556: MATHEMATICAL STATISTICS I

ASYMPTOTIC APPROXIMATIONS AND THE DELTA METHOD

To approximate the distribution of elements in sequence of random variables $\{X_n\}$ for large n, we attempt to find sequences of constants $\{a_n\}$ and $\{b_n\}$ such that

$$Z_n = a_n X_n + b_n \xrightarrow{d} Z$$

where Z has some distribution characterized by cdf F_Z. Then, for large n, $F_{Z_n}(z) \simeq F_Z(z)$, so

$$F_{X_n}(x) = P[X_n \leq x] = P[a_n X_n + b_n \leq a_n x + b_n] = F_{Z_n}(a_n x + b_n) \simeq F_Z(a_n x + b_n).$$

EXAMPLE Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. such that $X_i \sim Exp(1)$, and let $Y_n = \max\{X_1, X_2, \ldots, X_n\}$. Then by a previous result, for $y > 0$,

$$F_{Y_n}(y) = \{F_X(y)\}^n = \{1 - e^{-y}\}^n \longrightarrow 0$$

and there is no limiting distribution. However, if we take $a_n = 1$ and $b_n = -\log n$, and set $Z_n = a_n Y_n + b_n$, then as $n \longrightarrow \infty$,

$$\begin{aligned} F_{Z_n}(z) &= \mathrm{P}[Z_n \leq z] = \mathrm{P}[Y_n \leq z + \log n] = \{1 - e^{-z-\log n}\}^n \longrightarrow \exp\{-e^{-z}\} = F_Z(z), \\ \therefore \quad F_{Y_n}(y) &= P[Y_n \leq y] = P[Z_n \leq y - \log n] \simeq F_Z(y - \log n) = \exp\{-e^{-y+\log n}\} = \exp\{-ne^{-y}\} \end{aligned}$$

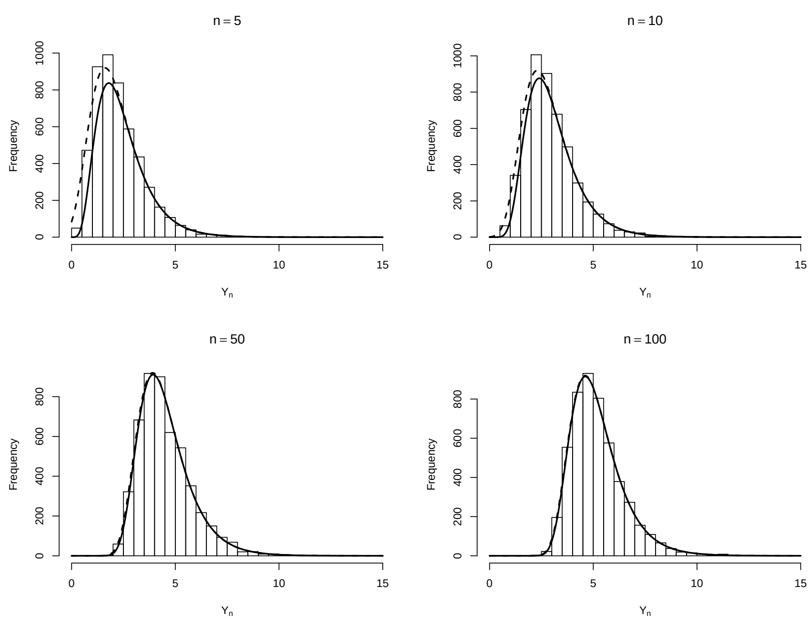
and by differentiating, for $y > 0$

$$f_{Y_n}(y) \simeq ne^{-y} \exp\{-ne^{-y}\}.$$

This can be compared with the exact version, for $y > 0$

$$f_{Y_n}(y) = ne^{-y}(1 - e^{-y})^{n-1}.$$

The figure below compares the approximations for $n = 50, 100, 500, 1000$. Solid lines use the exact formula, dotted lines use the approximation, histograms are 5000 simulated values.



DEFINITION (Asymptotic Normality)
A sequence of random variables $\{X_n\}$ is **asymptotically normally distributed** as $n \longrightarrow \infty$ if there exist sequences of real constants $\{\mu_n\}$ and $\{\sigma_n\}$ (with $\sigma_n > 0$) such that

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} Z \sim \mathcal{N}(0,1).$$

The notation $X_n \approx \mathcal{N}(\mu_n, \sigma_n^2)$ or $X_n \sim \mathcal{AN}(\mu_n, \sigma_n^2)$ as $n \longrightarrow \infty$ is commonly used.

