# MATH 556: RANDOM VARIABLES & PROBABILITY DISTRIBUTIONS

## 1.8 RANDOM VARIABLES & PROBABILITY MODELS

A **random variable (r.v.)** $X$ is a function defined on a space $(\Omega, \mathcal{F})$ that associates a *real number* $X(\omega) = x$ with each "event" $A \in \mathcal{F}$.

Formally, we regard $X$ as a (possibly many-to-one) mapping from $\Omega$ to $\mathbb{R}$

$$X: \quad \begin{aligned} \mathcal{F} &\longrightarrow \mathbb{R} \\ A &\longmapsto x \end{aligned}$$

We note the equivalence of events in $\mathcal{F}$ and their images under $X$. Consider the space $(\mathbb{R}, \mathcal{B})$, that is the real line $\mathbb{R}$, and the sigma-algebra $\mathcal{B}$ of sets generated by all open, half-open or closed subsets of the reals, that is, sets of the form

$$(a, b) \qquad (a, b] \qquad [a, b) \qquad [a, b]$$

and countable unions and intersections of these sets. Consider $B \in \mathcal{B}$, and consider the pre-image

$$B^{-1} = \{\omega \in \Omega : X(\omega) \in B\}$$

For $X$ to be a valid random variable, we require that $B^{-1} \in \mathcal{F}$. Strictly, when referring to random variables, we should make explicit the connection to original sample space $\Omega$ and write $X(\omega)$ for individual sample outcomes, and

$$P_X[X \in B] = P_X(\{\omega : X(\omega) \in B\})$$

for events. However, generally, we will suppress this and merely refer to $X$.

In deducing the probability measure for $X$, $P_X$, we must have

$$P(B^{-1}) = P_X[X \in B] = P_X(B).$$

for all $B \in \mathcal{B}$, where $B^{-1}$ is the pre-image of $B$.

**Probability Functions**
Consider the real function of a real argument, $F_X$, defined by

$$F_X(x) = P_X((-\infty, x]) = \int_{(-\infty,x]} P_X(dx)$$

for real values of $x$. Note that $X \in (-\infty, x]$ is equivalent to $X \leq x$, and, by definition,

$$F_X(x) = \int_{-\infty}^{x} dF_X(t).$$

$F_X$ defines the **probability distribution** of $X$. The nature of $F_X$ determines how we can manipulate this function. There are three cases to consider:

1. $F_X$ is a *step-function*, that is, $F_X$ changes only at a certain (countable) set of $x$ values.
2. $F_X$ is a *continuous* function.
3. $F_X$ is a mixture of 1. and 2.

### 1.8.1 DISCRETE RANDOM VARIABLES

**Definition** A random variable $X$ is **discrete** if $F_X$ is a step-function.

If $X$ is discrete then the set of all values at which $F_X$ changes, to be denoted $\mathbb{X}$, is **countable**, that is

- $\mathbb{X} \equiv \{x_1, x_2, \ldots, x_n\}$ (that is, a **finite** list)
- $\mathbb{X} \equiv \{x_1, x_2, \ldots\}$ (that is, a countably **infinite** list).

If $X$ is discrete, then it follows the event $X \in B$ can be decomposed

$$X \in B \qquad \Longleftrightarrow \qquad X \in \{x_i : x_i \in B\}$$

so that $F_X$ can be represented as a sum of probabilities

$$F_X(x) = \sum_{x_i \leq x} P_X(X \in \{x_i\}) = \sum_{x_i \leq x} P_X[X = x_i].$$

**Definition PROBABILITY MASS FUNCTION**

The function $f_X$, defined on $\mathbb{X}$ by

$$f_X(x) = P_X[X = x] \qquad x \in \mathbb{X}$$

that assigns probability to each $x \in \mathbb{X}$ is the (discrete) **probability mass function**, or **pmf**.

