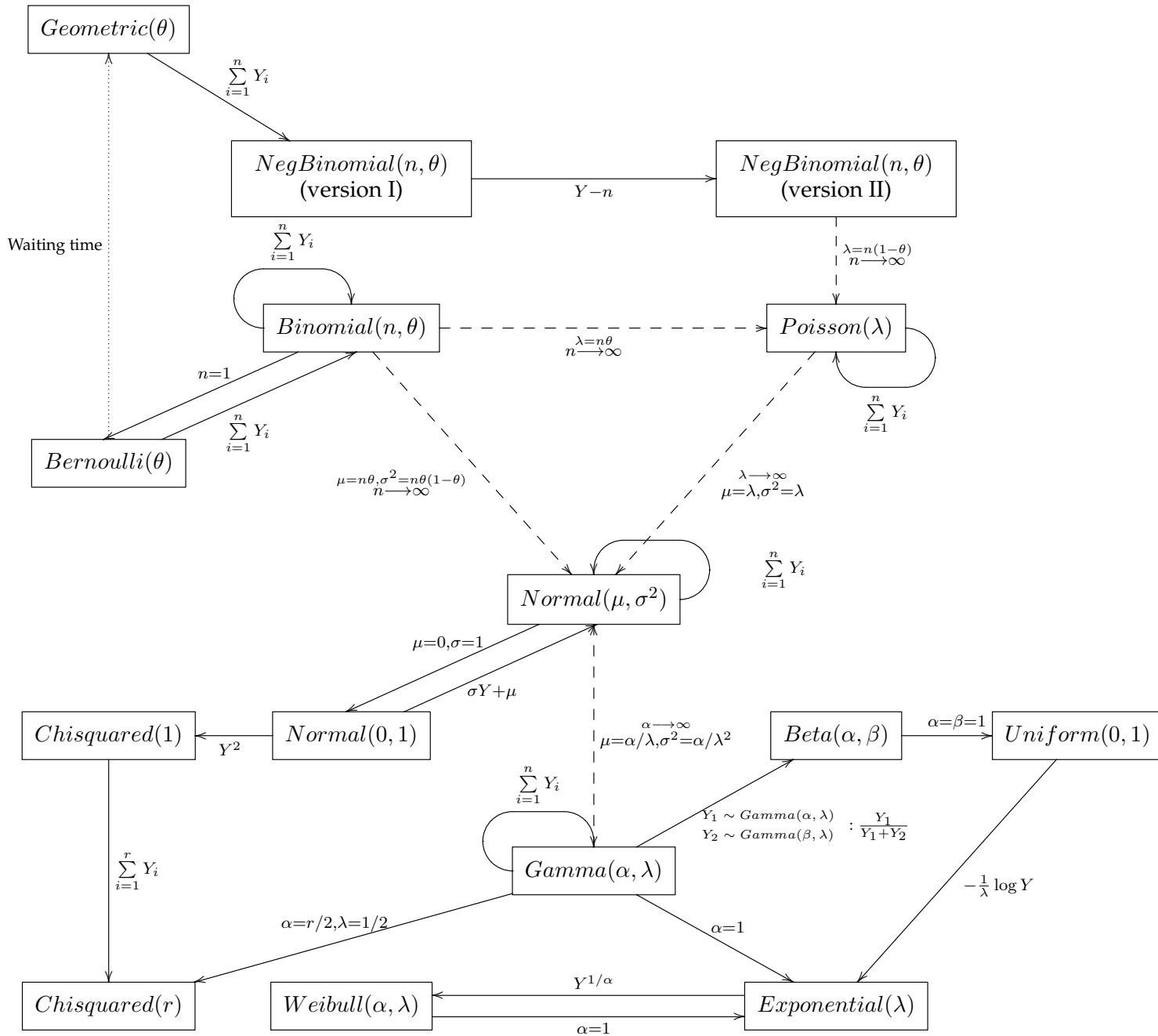


MATH 556: MATHEMATICAL STATISTICS I

A MAP OF THE DISTRIBUTIONS



Direct link Special case, transformation or summation
 Indirect link Limit case

After diagram in *Statistical Inference*, by G Casella and RL Berger.

DISCRETE DISTRIBUTIONS

Models based on an independent sequence of identical binary trials with success probability θ .