DEFINITION (Stochastic Order Notation)

- For random variable Z, we write $Z = \mathrm{O}_p(1)$ if for all $\epsilon > 0$, there exists $M < \infty$ such that

$$P[|Z| \geq M] \leq \epsilon.$$

- For sequence $\{Z_n\}$, write $Z_n = \mathrm{O}_p(1)$ if for all n, $P[|Z_n| \geq M] \leq \epsilon$, and write $Z_n = \mathrm{O}_p(S_n)$ for sequence of random variables $\{S_n\}$ if

$$\frac{|Z_n|}{|S_n|} = \mathrm{O}_p(1).$$

  Note that this includes the case where S_n is a sequence of reals, rather than random variables. Finally, write $Z_n = \mathrm{o}_p(1)$ if $Z_n \xrightarrow{p} 0$, and $Z_n = \mathrm{o}_p(S_n)$ if

$$\frac{|Z_n|}{|S_n|} = \mathrm{o}_p(1).$$

  Note that $\mathrm{O}_p(1)\mathrm{o}_p(1) = \mathrm{o}_p(1)$ and $\mathrm{O}_p(1) + \mathrm{o}_p(1) = \mathrm{O}_p(1)$.

LEMMA
Suppose $\{X_n\}$ are a sequence of rvs, and that for real sequence $\{a_n\}$ with $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$,

(i) for real constant x_0 and random variable V, $a_n(X_n - x_0) \xrightarrow{d} V$;
(ii) real function g is differentiable at x_0, with derivative $\dot{g}$.

Then

$$a_n(g(X_n) - g(x_0)) \xrightarrow{d} \dot{g}(x_0)V$$

Proof. Note first that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| \leq \delta \quad \Longrightarrow \quad |g(x) - g(x_0) - \dot{g}(x_0)(x - x_0)| \leq \epsilon |x - x_0|$$

Now, from (i) we have

$$a_n(X_n - x_0) = \mathrm{O}_p(1) \qquad \Longrightarrow \qquad X_n - x_0 = \mathrm{O}_p(a_n^{-1}) = \mathrm{o}_p(1)$$

as $a_n \longrightarrow \infty$. Therefore, by definition, for every $\delta > 0$, $P[|X_n - x_0| \leq \delta] \longrightarrow 1$, and therefore from above, for every $\epsilon > 0$,

$$P[|g(X_n) - g(x_0) - \dot{g}(x_0)(X_n - x_0)| \leq \epsilon |X_n - x_0|] \longrightarrow 1.$$

Hence

$$a_n(g(X_n) - g(x_0) - \dot{g}(x_0)(X_n - x_0)) = \mathrm{o}_p(a_n(X_n - x_0)) = \mathrm{o}_p(1)$$

Therefore

$$a_n(g(X_n) - g(x_0)) = \dot{g}(x_0)\{a_n(X_n - x_0)\} + \mathrm{o}_p(1)$$

and hence

$$a_n(g(X_n) - g(x_0)) \xrightarrow{d} \dot{g}(x_0)V.$$

THEOREM (The Delta Method)
Consider sequence of random variables $\{X_n\}$ such that

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} X.$$

Suppose that $g(.)$ is a function such that first derivative $\dot{g}(.)$ is continuous in a neighbourhood of μ, with $\dot{g}(\mu) \neq 0$. Then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \dot{g}(\mu)X.$$

In particular, if

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2).$$

then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \dot{g}(\mu)X \sim \mathcal{N}(0, \{\dot{g}(\mu)\}^2\sigma^2).$$

Proof. Using the Lemma above, with $a_n = \sqrt{n}, x_0 = \mu, V = X$, we have that

$$\sqrt{n}(g(X_n) - g(\mu)) = \dot{g}(\mu)\sqrt{n}(X_n - \mu) \xrightarrow{d} \dot{g}(\mu)X$$

and if $X \sim \mathcal{N}(0, \sigma^2)$, it follows from the properties of the Normal distribution that

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \mathcal{N}(0, \{\dot{g}(\mu)\}^2\sigma^2).$$

Note that this method does not give a useful result if $\dot{g}(\mu) = 0$.

