# MATH 556 - EXERCISES 1–SOLUTIONS

1. (i) The density must integrate to 1, so we must have $c = \lambda/2$

   (ii) The cdf takes slightly different forms either side of $\theta$. The density is symmetric about $\theta$ so we have

$$F_X(x) = \begin{cases} \frac{1}{2} - \frac{1}{2}(1 - \exp\{\lambda(x-\theta)\} & x \le \theta \\ \frac{1}{2} + \frac{1}{2}(1 - \exp\{-\lambda(x-\theta)\} & x > \theta \end{cases}$$

   (iii) The quantile function takes slightly different forms either side of $p = 1/2$. Again by symmetry, we have

$$Q_X(p) = \begin{cases} \theta + \dfrac{1}{\lambda}\log(2p) & p \le 1/2 \\ \theta - \dfrac{1}{\lambda}\log(2(1-p)) & p > 1/2 \end{cases}$$

   (iv) By symmetry, and the fact that the expectation is finite, we conclude that $\mathbb{E}_X[X] = \theta$;

   (v) The variance of $X$ is equal to the variance of $Z = X - \theta$, so using

$$f_Z(z) = \frac{\lambda}{2}\exp\{-\lambda|z|\} \qquad z \in \mathbb{R}$$

   we have

$$\begin{aligned} \mathbb{E}_Z[Z^2] &= \frac{\lambda}{2}\int_{-\infty}^{0} z^2 \exp\{\lambda z\}\,\mathrm{d}z + \frac{\lambda}{2}\int_{0}^{\infty} z^2 \exp\{-\lambda z\}\,\mathrm{d}z \\ &= \lambda \int_0^\infty z^2 \exp\{-\lambda z\}\,\mathrm{d}z = \lambda\frac{\Gamma(3)}{\lambda^3} = \frac{2}{\lambda^2} \end{aligned}$$

   by the fact that in the integral the integrand is proportional to a Gamma pdf.

2. For joint density defined on the unit cube $(0,1)^3$.

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = c(1 - \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3))$$

   and zero otherwise, for some constant $c$.

   (a) We have for $0 < x_1, x_2 < 1$

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \int_0^1 c(1 - \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3))\,\mathrm{d}x_3 \\ &= c - c\sin(2\pi x_1)\sin(2\pi x_2)\int_0^1 \sin(2\pi x_3)\,\mathrm{d}x_3 \\ &= c \end{aligned}$$

   Thus $c = 1$, and $X_1$ and $X_2$ are marginally uniform, and independent.

   (b) $(X_1, X_2, X_3)$ are not independent as the density does not factorize into the product of marginals, which is a necessary condition for independence.

3. We have that

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \left(\frac{1}{2\pi}\right)^{n/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} x_i^2\right\}$$

Let $V_i = X_i^2, i = 1,\ldots,n$. Then by univariate transformation methods

$$f_{V_1,\ldots,V_n}(v_1,\ldots,v_n) = \left(\frac{1}{2\pi}\right)^{n/2} \prod_{i=1}^{n} v_i^{-1/2} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} v_i\right\}$$

for $(v_1,\ldots,v_n)$ on $\{\mathbb{R}^+\}^n$. Now if

$$S = \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} V_i \qquad T_i = \frac{X_i^2}{S} = V_i/S \qquad i = 1,\ldots,n$$

then we know that

$$\sum_{i=1}^{n} T_i = 1.$$

Hence consider the transform $(V_1,\ldots,V_n) \longrightarrow (S, T_1,\ldots,T_{n-1})$, and the inverse transform

$$V_i = ST_i \quad i = 1,\ldots,n-1 \qquad\qquad V_n = S\left(1 - \sum_{i=1}^{n-1} T_i\right)$$

