556: MATHEMATICAL STATISTICS I

SOME MATHEMATICAL DEFINITIONS AND RESULTS

Definition: Limits of sequences of reals
Sequence $\{a_n\}$ has limit a as $n \longrightarrow \infty$, written

$$\lim_{n \longrightarrow \infty} a_n = a$$

if, for every $\epsilon > 0$, there exists an $N = N(\epsilon)$ such that $|a_n - a| < \epsilon$ for all $n > N$. We say that $\{a_n\}$ is a **convergent** sequence, and that $\{a_n\}$ **converges** to a.

Definition: Limits of functions
Let f be a real-valued function of real argument x.

- Limit as $x \longrightarrow \infty$:

  $$f(x) \longrightarrow a \qquad \text{or} \qquad \lim_{x \longrightarrow \infty} f(x) = a$$

  as $x \longrightarrow \infty$ if, every $\epsilon > 0$, $\exists M = M(\epsilon)$ such that $|f(x) - a| < \epsilon$, $\forall\, x > M$

- Limit as $x \longrightarrow x_0^{\pm}$:

  $$f(x) \longrightarrow a \qquad \text{or} \qquad \lim_{x \longrightarrow x_0^{\pm}} f(x) = a$$

  as $x \longrightarrow x_0^{\pm}$ (that is, $x \longrightarrow x_0^{-}$ means "from below" and $x \longrightarrow x_0^{+}$ means "from above") if, for all $\epsilon > 0$, $\exists\, \delta$ such that $|f(x) - a| < \epsilon$, $\forall\, x_0 < x < x_0 + \delta$ (or, respectively $x_0 - \delta < x < x_0$).

- Left/Right Limit as $x \longrightarrow x_0$:

  $$f(x) \longrightarrow a \qquad \text{or} \qquad \lim_{x \longrightarrow x_0} f(x) = a$$

  as $x \longrightarrow x_0$ if

  $$\lim_{x \longrightarrow x_0^{+}} f(x) = \lim_{x \longrightarrow x_0^{-}} f(x) = a.$$

Definition: Continuity

Consider function $f(x)$ with domain $\mathcal{X} \subseteq \mathbb{R}$.

- $f(x)$ is *continuous* at x_0 if

  $$\lim_{x \longrightarrow x_0^{+}} f(x) = \lim_{x \longrightarrow x_0^{-}} f(x) = f(x_0)$$

  and all limits exist. That is, for all $\epsilon > 0$, $\exists \delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

- $f(x)$ is *uniformly continuous* on $\mathcal{X}$ if, **for all** $x_1, x_2 \in \mathcal{X}$, $\exists \delta > 0$ such that $\forall \epsilon > 0$

  $$|x_2 - x_1| < \delta \quad \Longrightarrow \quad |f(x_2) - f(x_1)| < \epsilon$$

- $f(x)$ is *absolutely continuous* on $\mathcal{X}$ if, for all $\epsilon > 0$, $\exists \delta > 0$ such that for any finite sequence of disjoint sub-intervals (x_{k1}, x_{k2}) for $k = 1, \ldots, K$ with

  $$\sum_{k=1}^{K} (x_{k2} - x_{k1}) < \delta \quad \text{then} \quad \sum_{k=1}^{K} |f(x_{k2}) - f(x_{k1})| < \epsilon$$

Definition: Supremum and Infimum
A set of real values S is **bounded above (bounded below)** if there exists a real number a (b) such that, for all $x \in S$, $x \leq a$ ($x \geq b$). The quantity a (b) is an **upper bound (lower bound)**. A real value a_L (b_U) is a **least upper bound (greatest lower bound)** if it is an upper bound (a lower bound) of S, and no other upper (lower) bound is smaller (larger) than a_L (b_U). We write

$$a_L = \sup S \qquad b_U = \inf S$$

for the a_L, the **supremum**, and b_U, the **infimum** of S.

If S comprises a sequence of elements $\{x_n\}$, then we can write

$$a_L = \sup_{x_n \in S} x_n \equiv \sup_n x_n \qquad b_U = \inf_{x_n \in S} x_n \equiv \inf_n x_n.$$

A sequence that is both bounded above and bounded below is termed **bounded**. Any bounded, monotone real sequence is **convergent**.

Definition: Limit Superior and Limit Inferior
Suppose that $\{x_n\}$ is a bounded real sequence. Define sequences $\{y_k\}$ and $\{z_k\}$ by

$$y_k = \inf_{n \geq k} x_n \qquad z_k = \sup_{n \geq k} x_n$$

Then $\{y_k\}$ is **bounded non-decreasing** and $\{z_k\}$ is **bounded non-increasing**, and

$$\lim_{k \to \infty} y_k = \sup_k y_k \qquad \text{and} \qquad \lim_{k \to \infty} z_k = \inf_k z_k$$

and we can consider the limits of these convergent sequences, known as the **lim sup** and **lim inf**:

- **lim sup** is the **limiting least upper bound**
- **lim inf** is the **limiting greatest lower bound**

Specifically, we define the **limit superior** (or **upper** limit, or **lim sup**) and the **limit inferior** (or **lower** limit, or **lim inf**) by

$$\limsup_{n \longrightarrow \infty} x_n = \lim_{k \to \infty} \sup_{n \geq k} x_n = \inf_k \sup_{n \geq k} x_n = \overline{\lim}\, x_n$$

$$\liminf_{n \longrightarrow \infty} x_n = \lim_{k \to \infty} \inf_{n \geq k} x_n = \sup_k \inf_{n \geq k} x_n = \underline{\lim}\, x_n$$

Then we have $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$ and $\lim x_n = x$ if and only if $\underline{\lim}\, x_n = x = \overline{\lim}\, x_n$.