Multivariate Version: Consider a sequence of random vectors $\{\mathbf{X}_n\}$ such that

$$\sqrt{n}(\mathbf{X}_n - \boldsymbol{\mu}) \xrightarrow{d} \mathbf{X}.$$

and $\mathbf{g} : \mathbb{R}^k \longrightarrow \mathbb{R}^d$ is a vector-valued function with first derivative matrix $\dot{\mathbf{g}}(.)$ which is continuous in a neighbourhood of $\boldsymbol{\mu}$, with $\dot{g}(\boldsymbol{\mu}) \neq \mathbf{0}$. Note that $\mathbf{g}$ can be considered as a $d \times 1$ vector of scalar functions.

$$\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_d(\mathbf{x}))^\top.$$

Note that $\dot{\mathbf{g}}(\mathbf{x})$ is a $(d \times k)$ matrix with $(i, j)th$ element

$$\frac{\partial g_i(\mathbf{x})}{\partial x_j}$$

Under these assumptions, in general

$$\sqrt{n}(\mathbf{g}(\mathbf{X}_n) - \mathbf{g}(\boldsymbol{\mu})) \xrightarrow{d} \dot{\mathbf{g}}(\boldsymbol{\mu})\mathbf{X}.$$

and in particular, if

$$\sqrt{n}(\mathbf{X}_n - \boldsymbol{\mu}) \xrightarrow{d} \mathbf{X} \sim \mathcal{N}(\mathbf{0}, \Sigma).$$

where Σ is a positive definite, symmetric $k \times k$ matrix, then

$$\sqrt{n}(\mathbf{g}(\mathbf{X}_n) - \mathbf{g}(\boldsymbol{\mu})) \xrightarrow{d} \dot{\mathbf{g}}(\boldsymbol{\mu})X \sim \mathcal{N}\left(\mathbf{0}, \dot{\mathbf{g}}(\boldsymbol{\mu})\Sigma\dot{\mathbf{g}}(\boldsymbol{\mu})^\top\right).$$

THEOREM (The Second Order Delta Method: Normal case)
Consider sequence of random variables $\{X_n\}$ such that

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Suppose that $g(.)$ is a function such that first derivative $\dot{g}(.)$ is continuous in a neighbourhood of μ, with $\dot{g}(\mu) = 0$, but second derivative exists at μ with $\ddot{g}(\mu) \neq 0$. Then

$$n(g(X_n) - g(\mu)) \xrightarrow{d} \sigma^2 \frac{\ddot{g}(\mu)}{2} X$$

where $X \sim \chi^2_1$.

Proof. Uses a second order Taylor approximation; informally

$$g(X_n) = g(\mu) + \dot{g}(\mu)(X_n - \mu) + \frac{\ddot{g}(\mu)}{2}(X_n - \mu)^2 + \mathrm{o}_p(1)$$

thus, as $\dot{g}(\mu) = 0$,

$$g(X_n) - g(\mu) = \frac{\ddot{g}(\mu)}{2}(X_n - \mu)^2 + \mathrm{o}_p(1)$$

and thus

$$n(g(X_n) - g(\mu)) = \frac{\ddot{g}(\mu)}{2}\{\sqrt{n}(X_n - \mu)\}^2 \xrightarrow{d} \sigma^2 \frac{\ddot{g}(\mu)}{2} Z^2$$

where $Z^2 \sim \chi^2_1$.

EXAMPLES

1. Under the conditions of the Central Limit Theorem, for random variables $X_1, \ldots, X_n$ and their sample mean random variable $\overline{X}_n$

   $$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2).$$

   Consider $g(x) = x^2$, so that $\dot{g}(x) = 2x$, and hence, if $\mu \neq 0$,

   $$\sqrt{n}({\overline{X}_n}^2 - \mu^2) \xrightarrow{d} X \sim \mathcal{N}(0, 4\mu^2\sigma^2)$$

   and

   $${\overline{X}_n}^2 \sim \mathcal{AN}(\mu^2, 4\mu^2\sigma^2/n)$$

   If $\mu = 0$, we proceed by a different route to compute the approximate distribution of ${\overline{X}_n}^2$; note that, if $\mu = 0$,

   $$\sqrt{n}\overline{X}_n \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2)$$

   so therefore

   $$n{\overline{X}_n}^2 = (\sqrt{n}\overline{X}_n)^2 \xrightarrow{d} X^2 \sim \text{Gamma}(1/2, 1/(2\sigma^2))$$