**NOTE**: For completeness, we define

$$f_X(x) = 0 \qquad x \notin \mathbb{X}$$

so that $f_X$ is defined for all $x \in \mathbb{R}$. Thus $\mathbb{X}$ is the *support* of random variable $X$, that is, the set of $x \in \mathbb{R}$ such that $f_X(x) > 0$

**PROPERTIES OF MASS FUNCTION $f_X$**

Elementary properties of the mass function are straightforward to establish using the probability axioms.

A function $f_X$ is a probability mass function for discrete random variable $X$ with support $\mathbb{X}$ of the form $\{x_1, x_2, \ldots\}$ if and only if

$$\text{(i) } f_X(x_i) \geq 0 \qquad\qquad \text{(ii) } \sum_i f_X(x_i) = 1$$

Clearly as $f_X(x) = P_X[X = x]$, we must have, for all $x \in \mathbb{R}$,

$$0 \leq f_X(x) \leq 1.$$

**Definition DISCRETE CUMULATIVE DISTRIBUTION FUNCTION**

The **cumulative distribution function**, or **cdf**, $F_X$ of a discrete r.v. $X$ is defined by

$$F_X(x) = P_X[X \leq x] \qquad x \in \mathbb{R}.$$

**Connection between $F_X$ and $f_X$:** Let $X$ be a discrete random variable with support $\mathbb{X} \equiv \{x_1, x_2, \ldots\}$, where $x_1 < x_2 < \ldots$, and pmf $f_X$ and cdf $F_X$. For any real value $x$, if $x < x_1$, then $F_X(x) = 0$, and for $x \geq x_1$,

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i)$$

so that, for $i = 2, 3, \ldots$,

$$f_X(x_i) = F_X(x_i) - F_X(x_{i-1})$$

with, for completeness, $f_X(x_1) = F_X(x_1)$ .

### 1.8.2 PROPERTIES OF DISCRETE CDF $F_X$

(i) In the limiting cases,

$$\lim_{x \to -\infty} F_X(x) = 0 \qquad\qquad \lim_{x \to \infty} F_X(x) = 1.$$

(ii) $F_X$ is **continuous from the right** (but not continuous) on $\mathbb{R}$ that is, for $x \in \mathbb{R}$,

$$\lim_{h \to 0^+} F_X(x+h) = F_X(x)$$

but, if $x \in \mathbb{X}$,

$$\lim_{h \to 0^-} F_X(x+h) \neq F_X(x)$$

that is, the "left limit" is not equal to the "right limit" at $x$ values in $\mathbb{X}$.

(iii) $F_X$ is **non-decreasing**, that is

$$a < b \Longrightarrow F_X(a) \leq F_X(b)$$

(iv) For $a < b$,

$$P[a < X \leq b] = F_X(b) - F_X(a)$$

**Notes:**

- The functions $f_X$ and/or $F_X$ can both be used to describe the probability distribution of random variable $X$.
- The function $f_X$ is non-zero only at the elements of $\mathbb{X}$.
- The function $F_X$ is a **step-function**, which takes the value zero at minus infinity, the value one at infinity, and is non-decreasing with points of discontinuity at the elements of $\mathbb{X}$.
- The right-continuity of $F_X$ is denoted in plots by the use of a filled circle, $\bullet$, as in the example below.
- In the discrete case, $F_X$ is **not differentiable** for all $x \in \mathbb{R}$; at points of continuity (that is, for $x \notin \mathbb{X}$), it is differentiable, and the derivative is zero.

**Example** Consider a coin tossing experiment where a fair coin is tossed repeatedly under identical experimental conditions, with the sequence of tosses independent, until a Head is obtained. For this experiment, the sample space, $\Omega$ is then the set of sequences

$$(\{H\}, \{TH\}, \{TTH\}, \{TTTH\} \ldots)$$

with associated probabilities 1/2, 1/4, 1/8, 1/16, ....

