# MATH 556 - ASSIGNMENT 1 – SOLUTIONS

1. (a) We have in general that

$$F_X(x) = 1 - \exp\{-\beta x^\alpha\} \qquad x > 0$$

with $F_X(0) = 0$ for $x \leq 0$. Therefore, by direct calculation

$$Q_X(p) = \left\{-\frac{1}{\beta}\log(1-p)\right\}^{1/\alpha} \qquad 0 < p < 1$$

2 Marks

(b) We deduce directly that $c = 1/10$, and hence that

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \dfrac{\lfloor \min\{x, 10\} \rfloor}{10} & x \geq 1 \end{cases}.$$

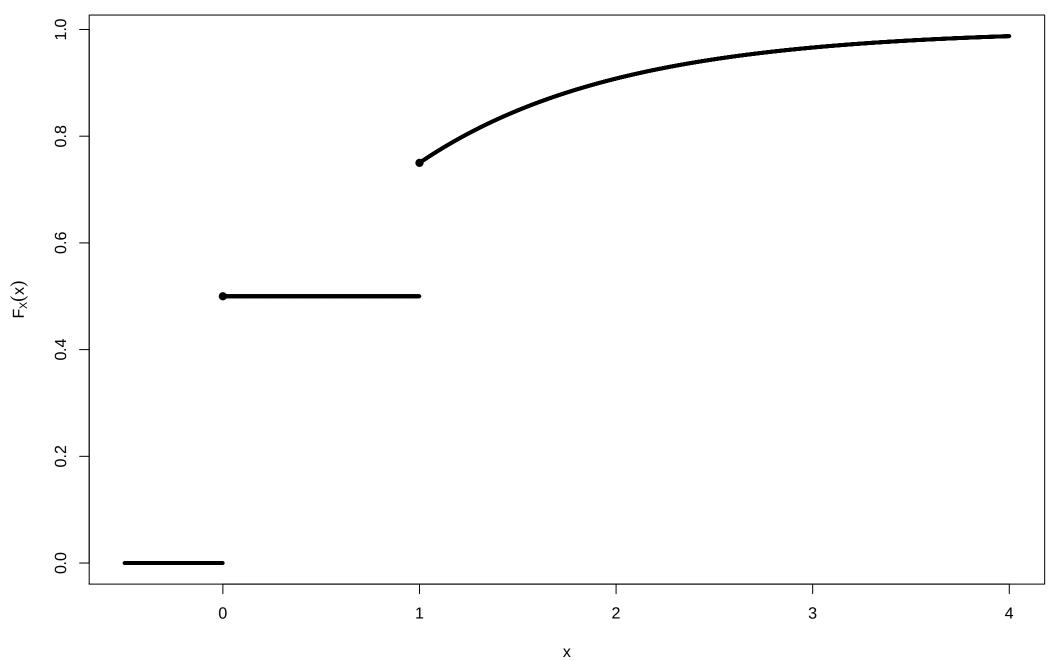
Hence

$$Q_X(p) = \lceil 10p \rceil \qquad 0 < p < 1.$$

2 Marks

(c) By right-continuity at $x = 1$ we must have

$$\frac{3}{4} = 1 - c \qquad \Longrightarrow \qquad c = \frac{1}{4}.$$




so therefore

$$Q_X(p) = \begin{cases} 0 & 0 < p \leq 0.5 \\ 1 & 0.5 < p \leq 0.75 \\ 1 - \log(4(1-p)) & 0.75 < p < 1 \end{cases}$$

4 Marks

2. (a) For $y > 0$

$$F_Y(y) = P_Y[Y \leq y] = P_X[X^2 \leq y] = P_Y[-\sqrt{y} \leq X \leq \sqrt{y}] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

so therefore by differentiation, for $y > 0$

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) = \frac{1}{\sqrt{y}} f_X(\sqrt{y})$$

as $f_X(.)$ is symmetric around zero. That is

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{y}{2}\right\} \qquad y > 0$$

and zero otherwise.

1 Mark

(b) For $y > 0$

$$F_Y(y) = P_Y[Y \leq y] = P_X[|X| \leq y] = P_Y[-y \leq X \leq y] = F_X(y) - F_X(-y)$$

so therefore by differentiation, for $y > 0$

$$f_Y(y) = f_X(y) + f_X(-y) = 2f_X(y)$$

as $f_X(.)$ is symmetric around zero. That is

$$f_Y(y) = \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \qquad y > 0$$

and zero otherwise.

1 Mark

(c) We have

$$F_Y(y) = P_Y[Y \leq y] = P_X[2X - X^2 \leq y] = P_X[X^2 - 2X + y \geq 0] = P_X[(X - a_1(y))(X - a_2(y)) \geq 0]$$

say, where

$$(a_1(y), a_2(y)) = \frac{2 \pm \sqrt{4(1-y)}}{2} = 1 \pm \sqrt{1-y}$$

provided $y \leq 1$; if $y > 1$, $P_X[X^2 - 2X + y \geq 0] = 1$. Thus for $y < 1$,

$$F_Y(y) = P_X[X \leq a_1(y)] + P_X[X \geq a_2(y)] = F_X(a_1(y)) + 1 - F_X(a_2(y))$$

and hence

$$f_Y(y) = \frac{1}{2\sqrt{1-y}} f_X(1 - \sqrt{1-y}) + \frac{1}{2\sqrt{1-y}} f_X(1 + \sqrt{1-y})$$

