MATH 556 - EXERCISES 6

Not for Assessment.

1. Suppose that $X_1, \ldots, X_r$ are independent random variables such that, for each i, $X_i \sim N(\mu_i, 1)$, for fixed constants $\mu_1, \ldots, \mu_r$.
   (a) Find the mgf of random variable Y defined by
   $$Y = \sum_{i=1}^{r} X_i^2.$$
   (b) Find the skewness of Y, ς, where
   $$\varsigma = \frac{\mathbb{E}_Y[(Y-\mu)^3]}{\sigma^3}$$
   where μ and σ^2 are the expectation and variance of f_Y.

2. Consider the three-level hierarchical model:
   $$\begin{array}{lll} \text{LEVEL 3}: & \lambda > 0, r \in \{1, 2, \ldots\} & \text{Fixed parameters} \\ \text{LEVEL 2}: & N \sim Poisson(\lambda) & \\ \text{LEVEL 1}: & X|N = n \sim Gamma(n + r/2, 1/2) & \end{array}$$
   Find
   (a) The expectation of X, $\mathbb{E}_X[X]$,
   (b) The mgf of X, $M_X(t)$.

3. Consider the three-level hierarchical model:
   $$\begin{array}{lll} \text{LEVEL 3}: & \mu \in \mathbb{R}, \tau, \sigma > 0 & \text{Fixed parameters} \\ \text{LEVEL 2}: & M \sim Normal(\mu, \tau^2) & \\ \text{LEVEL 1}: & X_1, X_2|M = m \sim Normal(m, \sigma^2) & \end{array}$$
   where X_1 and X_2 are conditionally independent given M, denoted
   $$X_1 \perp X_2 \mid M.$$
   Using the law of iterated expectation, find the (marginal) covariance and correlation between X_1 and X_2. Are X_1 and X_2 (marginally) independent ? Justify your answer.

4. In a branching process model, the total number of individuals in successive generations are random variables $S_0, S_1, S_2, \ldots$. Suppose that, in the passage from generation i to generation $i+1$, each of the s_i individuals observed in generation i gives rise to N_{ij} offspring for $j = 1, \ldots, s_i$ according to a pmf with corresponding pgf G_N.

   In addition to the production of offspring, suppose that at each generation, immigration into the population is allowed, and that at generation i, K_i new individuals enter the population to go forward to the $i+1$st generation, so that
   $$S_{i+1} = \sum_{j=1}^{s_i} N_{ij} + K_i$$
   where $K_0, K_1, K_2, \ldots$ are iid random variables, with pgf G_K, that are independent of all N_{ij}.

   Find the pgf of S_{i+1} in terms of the pgf of random variable S_i and G_K.