Define discrete random variable $X$ by $X(\omega) = x \Longleftrightarrow$ first H on toss $x$. Then

$$f_X(x) = P_X[X = x] = \left(\frac{1}{2}\right)^x \qquad x = 1, 2, 3, \ldots$$

and zero otherwise. For $x \geq 1$, let $\lfloor x \rfloor$ be the largest integer not greater than $x$. Then

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i) = \sum_{i=1}^{\lfloor x \rfloor} f_X(i) = 1 - \left(\frac{1}{2}\right)^{\lfloor x \rfloor}$$

and $F_X(x) = 0$ for $x < 1$.

Graphs of the probability mass function (top) and cumulative distribution function (bottom) are shown in Figure 3. Note that the mass function is only non-zero at points that are elements of $X$, and that the cdf is defined for all real values of $x$, but is only continuous from the right. $F_X$ is therefore a step-function.

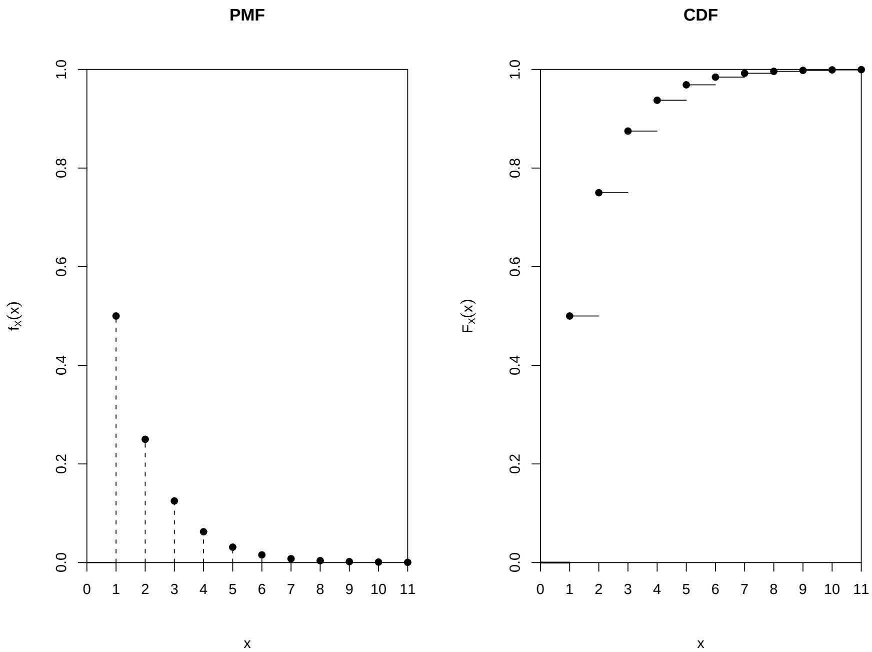


Figure 3: PMF $f_X(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \ldots$ and CDF $F_X(x) = 1 - \left(\frac{1}{2}\right)^{\lfloor x \rfloor}$

### 1.8.3 CONTINUOUS RANDOM VARIABLES

**Definition** A random variable $X$ is **continuous** if the function $F_X$ defined on $\mathbb{R}$ by

$$F_X(x) = P[X \leq x]$$

for $x \in \mathbb{R}$ is a **continuous** function on $\mathbb{R}$ , that is, for $x \in \mathbb{R}$,

$$\lim_{h \to 0} F_X(x + h) = F_X(x).$$

**Definition CONTINUOUS CUMULATIVE DISTRIBUTION FUNCTION**
The **cumulative distribution function**, or **cdf**, $F_X$ of a continuous r.v. $X$ is defined by

$$F_X(x) = P[X \leq x] \qquad x \in \mathbb{R}.$$

**Definition PROBABILITY DENSITY FUNCTION**
A random variable is **absolutely continuous** if the cumulative distribution function $F_X$ can be written

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

for some function $f_X$, termed the **probability density function**, or **pdf**, of $X$. For any suitable set $B$,

$$P_X[X \in B] = \int_B f_X(x)\, dx$$

Directly from the definition, at values of $x$ where $F_X$ is differentiable $x$,

$$f_X(x) = \frac{d}{dt}\{F_X(t)\}_{t=x}$$

**PROPERTIES OF CONTINUOUS $F_X$ AND $f_X$**

(i) The pdf $f_X$ need not exist, but continuous r.v.s where $f_X$ cannot be defined in this way will be ignored. The function $f_X$ can be defined piecewise on intervals of $\mathbb{R}$ .

