MATH 556 - ASSIGNMENT 3 – SOLUTIONS

1.

$$P_{X,Y}\left[X < Y\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{X,Y}(x,y)\,dx\,dy = \int_{0}^{\infty}\int_{0}^{y} f_X(x)f_Y(y)\,dx\,dy$$

$$= \int_0^{\infty} F_X(y)f_Y(y)\,dy \qquad \text{by definition of } F_X$$

$$= \int_0^1 F_X(F_Y^{-1}(t))f_Y(F_Y^{-1}(t))\frac{dF_Y^{-1}(t)}{dt}\,dt \qquad \text{setting } t = F_Y(y)$$

$$= \int_0^1 F_X(F_Y^{-1}(t))\,dt$$

as

$$\frac{dF_Y^{-1}(t)}{dt} = \frac{1}{f_Y(F_Y^{-1}(t))}$$

using implicit differentiation. 6 Marks

2. The joint pdf of Z_1 and Z_2 is

$$f_{Z_1,Z_2}(z_1,z_2) = \mathbb{1}_{(0,\infty)}(z_1)\mathbb{1}_{(0,\infty)}(z_2)\exp\{-(z_1+z_2)\}.$$

The inverse transformations are given by

$$Z_1 = Y_1Y_2 \qquad Z_2 = (1-Y_1)Y_2$$

and Y_1 has support $(0,1)$, with Y_2 having support $(0,\infty)$. The Jacobian is therefore

$$\left|\begin{bmatrix} \dfrac{\partial z_1}{\partial y_1} & \dfrac{\partial z_1}{\partial y_2} \\ \dfrac{\partial z_2}{\partial y_1} & \dfrac{\partial z_2}{\partial y_2}\end{bmatrix}\right| = \left|\begin{bmatrix} y_2 & y_1 \\ -y_2 & 1-y_1\end{bmatrix}\right| = y_2$$

so therefore, using the transformation theorem, we have that

$$f_{Y_1,Y_2}(y_1,y_2) = \mathbb{1}_{(0,1)}(y_1)\mathbb{1}_{(0,\infty)}(y_2)\exp\left\{-(y_1y_2 + (1-y_1)y_2\right\} = \mathbb{1}_{(0,1)}(y_1)\mathbb{1}_{(0,\infty)}(y_2)y_2\exp\left\{-y_2\right\}.$$

We can then spot that

$$f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2) \qquad (y_1,y_2)\in\mathbb{R}^2$$

where

$$f_{Y_1}(y_1) = \mathbb{1}_{(0,1)}(y_1)$$

$$f_{Y_2}(y_2) = \mathbb{1}_{(0,\infty)}(y_2)y_2\exp\left\{-y_2\right\}$$

that is, $Y_1 \sim Uniform(0,1)$ and $Y_2 \sim Gamma(2,1)$.

Alternately, we can deduce using mgfs that $Y_2 \sim Gamma(2,1)$ as it is the sum of two independent $Exponential(1)$ rvs, and then note that by symmetry of form Y_1 and $1-Y_1$ must have the same distribution, which for a rv on $(0,1)$ ensures that the distribution of Y_1 is $Uniform(0,1)$. 5 Marks

Y_1 and Y_2 are independent, as the joint pdf factorizes into a product of the two marginal pdfs and the support of the joint pdf is the Cartesian production $(0,1)\times\mathbb{R}^+$. 1 Mark

3. We have

$$f_X(x;\psi,\gamma) = \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{1}{2\pi\gamma x^3}}\exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right\}$$

for $\psi, \gamma > 0$ and

(a) This is NOT a location-scale family. For the family to be a location-scale family, we must be able to make a transform of the form

$$Z = \frac{X-\mu}{\sigma}$$

with the result that the distribution of Z does not depend on any parameters. The presence of the $1/x$ term renders the required linear transformation impossible. 2 Marks

(b) This IS an Exponential Family distribution; we may write the transparent parameterization

$$f_X(x;\psi,\gamma) = h(x)\exp\left\{(c_1(\theta_1), c_2(\theta_2))\begin{pmatrix}T_1(x)\\ T_2(x)\end{pmatrix} - A(\theta)\right\}$$

where

- $h(x) = \mathbb{1}_{(0,\infty)}(x)x^{-3/2}(2\pi)^{-1/2}$
- $T_1(x) = x$, $T_2(x) = 1/x$.
- $c_1(\theta) = -\dfrac{1}{2}\psi^2\gamma$ and $c_2(\theta) = -\dfrac{1}{2\gamma}$.
- $A(\theta) = -\psi + \dfrac{1}{2}\log\gamma$.

2 Marks

(c) Using the score result,we see that

$$\mathbb{E}_X\left[\frac{\partial c_1(\theta)}{\partial\psi}X + \frac{\partial c_2(\theta)}{\partial\psi}\frac{1}{X}\right] = \frac{\partial A(\theta)}{\partial\psi}$$

and

$$\mathbb{E}_X\left[\frac{\partial c_1(\theta)}{\partial\gamma}X + \frac{\partial c_2(\theta)}{\partial\gamma}\frac{1}{X}\right] = \frac{\partial A(\theta)}{\partial\gamma}$$

or equivalently

$$\mathbb{E}_X\left[-\psi\gamma X + 0\frac{1}{X}\right] = -1 \qquad \therefore \qquad \mathbb{E}_X[X] = \frac{1}{\psi\gamma}$$

and

$$\mathbb{E}_X\left[-\frac{1}{2}\psi^2 X + \frac{1}{2\gamma^2}\frac{1}{X}\right] = \frac{1}{2\gamma} \qquad \therefore \qquad \mathbb{E}_X\left[\frac{1}{X}\right] = \gamma + \psi\gamma$$

4 Marks

Note that we may further rewrite the density

$$f_X(x;\phi_1,\phi_2) = \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{\phi_1}{2\pi x^3}}\exp\left\{-\frac{\phi_1}{2}\frac{(x-\phi_2)^2}{\phi_2^2 x}\right\}$$

where

$$\phi_1 = \frac{1}{\gamma} \qquad \phi_2 = \frac{1}{\psi\gamma}$$

rendering

$$\mathbb{E}_X[X] = \phi_2 \qquad \mathbb{E}_X\left[\frac{1}{X}\right] = \frac{1}{\phi_1} + \frac{1}{\phi_2}$$