The Jacobian matrix of partial derivatives is an $n \times n$ matrix taking the form

$$\begin{bmatrix} t_1 & s & 0 & \cdots & 0 \\ t_2 & 0 & s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ t_{n-1} & 0 & \cdots & 0 & s \\ \left(1 - \sum\limits_{i=1}^{n-1} t_i\right) & -s & -s & \cdots & -s \end{bmatrix}$$

The determinant of this matrix is identical to the determinant of the matrix formed by adding the first $n-1$ rows to the last, that is,

$$\begin{bmatrix} t_1 & s & 0 & \cdots & 0 \\ t_2 & 0 & s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ t_{n-1} & 0 & \cdots & 0 & s \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

which yields the determinant $s^{n-1}$. Hence the target joint density is

$$\begin{aligned} f_{S,T_1,\ldots,T_{n-1}}(s,t_1,\ldots,t_{n-1}) &= \left(\frac{1}{2\pi}\right)^{n/2} \left\{\prod_{i=1}^{n-1} (st_i)^{-1/2}\right\} \left(s\left(1 - \sum_{i=1}^{n-1} t_i\right)\right)^{-1/2} \exp\left\{-\frac{1}{2}s\right\} s^{n-1} \\ &= \left(\frac{1}{2\pi}\right)^{n/2} \left\{\prod_{i=1}^{n} t_i^{-1/2}\right\} \left(1 - \sum_{i=1}^{n-1} t_i\right)^{-1/2} s^{n/2-1} \exp\left\{-\frac{s}{2}\right\} \end{aligned}$$

The support of this density is readily seen to be $\mathbb{R}^+ \times \mathcal{S}_{n-1}$, where $\mathcal{S}_{n-1}$ is the $n-1$ dimensional simplex. Thus we have directly that for all $(s, t_1, \ldots, t_{n-1})$,

$$f_{S,T_1,\ldots,T_{n-1}}(s, t_1, \ldots, t_{n-1}) = f_S(s) f_{T_1,\ldots,T_{n-1}}(t_1, \ldots, t_{n-1})$$

where

$$f_S(s) = \frac{(1/2)^{n/2}}{\Gamma(n/2)} s^{n/2-1} \exp\left\{-\frac{s}{2}\right\} \qquad s > 0$$

and zero otherwise, that is, $S \sim Gamma(n/2, 1/2)$, and

$$f_{T_1,\ldots,T_{n-1}}(t_1, \ldots, t_{n-1}) = \frac{\Gamma(n/2)}{\pi^{n/2}} \left\{\prod_{i=1}^{n} t_i^{-1/2}\right\} \left(1 - \sum_{i=1}^{n-1} t_i\right)^{-1/2}$$

In fact, we have that

$$(T_1, \ldots, T_{n-1}) \sim Dirichlet(1/2, 1/2, \cdots, 1/2).$$

Thus $S$ and $(T_1, \ldots, T_{n-1})$ are independent. Finally, as $T_n$ is a deterministic function of $(T_1, \ldots, T_{n-1})$, $S$ is independent of $T_n$ also.

4. The cdf of the $Pareto(\theta, \alpha)$ distribution is

$$F_X(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha} \qquad x > 0$$

which yields the quantile function

$$Q_X(p) = \theta(\{(1-p)\}^{-1/\alpha} - 1)$$

Therefore we may set

$$X = \theta(\{(1 - (1 - \exp\{-Z\}))\}^{-1/\alpha} - 1)$$

or

$$X = \theta(\exp\{Z/\alpha\} - 1)$$

as we require

$$\mathrm{P}_Z[g(Z) \leq x] = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$$

for $x > 0$, but

$$\mathrm{P}_Z[g(Z) \leq x] \equiv \mathrm{P}_Z[Z \leq g^{-1}(x)] = 1 - \exp\{-g^{-1}(x)\}$$

dictates that

$$\exp\{-g^{-1}(x)\} = \left(\frac{\theta}{\theta + x}\right)^{\alpha}$$

or

$$g^{-1}(x) = -\alpha \log\theta + \alpha \log(\theta + x)$$

which yields the solution.