   by elementary transformation results. Hence, for large n,

   $${\overline{X}_n}^2 \stackrel{\cdot}{\sim} \text{Gamma}(1/2, n/(2\sigma^2))$$

2. Again under the conditions of the CLT, consider the distribution of $1/\overline{X}_n$. In this case, we have a function $g(x) = 1/x$, so $\dot{g}(x) = -1/x^2$, and if $\mu \neq 0$, the Delta method gives

   $$\sqrt{n}(1/\overline{X}_n - 1/\mu) \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2/\mu^4)$$

   or,

   $$\frac{1}{\overline{X}_n} \sim \mathcal{AN}(1/\mu, n^{-1}\sigma^2/\mu^4).$$

RESULT 1: If $Y_1, Y_2, \ldots, Y_{n+1} \sim$ Exponential (1) are independent random variables, and $S_1, S_2, \ldots, S_{n+1}$ are defined by

$$S_k = \sum_{j=1}^{k} Y_j \qquad k = 1, 2, \ldots, n+1$$

then the random variables

$$\left[\frac{S_1}{S_{n+1}}, \frac{S_2}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}}\right]$$

given that $S_{n+1} = s$, say, have the same distribution as the order statistics from a random sample of size n from the Uniform distribution on $(0, 1)$.

Proof: Let the Y_js be defined as above. Then the joint density for the Y_js is given by

$$\exp\left\{-\sum_{j=1}^{n+1} y_j\right\} \qquad y_1, y_2, \ldots, y_{n+1} > 0.$$

Now

$$\left.\begin{array}{rcl} S_1 & = & Y_1 \\ S_2 & = & Y_1 + Y_2 \\ S_3 & = & Y_1 + Y_2 + Y_3 \\ \vdots & & \vdots \\ S_n & = & \sum\limits_{j=1}^{n} Y_j \\ S_{n+1} & = & \sum\limits_{j=1}^{n+1} Y_j \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} Y_1 & = & S_1 \\ Y_2 & = & S_2 - S_1 \\ Y_3 & = & S_3 - S_2 \\ \vdots & & \vdots \\ Y_n & = & S_n - S_{n-1} \\ Y_{n+1} & = & S_{n+1} - S_n \end{array}\right.$$

and so the Jacobian of the transformation from $(Y_1, \ldots, Y_{n+1}) \longrightarrow (S_1, \ldots, S_{n+1})$ is 1, and hence the joint density for $(S_1, \ldots, S_{n+1})$ is given by

$$\exp\{-s_{n+1}\} \qquad 0 < s_1 < s_2 < \ldots < s_{n+1} < \infty.$$

The marginal distribution for S_{n+1} is Gamma $(n+1, 1)$ and thus the conditional distribution of $(S_1, \ldots, S_n)$ given $S_{n+1} = s$ is

$$\frac{\exp\{-s\}}{\dfrac{1}{\Gamma(n+1)} s^n \exp\{-s\}} = \frac{n!}{s^n} \qquad 0 < s_1 < s_2 < \ldots < s < \infty.$$

Finally, conditional on $S_{n+1} = s$, define the joint transformation

$$V_j = \frac{S_j}{s} \Longleftrightarrow S_j = sV_j \qquad j = 1, 2, \ldots, n$$

which has Jacobian s^n. Then, conditional on $S_{n+1} = s$, $(V_1, \ldots, V_n)$ have joint pdf equal to $n!$ for $0 < v_1 < v_2 < \ldots < v_n < 1$. Finally, if $U_1, \ldots, U_n$ are independent random variables each having a Uniform distribution on $(0, 1)$, then $(U_1, \ldots, U_n)$ have joint pdf equal to 1 on the unit hypercube in n dimensions, and thus the corresponding order statistics $U_{(1)}, \ldots, U_{(n)}$ also have joint pdf equal to

$$n! \qquad 0 < u_1 < u_2 < \ldots < u_n < 1.$$

RESULT 2: Let the S_k be defined as in Result 1. Then

$$\sqrt{k}\left(\frac{S_k}{k}-1\right) \stackrel{d}{\longrightarrow} \mathcal{N}(0,1) \text{ as } k \longrightarrow \infty$$

Proof: We have that S_k is the sum of k independent and identically distributed Exponential(1) random variables, $Y_1, \ldots, Y_k$, so that $\mathbb{E}[Y_j] = \text{Var}[Y_j] = 1$. Thus result follows via the Central Limit Theorem.

