MATH 556 - ASSIGNMENT 2 – SOLUTIONS

1. By direct calculation

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} dF_X(x) = \int_{-\infty}^{\infty} e^{itx} d\left(\sum_{k=1}^{K} \omega_k F_k(x)\right) = \sum_{k=1}^{K} \omega_k \left(\int_{-\infty}^{\infty} e^{itx} dF_k(x)\right)$$

(writing out the sum/integral in full) so therefore

$$\varphi_X(t) = \sum_{k=1}^{K} \omega_k \varphi_k(t).$$

4 Marks

2. We have seen in lectures the cf of $X \sim Exponential(1)$, which is absolutely continuous with cdf

$$F_X(x) = 1 - e^{-x} \qquad x > 0$$

and zero otherwise, takes the form

$$\varphi_X(t) = \frac{1}{1 - it} \qquad t \in \mathbb{R}.$$

Here we have that

$$|\varphi_X(t)| = \frac{1}{\sqrt{1 + t^2}}$$

and which provides the counterexample, as this function integrates to infinity on $\mathbb{R}$.

A strategy for funding such a counterexample involves (a) finding any suitable real-valued function that integrates to infinity, then (b) proving it is a cf; the second step can be achieved either by looking the table of cfs (or Fourier transforms), applying Bochner's Theorem, or checking the sufficient conditions generated by Polya's Theorem.

4 Marks

3. The function

$$\varphi_X(t) = \frac{1}{2}\left(\cos(t) + \cos(\pi t)\right)$$

exhibits the behaviour that

$$\limsup_{|t| \longrightarrow \infty} |\varphi_X(t)| = 1$$

due to the $\cos$ terms. Therefore X is discrete by the result from lectures. Furthermore, $\varphi_X(t)$ is entirely real, and therefore must correspond to a distribution with mass function that is symmetric about zero. From the cf definition, and the nature of the $\cos$ function, we deduce that the pmf can only have support on values $\{-\pi, -1, 1, \pi\}$, and hence

$$f_X(x) = \frac{1}{4} \qquad x \in \{-\pi, -1, 1, \pi\}$$

and zero otherwise. Note that $\cos(-t) = \cos(t)$ and $\cos(-\pi t) = \cos(\pi t)$.

(a) No, it is the discrete distribution on the finite support identified above. 2 Marks

(b) As the distribution has finite support, moments of all orders are finite. 2 Marks

4. Using the formula sheet

   (a) If $X \sim Poisson(\lambda)$, then $M_X(t) = \exp\{\lambda(e^t - 1)\}$ for $t \in \mathbb{R}$, so

   $$K_X(t) = \lambda(e^t - 1) = \lambda \sum_{j=1}^{\infty} \frac{t^j}{j!}$$

   Therefore all the cumulants are equal to λ. 2 Marks

   (b) If $X \sim Normal(\mu, \sigma^2)$, then $M_X(t) = \exp\{\mu t + \sigma^2 t^2/2\}$ for $t \in \mathbb{R}$, so

   $$K_X(t) = \mu t + \sigma^2 \frac{t^2}{2}.$$

   Hence

   $$\kappa_{X1} = \mu \qquad \kappa_{X2} = \sigma^2 \qquad \kappa_{X3} = 0.$$

   2 Marks

5. We have that

   $$\varphi(t) = \frac{1}{(1 + 2t^2 + t^4)} = \frac{1}{(1 + t^2)^2} = \{\varphi_X(t)\}^2$$

   where, from lectures, $\varphi_X(t)$ is the cf for $X \sim Laplace$ with pdf

   $$f_X(x) = \frac{1}{2} e^{-|x|} \qquad x \in \mathbb{R}.$$

   Hence $\varphi(t)$ is the cf of a continuous random variable, Y say, which can be decomposed as the sum of two independent *Laplace* or *double exponential* random variables. 4 Marks

   We can diagnose that $\varphi(t)$ is the cf of an absolutely continuous rv as

   $$\limsup_{|t| \longrightarrow \infty} |\varphi(t)| = 0$$

   and we also have here that

   $$\int_{-\infty}^{\infty} |\varphi(t)| dt < \infty$$

   so in principle we can use the continuous version of the inversion formula to find the corresponding pdf.