(r/2)^{r/2}}{\Gamma(r/2)}\frac{\Gamma((r-k)/2)}{(r/2)^{(r-k)/2}} = \frac{\Gamma((r-k)/2)}{\Gamma(r/2)}(r/2)^{k/2} \end{aligned}$$

provided $r > k$. For $k = 2$

$$E_{f_V}[V^{-1}] = \frac{\Gamma(r/2-1)}{\Gamma(r/2)}(r/2) = \frac{r/2}{r/2-1} = \frac{r}{r-2}.$$

For $k = 4$

$$E_{f_V}[V^{-2}] = \frac{\Gamma(r/2-2)}{\Gamma(r/2)}(r/2)^2 = \frac{(r/2)^2}{(r/2-1)(r/2-2)} = \frac{r^2}{(r-2)(r-4)}$$

and thus the kurtosis is

$$\kappa = \frac{E_{f_X}[(X-\mu)^4]}{\sigma^4} = \frac{E_{f_X}[X^4]}{\{E_{f_X}[X^2]\}^2} = \frac{3(r-2)}{(r-4)}$$

provided $r > 4$.

15 MARKS

6. (a) For the cdf of a maximum order statistic

$$F_{Y_n}(y) = P[Y \le y] = P[\max\{X_1, ..., X_n\} \le y] = \prod_{i=1}^{n} \{F_X(y)\} = \{F_X(y)\}^n$$

5 MARKS

(i) As $n \to \infty$, for $x \in \mathbb{R}$

$$\left(\frac{x^2}{1+x^2}\right) < 1 \quad \therefore \quad F_{X_n}(x) \to 0$$

and so the limiting function is not a cdf, and no limiting distribution exists.

3 MARKS

(ii) If $Y_n = X_n/\sqrt{n}$. Then $\mathbb{Y} \equiv (0, \infty)$ and the cdf of Y_n is, for $y > 0$,

$$F_{Y_n}(y) = P[Y_n \le y] = P\left[X_n/\sqrt{n} \le y\right] = P\left[X_n \le \sqrt{n}y\right] = F_{X_n}(\sqrt{n}y) = \left(\frac{(\sqrt{n}y)^2}{1+(\sqrt{n}y)^2}\right)^n$$

and so

$$F_{Y_n}(y) = \left(\frac{ny^2}{1+ny^2}\right)^n = \left(1 - \frac{1}{ny^2}\right)^n.$$

Thus as $n \to \infty$, for all $y > 0$

$$F_{Y_n}(v) \to \exp\left\{-1/y^2\right\} \qquad \therefore \qquad F_{Y_n}(y) \to F_Y(y) = \exp\left\{-1/y^2\right\}$$

and the limiting distribution of Y_n does exist, and is continuous on $\mathbb{Y} \equiv \mathbb{X}$.

7 MARKS

(b) (i) The Central Limit Theorem gives that for the iid $\{X_i\}$ collection

$$\frac{\sum\limits_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} Z \sim N(0,1)$$

Here

$$\mu = E_{f_X}[X_i] = a \times \frac{1}{2} + (-a) \times \frac{1}{2} = 0$$

$$\sigma^2 = Var_{f_X}[X_i] = (a)^2 \times \frac{1}{2} + (-a)^2 \times \frac{1}{2} - E_{f_X}[X_i] = a^2$$

and thus

$$\sum_{i=1}^{n} X_i \sim AN\left(0, na^2\right)$$

and

$$Y_n \sim AN\left(x_0, na^2\right)$$

where AN denotes Asymptotically Normal (as $n \to \infty$).

4 MARKS

(i) This is an elementary application of the Chebychev Inequality to the variable Y_n and its distribution. The (exact) bound to the probability is given in general, for any $k > 0$, by

$$P\left[|Y_n - x_0| \ge k\sigma_n\right] \le \frac{1}{k^2}$$

as for any n

$$E[Y_n] = x_0 \qquad Var_{f_{Y_n}}[Y_n] = n(a)^2 = \sigma_n^2$$

Here, we need $k = 2$, and the result follows.

6 MARKS