We can define the same concepts for real functions; we write

$$\limsup_{x \longrightarrow \infty} f(x) = \lim_{y \longrightarrow \infty} \left\{ \sup_{x \geq y} \{f(x)\} \right\} \qquad \liminf_{x \longrightarrow \infty} f(x) = \lim_{y \longrightarrow \infty} \left\{ \inf_{x \geq y} \{f(x)\} \right\}$$

and the limit as $x \longrightarrow \infty$ exists if and only if

$$\limsup_{x \longrightarrow \infty} f(x) = \liminf_{x \longrightarrow \infty} f(x) = \lim_{x \longrightarrow \infty} f(x).$$

For example, the function $f(x) = \cos(x)$ does not converge to any limit as $x \longrightarrow \infty$. But

$$\sup_{x \geq y} \{\cos(x)\} = 1 \qquad \Longrightarrow \qquad \limsup_{x \longrightarrow \infty} f(x) = \lim_{y \longrightarrow \infty} \left\{ \sup_{x \geq y} \{\cos(x)\} \right\} = \lim_{y \longrightarrow \infty} \{1\} = 1$$

and similarly $\liminf_{x \longrightarrow \infty} f(x) = -1$

Definition: Order Notation ("little oh" and "big oh")
Consider $x \longrightarrow x_0$ where x_0 is possibly $\pm\infty$. Then we write

$$f(x) \sim g(x) \qquad \text{if} \qquad \frac{f(x)}{g(x)} \longrightarrow 1 \quad \text{as} \quad x \longrightarrow x_0$$

$$f(x) = \mathrm{o}(g(x)) \qquad \text{if} \qquad \frac{f(x)}{g(x)} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow x_0$$

$$f(x) = \mathrm{O}(g(x)) \qquad \text{if} \qquad \frac{f(x)}{g(x)} \longrightarrow b \quad \text{as} \quad x \longrightarrow x_0, \text{for some } b$$

with similar notation for real sequences. For example

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = x + \mathrm{o}(x)$$

as $x \longrightarrow 0$, and

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1 = x^3 + \mathrm{o}(x^3) = \mathrm{o}(x^4)$$

as $x \longrightarrow \infty$.

Leibniz's Rule: Let $f(x,t)$ be a real-valued function that is continuous in t and x at least on the closed region $\mathcal{R} \in \mathbb{R}^2$

$$\mathcal{R} \equiv \{(x,t) \in \mathbb{R}^2 : a(t) \leq x \leq b(t), t_0 \leq t \leq t_1\}$$

where $a(.)$ and $b(.)$ are continuous functions of t with continuous derivatives wrt t for $t_0 \leq t \leq t_1$. Suppose also that the partial derivative

$$\frac{\partial f(x,t)}{\partial t}$$

is also continuous in x and t at least on $\mathcal{R}$. Then for $t_0 \leq t \leq t_1$ we have that

$$\frac{d}{dt}\left\{\int_{a(t)}^{b(t)} f(x,t)\, dx\right\} = f(b(t),t)\frac{db(t)}{dt} - f(a(t),t)\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\, dx.$$

Note that if $a(t) = a$ and $b(t) = b$ are constant functions, then

$$\frac{d}{dt}\left\{\int_a^b f(x,t)\, dx\right\} = \int_a^a \frac{\partial f(x,t)}{\partial t}\, dx.$$

- **Series Summations:**

$$\text{GEOMETRIC} \qquad \frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{k=0}^{\infty} z^k \qquad |z| < 1$$

$$\text{EXPONENTIAL} \qquad e^z = 1 + z + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad z \in \mathbb{R}$$

$$\text{BINOMIAL} \quad (n = 1, 2, \cdots) \qquad (1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \cdots + \alpha z^{n-1} + z^n = \sum_{k=0}^{n} \binom{n}{k} z^k$$

$$\text{BINOMIAL} \quad (\alpha > 0) \qquad (1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \cdots = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$$

$$\text{NEG. BINOMIAL} \quad (\alpha > 0) \qquad \frac{1}{(1-z)^\alpha} = 1 + \alpha z + \frac{\alpha(\alpha+1)}{2!} z^2 + \cdots = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} z^k \qquad |z| < 1$$

$$\text{LOGARITHMIC} \qquad -\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots = \sum_{k=1}^{\infty} \frac{z^k}{k} \qquad |z| < 1$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k} \qquad |z| < 1$$

where, if $\Gamma(.)$ is the **gamma function**, in general

$$\binom{\theta}{x} = \frac{\Gamma(\theta+1)}{\Gamma(x+1)\Gamma(\theta-x+1)}.$$

- **Exponential Function:** For real $x > 0$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^{-n} = e^x \qquad \lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{-n} = e^{-x}$$

- **Taylor Series:** For real-valued scalar function f and real number x_0, under mild regularity assumptions

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} f^k(x_0) = \sum_{k=0}^{r} \frac{(x-x_0)^k}{k!} f^k(x_0) + \mathrm{o}((x-x_0)^r)$$

where the approximation holds as $x \longrightarrow x_0$, and

$$f^k(x_0) = \frac{d^k}{dx^k} \{f(x)\}_{x=x_0}$$

if this derivative exists.