MATH 556 - EXERCISES 4

Not for Assessment.

1. Suppose that $X_1 \sim Geometric(\theta_1)$ and $X_2 \sim Geometric(\theta_2)$ are independent random variables. Find the pmf of random variable Y where $Y = X_1 + X_2$.

2. Suppose that X_1 and X_2 are random variables with joint pdf given by

$$f_{X_1,X_2}(x_1, x_2) = c|x_1| \exp\left\{-|x_1| - \frac{x_1^2 x_2^2}{2}\right\} \qquad (x_1, x_2) \in \mathbb{R}^2$$

   Find the marginal pdf of X_1, and the conditional pdf of X_2 given $X_1 = x_1$, for appropriate values of x_1. *Take care to define the pdfs for all real values of their arguments. Compute the value of constant c*

3. The radius of a circle, R, is a continuous random variable with density function given by

$$f_R(r) = 6r(1-r) \qquad 0 < r < 1$$

   and zero otherwise. Find the joint and marginal pdfs of X_1, the circumference of the circle, and X_2, the area of the circle.

4. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x, y) = cx(1-y) \qquad 0 < x < 1,\ 0 < y < 1$$

   and zero otherwise for some constant c. Are X and Y independent random variables ?

   Find the value of c, and, for the set $A \equiv \{(x, y) : 0 < x < y < 1\}$, the probability

$$P_{X,Y}[X < Y] = \iint_A f_{X,Y}(x, y)\, dxdy$$

5. Suppose that X and Y are continuous random variables with joint pdf given by

$$f_{X,Y}(x, y) = \frac{1}{2x^2 y} \qquad 1 \le x < \infty, 1/x \le y \le x$$

   and zero otherwise. Derive

   (i) the marginal pdfs of X and Y
   (ii) the conditional pdf of X given $Y = y$, and the conditional pdf of Y given $X = x$.
   (iii) the expectation of Y, $\mathbb{E}_Y[Y]$.

6. Suppose that X and Y have joint pdf that is constant with support $\mathbb{X}^{(2)} \equiv (0, 1) \times (0, 1)$.

   (i) Find the marginal pdf of random variables $U = X/Y$ and $V = -\log(XY)$, stating clearly the range of the transformed random variable in each case.
   (ii) Find the pdf and cdf of $Z = X - Y$.

7. (a) Consider random variable X with probability function P_X and cdf F_X. The indicator random variable for set B, $\mathbb{1}_B(.)$, is a transformation of X, and is defined by

$$\mathbb{1}_B(X) = \begin{cases} 1 & X \in B \\ 0 & X \notin B \end{cases}$$

Find the pmf/pdf and the expectation of rv $\mathbb{1}_B(X)$.

(b) The expectation of any random variable with pmf/pdf f_X can be approximated to arbitrary accuracy (under mild conditions) by a *Monte Carlo* simulation procedure: a large sample of simulated values $x_1, \ldots, x_N$ are generated from f_X, and then the expectation is approximated by the sample mean to produce the approximation $\widehat{\mathbb{E}}_X[X]$.

$$\widehat{\mathbb{E}}_X[X] = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Using the result in (a), and a Monte Carlo procedure, to approximate the probability

$$P_{\mathbf{X}}[\mathbf{X} \in B]$$

if $\mathbf{X}$ has a three-dimensional multivariate Normal distribution, $\mathbf{X} \sim Normal_3(\mathbf{0}, \Sigma)$, with

$$\Sigma = \begin{bmatrix} 1.0 & 0.2 & -0.5 \\ 0.2 & 2.0 & -0.1 \\ -0.5 & -0.1 & 2.0 \end{bmatrix}$$

and B is the set $\{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq x_1 + x_3\}$. Tabulate the results of five replicate Monte Carlo runs with $N = 10000$.

`R` functions for simulating the multivariate Normal distribution include `mvrnorm` from the `MASS` library and `rmvn` from the `mvnfast` library. Note also that if $Z_1, \ldots, Z_n$ are independent standard Normal variables, and $\mathbf{L}$ is an $n \times n$ matrix, then

$$\mathbf{Y} = \mathbf{L}\mathbf{Z} \sim Normal(0, \mathbf{L}\mathbf{L}^\top).$$

Therefore if $\mathbf{L}$ is the lower-triangular matrix termed the *Cholesky factor* for Σ, defined by

$$\mathbf{L}\mathbf{L}^\top = \Sigma$$

then we can generate $\mathbf{X}$ as $\mathbf{L}\mathbf{Z}$. The Cholesky factor can be computed in `R` using the function `chol`.

```
Sigma<-matrix(c(1,0.2,-0.5,0.2,2,-0.1,-0.5,-0.1,2.0),3,3)
library(mvnfast)
N<-10000
X<-rmvn(N,mu=c(0,0,0),Sigma)
cov(X)          #computes the sample covariance matrix for X

Z<-matrix(rnorm(N*3),nrow=N,ncol=3)
L<-t(chol(Sigma)) #chol produces the transpose of L
X<-t(L %*% t(Z))
cov(X)
```