# 556: MATHEMATICAL STATISTICS I

## MULTIVARIATE TRANSFORMATIONS: THE CAUCHY DISTRIBUTION

The Cauchy distribution is a symmetric distribution on $(-\infty, \infty)$ with pdf

$$f_X(x;\theta,\sigma) = \frac{1}{\pi}\frac{1}{\sigma}\cdot\frac{1}{1+\left(\dfrac{x-\theta}{\sigma}\right)^2} = \frac{1}{\pi}\cdot\frac{\sigma}{\sigma^2+(x-\theta)^2}$$

The standard case is $\theta = 0$, $\sigma = 1$.

The Cauchy distribution arises as the ratio of two independent Gaussian random variables. Suppose that $X, Y \sim Normal(0,1)$. We then proceed by

(i) defining the transformation $U = X/Y$ and $V = |Y|$,

(ii) finding the joint pdf $f_{U,V}(u,v)$, and

(iii) integrating out $V$ to obtain the marginal pdf of $U$.

The mapping $U = X/Y$ and $V = |Y|$ is not 1-1: the two points $(x,y)$ and $(-x,-y)$ map to the same $(u,v)$. However, we may partition the support of $(X,Y)$ into $A_0, A_1, A_2$ such that the mapping from $A_i$ to $(U,V)$ is one-to-one.

1. $A_0 = \{(X,Y) : Y = 0\}$: we can ignore this case as $P_Y[Y=0] = 0$ when $Y \sim Normal(0,1)$.
2. $A_1 = \{(X,Y) : Y > 0\}$: The mapping $U = X/Y$, $V = |Y|$ is 1-1, and the inverse mappings are $h_{11}(u,v) = uv$, $h_{21}(u,v) = v$.
3. $A_2 = \{(X,Y) : Y < 0\}$: The mapping $U = X/Y$, $V = |Y|$ is one-to-one, and the inverse mappings are $h_{12}(u,v) = -uv$, $h_{22}(u,v) = -v$.

In cases (ii) and (iii) we have the following Jacobians:

$$J_1 = \begin{vmatrix} \dfrac{\partial h_{11}(u,v)}{\partial u} & \dfrac{\partial h_{11}(u,v)}{\partial v} \\ \dfrac{\partial h_{21}(u,v)}{\partial u} & \dfrac{\partial h_{21}(u,v)}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial(uv)}{\partial u} & \dfrac{\partial(uv)}{\partial v} \\ \dfrac{\partial v}{\partial u} & \dfrac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

$$J_2 = \begin{vmatrix} \dfrac{\partial h_{12}(u,v)}{\partial u} & \dfrac{\partial h_{12}(u,v)}{\partial v} \\ \dfrac{\partial h_{22}(u,v)}{\partial u} & \dfrac{\partial h_{22}(u,v)}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial(-uv)}{\partial u} & \dfrac{\partial(-uv)}{\partial v} \\ \dfrac{\partial(-v)}{\partial u} & \dfrac{\partial(-v)}{\partial v} \end{vmatrix} = \begin{vmatrix} -v & -u \\ 0 & -1 \end{vmatrix} = v$$

We have that

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}}\exp\{-x^2/2\}\frac{1}{\sqrt{2\pi}}\exp\{-y^2/2\} = \frac{1}{2\pi}\exp\left\{-\frac{(x^2+y^2)}{2}\right\}$$

so therefore

$$f_{U,V}(u,v) = \mathbb{1}_{A_1}(u,v) f_{X,Y}(h_{11}(u,v),h_{21}(u,v))|J_1| + \mathbb{1}_{A_2}(u,v) f_{X,Y}(h_{12}(u,v),h_{22}(u,v))|J_2|$$

$$= \frac{\mathbb{1}_{A_1}(u,v)}{2\pi}\exp\left(-\frac{(uv)^2+v^2}{2}\right)|v| + \frac{\mathbb{1}_{A_2}(u,v)}{2\pi}\exp\left(-\frac{(-uv)^2+(-v)^2}{2}\right)|v|$$

$$= \frac{v}{\pi}\exp\left(-\frac{v^2(u^2+1)}{2}\right), \qquad u \in \mathbb{R}, v \in \mathbb{R}^+$$

and hence, on marginalization

$$\begin{aligned}
f_U(u) &= \int_0^\infty \frac{v}{\pi} \exp\left\{-\frac{v^2(u^2+1)}{2}\right\} dv && \text{integrating out } v \\
&= \int_0^\infty \frac{1}{2\pi} \exp\left\{-\frac{(u^2+1)}{2} z\right\} dz && \text{setting } z = v^2 \text{ and } dz = 2v dv \\
&= \frac{1}{2\pi} \cdot \frac{2}{1+u^2} && \int_0^\infty \exp(-\alpha z) dz = \frac{1}{\alpha} \\
&= \frac{1}{\pi} \cdot \frac{1}{1+u^2}
\end{aligned}$$

The general $Cauchy(\theta, \sigma)$ form is generated using a location-scale transform: that is, if $Z \sim Cauchy(0, 1)$, then

$$X = \sigma Z + \theta$$

has a $Cauchy(\theta, \sigma)$ distribution.

The second (equivalent) construction of the standard Cauchy distribution is as a *scale mixture*. Suppose $X$ and $Y$ have a joint distribution specified as

$$Y \sim \chi^2_1 \equiv Gamma(1/2, 1/2)$$

$$X|Y = y \sim Normal(0, y^{-1})$$

that is, the variance of $X$ given $Y = y$ is $1/y$. Then we have that

$$\begin{aligned}
f_X(x) = \int_{-\infty}^\infty f_{X,Y}(x, y)\, dy &= \int_0^\infty f_{X|Y}(x|y) f_Y(y)\, dy \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} y^{1/2} \exp\left\{-\frac{y}{2} x^2\right\} \frac{(1/2)^{1/2}}{\Gamma(1/2)} y^{-1/2} \exp\left\{-\frac{y}{2}\right\}\, dy \\
&= \frac{1}{2\pi} \int_0^\infty \exp\left\{-\frac{y}{2}(1+x^2)\right\}\, dy \\
&= \frac{1}{\pi} \frac{1}{1+x^2}
\end{aligned}$$

as $\Gamma(1/2) = \sqrt{\pi}$.