(ii) For the cdf of a continuous r.v.,

$$\lim_{x\to-\infty} F_X(x) = 0 \qquad \lim_{x\to\infty} F_X(x) = 1$$

(iii) If $X$ is continuous,

$$f_X(x) \neq P_X[X = x] = \lim_{h\to 0}[F_X(x+h) - F_X(x)] = 0$$

(iv) For $a < b$,

$$P_X[a < X \leq b] = P_X[a \leq X < b] = P_X[a \leq X \leq b] = P_X[a < X < b] = F_X(b) - F_X(a)$$

It follows that a function $f_X$ is a pdf for a continuous random variable $X$ **if and only if**

$$\text{(i) } f_X(x) \geq 0 \qquad \text{(ii) } \int_{-\infty}^{\infty} f_X(x)dx = 1$$

This result follows direct from definitions and properties of $F_X$. Note that in the continuous case, there is no requirement that $f_X$ is bounded above.

**Example** Consider an experiment to measure the length of time that an electrical component functions before failure. The sample space of outcomes of the experiment, $\Omega$ is $^+$, and if $A_x$ is the event that the component functions for longer than $x > 0$ time units, suppose that

$$P(A_x) = \exp\left\{-x^2\right\}.$$

Define continuous random variable $X$ by $X(\omega) = x \iff$ component fails at time $x$. Then, if $x > 0$,

$$F_X(x) = P_X[X \leq x] = 1 - P_X(A_x) = 1 - \exp\left\{-x^2\right\}$$

and $F_X(x) = 0$ if $x \leq 0$. Hence if $x > 0$,

$$f_X(x) = \frac{d}{dt}\{F_X(t)\}_{t=x} = 2x\exp\left\{-x^2\right\}$$

and zero otherwise.

Graphs of the probability density function (top) and cumulative distribution function (bottom) are shown in Figure 4. Note that both the pdf and cdf are defined for all real values of $x$, and that both are continuous functions.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_{0}^{x} f_X(t)dt$$

as $f_X(x) = 0$ for $x \leq 0$, and also that

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{0}^{\infty} f_X(x)dx = 1.$$

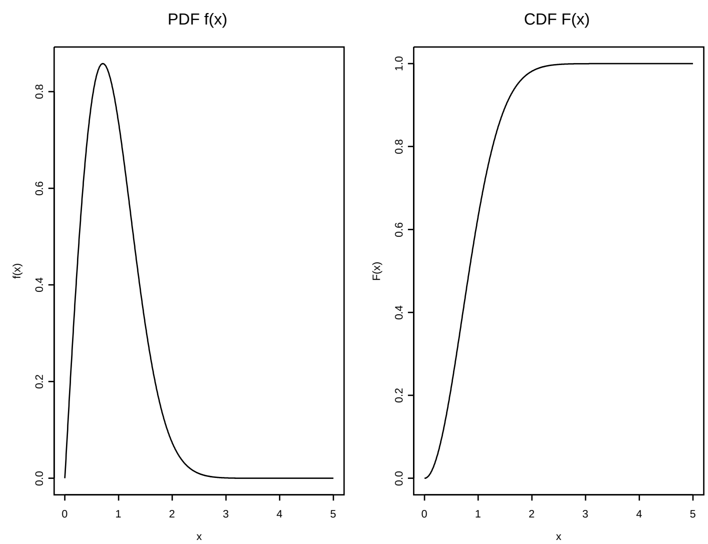

Figure 4: PDF $f_X(x) = 2x \exp\left\{-x^2\right\},\ x > 0$, and CDF $F_X(x) = 1 - \exp\left\{-x^2\right\},\ x > 0$