2 Marks

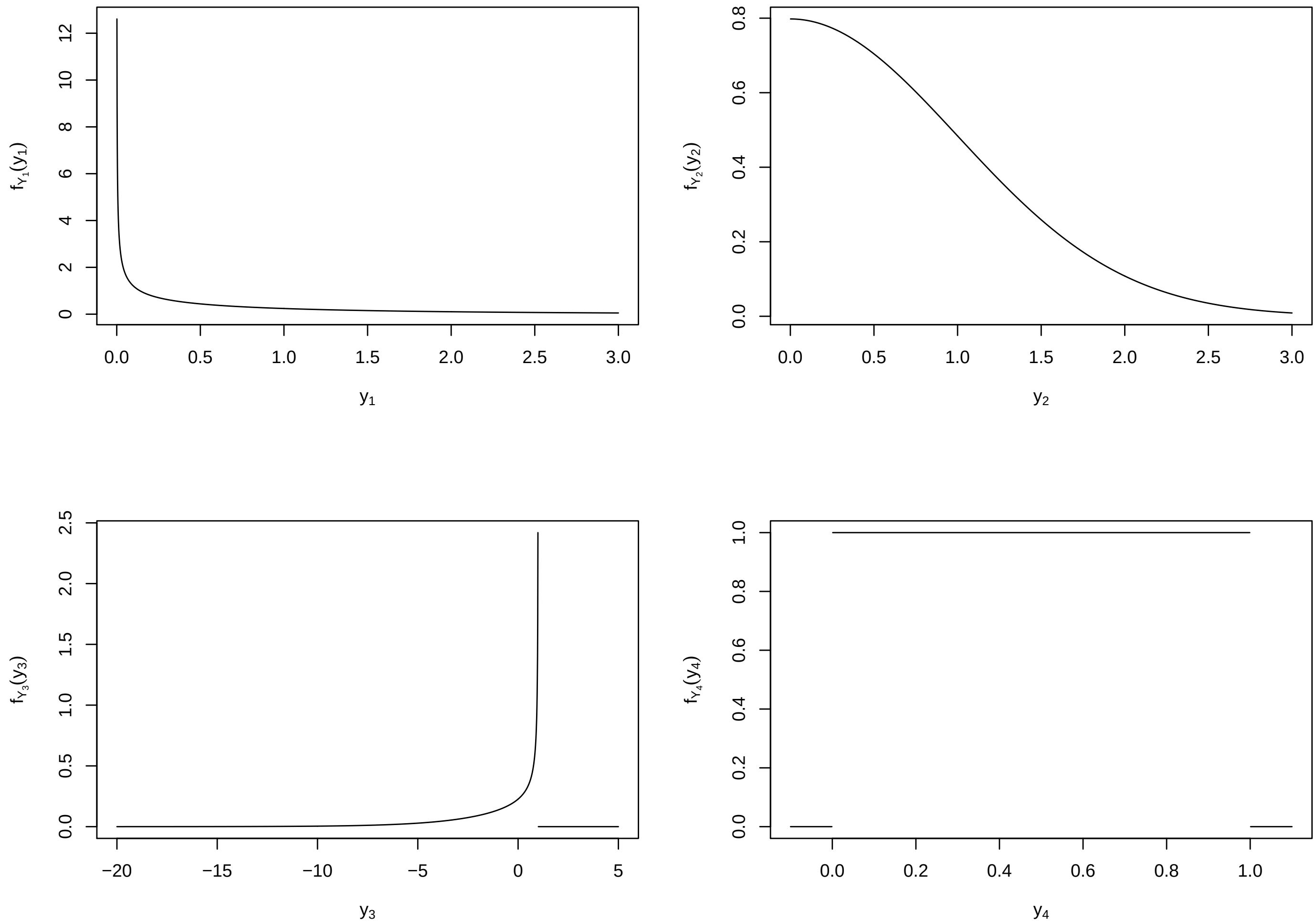
(d) The function $F_X(.)$ maps onto $(0, 1)$, so for $0 < y < 1$

$$F_Y(y) = P_Y[Y \leq y] = P_X[F_X(X) \leq y] = P_X[X \leq F_X^{-1}(y)] = F_X(F_X^{-1}(y)) = y$$

so therefore

$$f_Y(y) = 1 \qquad 0 < y < 1$$

and zero otherwise. 2 Marks




3. We have

$$\mathbb{E}_Y[Y] \equiv \mathbb{E}_X[\{F_X(X)\}^k] = \int_{-\infty}^{\infty} \{F_X(x)\}^k f_X(x)\; dx = \left[\frac{1}{k+1}\{F_X(x)\}^{k+1}\right]_{-\infty}^{\infty} = \frac{1}{k+1}.$$

3 Marks

4. We have that the area is a continuous random variable $Z$ given by $Z = XY$. Then, by first principles of expectations

$$\begin{aligned}
\mathbb{E}_Z[Z] &= \int_0^{\infty} z f_Z(z)\; dz \\
&\equiv \int_0^{\infty}\int_0^{\infty} xy f_{X,Y}(x,y)\; dx\; dy \\
&= \left\{\int_0^{\infty} x f_X(x)\; dx\right\}\left\{\int_0^{\infty} y f_Y(y)\; dy\right\} \qquad \text{by independence} \\
&= \mathbb{E}_X[X]\mathbb{E}_Y[Y] = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

3 Marks