MATH 556 - EXERCISES 3

Not for Assessment.

1. Suppose X is a random variable, with mgf $M_X(t)$ defined on $(-h, h)$ for some $h >$. Show that

$$P_X[X \geq a] \leq e^{-at} M_X(t) \qquad \text{for } 0 < t < \delta$$

For the *cumulant generating function* $K_X(t) = \log M_X(t)$, verify that

$$\frac{d}{dt}\{K_X(t)\}_{t=0} = \mathbb{E}_X[X] \qquad\qquad \frac{d^2}{dt^2}\{K_X(t)\}_{t=0} = \text{Var}_X[X]$$

2. The *non-central chi-square* distribution arises as the distribution of the square of a normal random variable. That is, if $X \sim \text{Normal}(\mu, 1)$, then $Y = X^2$ has the non-central chi-square distribution with one degree of freedom and non-centrality parameter λ, denoted $Y \sim \chi^2_\nu(\lambda)$, where $\nu = 1$ and $\lambda = \mu^2$. In this setting,

   (a) Find the pdf of Y, and show that it can be expressed in the form

   $$f_Y(y) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} f_{Z_{2j+k}}(y) \qquad y > 0$$

   where f_{Z_m} is the pdf of a random variable Z_m which has a chi-square distribution with m degrees of freedom (that is, $Z_m \sim Gamma(m/2, 1/2)$).

   (b) Find the characteristic function $\varphi_Y(t)$.

   (c) Find the *Laplace transform* $\mathcal{L}_Y(t)$, defined for $t \geq 0$ by

   $$\mathcal{L}_Y(t) = \int_0^\infty e^{-ty}\, dF_Y(y) = \mathbb{E}_Y[e^{-tY}].$$

   Note that $\mathcal{L}_Y(t)$ is well-defined provided $Y \geq 0$ with probability 1.

   (d) Find the expectation and variance of Y.

   (e) Find the distribution of

   $$S = \sum_{i=1}^{n} Y_i$$

   where $Y_1, \ldots, Y_n$ are independent, with $Y_i \sim \chi^2_{\nu_i}(\lambda_i)$, $i = 1, \ldots, n$.

3. If $\mathcal{L}_X(t)$ is the Laplace transform (see question above) of a nonnegative random variable X, show that for $r = 1, 2, \ldots$

$$(-1)^r \frac{d^r}{dt^r}\{\mathcal{L}_\mathcal{X}(t)\} \geq 0 \qquad t \geq 0.$$

If F_X is the corresponding cdf, show that

$$\mathcal{L}_X(t) = t \int_0^\infty \exp\{-tx\} F_X(x) dx.$$

4. Suppose that $X_1 \sim Gamma(\alpha_1, \beta_1)$ and $X_2 \sim Gamma(\alpha_2, \beta_2)$ are independent random variables. Characterize the distribution of $Y = X_1 - X_2$.

5. Suppose that $\{\varphi_k(t)\}_{k=1}^n$ is a sequence of characteristic functions, and $\{c_k\}_{k=1}^n$ is a sequence of non-negative real valued constants, with

$$\sum_{k=1}^n c_k = 1.$$

Show that

$$\sum_{k=1}^n c_k \varphi_k(t)$$

is also a characteristic function, and identify the distribution to which it corresponds. Does the result extend to the case where $n \longrightarrow \infty$? Justify your answer.

6. If

$$\varphi_1(t) = \exp(-4t^2) \qquad \varphi_2(t) = (3 + \cos(t) + \cos(2t))/5$$

identify the distribution with cf

$$\frac{\varphi_1(t) + \varphi_2(t)}{2}.$$

7. Suppose X_1 and X_2 are independent random variables, and suppose also that X_1 and $X_1 - X_2$ are independent. Show that

$$P_{X_1}[\,X_1 = c\,] = 1$$

for some constant c.

Hint: write $X_2 = X_1 + (X_2 - X_1)$, and recall that if $\varphi(t)$ is an arbitrary cf, then $\varphi(t)$ is continuous for all t.

8. Suppose that mgf $M_X(t)$ is defined, for a suitable neighbourhood of zero $(-h, h)$, as

$$M_X(t) = \frac{9e^{-t}}{(3+2t)^2}.$$

Find an expression for $\mathbb{E}_X[X^r]$, for $r = 1, 2, \ldots,$.

9. Suppose that $X \sim$ Binomial(n, θ) for integer $n \geq 1$, and $0 < \theta < 1$. Let

$$Z_n = \frac{(X - n\theta)}{\sqrt{n\theta(1-\theta)}}.$$

Find the first two non-zero terms in the power series expansion of the cumulant generating function of Z_n, and the order of approximation (in terms of n) when truncating the expansion at the second term, for large n.

Recall that

$$\log\{(1+z)^n\} = n\{z - z^2/2 + \cdots\}.$$