- **BERNOULLI:** Y is the total number of successes in **one** trial.
- **BINOMIAL:** Y is the total number of successes in n trials.
- **GEOMETRIC:** Y is the total number of trials required to obtain **one** success.
- **NEGATIVE BINOMIAL:**
 - **Version I:** Y is the total number of trials required to obtain n successes.
 - **Version II:** Consider $X = Y - n$, to give a distribution on $\{0, 1, 2, \dots\}$.
- **POISSON:** The Poisson distribution is obtained as the limiting form of the *Binomial*(n, θ) distribution, with $n \rightarrow \infty$ but with $\lambda = n\theta$ held fixed. Y is the count of the number of events in a given (continuous) time interval.

Connections:

- Bernoulli/Binomial

$$Y_1, \dots, Y_n \sim \text{Bernoulli}(\theta) \quad \implies \quad Y = \sum_{i=1}^n Y_i \sim \text{Binomial}(n, \theta)$$

- Geometric/Negative Binomial

$$Y_1, \dots, Y_n \sim \text{Geometric}(\theta) \quad \implies \quad Y = \sum_{i=1}^n Y_i \sim \text{NegBinomial}(n, \theta)$$

- Binomial/Poisson

$$Y_n \sim \text{Binomial}(n, \theta) \longrightarrow Y \sim \text{Poisson}(\lambda)$$

where $\lambda = n\theta$ is held fixed and $n \rightarrow \infty$.

- Negative Binomial/Poisson

$$Y_n \sim \text{NegBinomial}(n, \theta) \quad X_n = Y_n - n \longrightarrow X \sim \text{Poisson}(\lambda)$$

where $\lambda = n(1 - \theta)$ is held fixed and $n \rightarrow \infty$.

Sums of Independent Random Variables: Proved using mgfs.

- Binomial

$$\left. \begin{array}{l} Y_1 \sim \text{Binomial}(m, \theta) \\ Y_2 \sim \text{Binomial}(n, \theta) \end{array} \right\} \implies Y = Y_1 + Y_2 \sim \text{Binomial}(m + n, \theta)$$

- Negative Binomial

$$\left. \begin{array}{l} Y_1 \sim \text{NegBinomial}(m, \theta) \\ Y_2 \sim \text{NegBinomial}(n, \theta) \end{array} \right\} \implies Y = Y_1 + Y_2 \sim \text{NegBinomial}(m + n, \theta)$$

- Poisson

$$\left. \begin{array}{l} Y_1 \sim \text{Poisson}(\lambda_1) \\ Y_2 \sim \text{Poisson}(\lambda_2) \end{array} \right\} \implies Y = Y_1 + Y_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

CONTINUOUS DISTRIBUTIONS

- Distributions on \mathbb{R}^+ : Begin with $Y \sim Uniform(0, 1)$:

- ▶ $U = -\frac{1}{\lambda} \log Y \sim Exponential(\lambda)$, for $\lambda > 0$.

- ▶ $X = U^{1/\alpha} \sim Weibull(\alpha, \lambda)$, for $\alpha, \lambda > 0$.

- ▶ If $X_1, \dots, X_n \sim Exponential(\lambda)$, independent, then $Z = \sum_{i=1}^n X_i \sim Gamma(n, \lambda)$.

- ▶ If $Y_1 \sim Gamma(\alpha_1, \lambda)$ and $Y_2 \sim Gamma(\alpha_2, \lambda)$ are independent, then

$$S = Y_1 + Y_2 \sim Gamma(\alpha_1 + \alpha_2, \lambda)$$

- Distributions on \mathbb{R} : The Normal distribution and connections

- ▶ Suppose $Y \sim N(0, 1)$. Then $X = \mu + \sigma Y \sim N(\mu, \sigma^2)$.

- ▶ Suppose $Y \sim N(0, 1)$. Then $U = Y^2 \sim Gamma(1/2, 2) \equiv Chisquared(1)$.

- ▶ If $Y_i \sim Gamma(\alpha_i/2, 2) \equiv Chisquared(\alpha_i)$ for $i = 1, \dots, n$ are independent, then

$$V = \sum_{i=1}^n Y_i \sim Gamma(\nu/2, 2) \equiv Chisquared(\nu)$$

where

$$\nu = \sum_{i=1}^n \alpha_i.$$

- ▶ If $Y_1 \sim N(\mu_1, \sigma_1^2)$ and $Y_2 \sim N(\mu_2, \sigma_2^2)$ are independent, then

$$Y = Y_1 + Y_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

- Distribution on $(0, 1)$: The Beta distribution

- ▶ If $Y_1 \sim Gamma(\alpha_1, \beta)$ and $Y_2 \sim Gamma(\alpha_2, \beta)$ are independent, then

$$Y = \frac{Y_1}{Y_1 + Y_2} \sim Beta(\alpha_1, \alpha_2)$$

This result follows by multivariate transformations.

All the summation results proved using mgfs.