RESULT 3: Let the S_k be defined as in Result 1. Then, if k_{1n} is a sequence of integers such that

$$k_{1n} \longrightarrow \infty \qquad \text{while} \qquad \frac{k_{1n}}{n} \longrightarrow p_1$$

for some p_1 with $0 < p_1 < 1$, it follows that

$$\sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1}-\frac{k_{1n}}{n+1}\right) \stackrel{d}{\longrightarrow} \mathcal{N}(0,p_1) \text{ as } n \longrightarrow \infty$$

Proof: We have

$$\sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1}-\frac{k_{1n}}{n+1}\right) = \sqrt{\frac{k_{1n}}{n+1}} \times \sqrt{k_{1n}}\left(\frac{S_{k_{1n}}}{k_{1n}}-1\right) \stackrel{d}{\longrightarrow} \sqrt{p_1} \times \mathcal{N}(0,1) \equiv \mathcal{N}(0,p_1)$$

as $n \longrightarrow \infty$ and $k_{1n} \longrightarrow \infty$.

Corollary: Using the same approach, if

$$\frac{k_{1n}}{n} \longrightarrow p_1 \qquad \text{and} \qquad \frac{k_{2n}}{n} \longrightarrow p_2$$

for $0 < p_1 < p_2 < 1$, then if $D_n = \sum_{j=k_{1n}+1}^{k_{2n}} Y_j$,

$$\begin{aligned}\sqrt{n+1}\left(\frac{(S_{k_{2n}}-S_{k_{1n}})}{n+1}-\frac{k_{2n}-k_{1n}}{n+1}\right) &= \sqrt{\frac{k_{2n}-k_{1n}}{n+1}}\sqrt{k_{2n}-k_{1n}}\left(\frac{D_n}{k_{2n}-k_{1n}}-1\right) \\ &\stackrel{d}{\longrightarrow} \sqrt{p_2-p_1} \times \mathcal{N}(0,1) \equiv \mathcal{N}(0,p_2-p_1).\end{aligned}$$

Similarly

$$\sqrt{n+1}\left(\frac{1}{n+1}(S_{n+1}-S_{k_{2n}})-\frac{n+1-k_{2n}}{n+1}\right) \stackrel{d}{\longrightarrow} \mathcal{N}(0,1-p_2)$$

where the limiting variables in the three cases are independent, as

$$\begin{aligned} S_{k_{1n}} &= \sum_{j=1}^{k_{1n}} Y_j \\ (S_{k_{2n}}-S_{k_{1n}}) &= \sum_{j=k_{1n}+1}^{k_{2n}} Y_j \\ (S_{n+1}-S_{k_{2n}}) &= \sum_{j=k_{2n}+1}^{n+1} Y_j \end{aligned}$$

are independent.

RESULT 4: Let

$$Z_1 = \frac{S_{k_{1n}}}{n+1} \qquad Z_2 = \frac{(S_{k_{2n}} - S_{k_{1n}})}{n+1} \qquad Z_3 = \frac{(S_{n+1} - S_{k_{2n}})}{n+1}$$

and suppose that

$$\sqrt{n}\left(\frac{k_{1n}}{n} - p_1\right) \longrightarrow 0 \qquad \text{and} \qquad \sqrt{n}\left(\frac{k_{2n}}{n} - p_2\right) \longrightarrow 0$$

as $n \longrightarrow \infty$. Then

$$\sqrt{n+1}\left(\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 - p_1 \\ 1 - p_2 \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

as $n \longrightarrow \infty$, where $\Sigma = \text{diag}(p_1, p_2 - p_1, 1 - p_2)$.

Proof: We have, as $n \longrightarrow \infty$,

$$\sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - p_1\right) - \sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - \frac{k_{1n}}{n+1}\right) = \sqrt{n+1}\left(\frac{k_{1n}}{n+1} - p_1\right) \longrightarrow 0$$

$$\therefore \sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - p_1\right) \qquad \text{and} \qquad \sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - \frac{k_{1n}}{n+1}\right)$$

have the same asymptotic distribution, and thus the result follows from Result 3. The proof is similar for the other two terms. Independence (that is, the diagonal nature of Σ) follows from the independence of $S_{k_{1n}}, (S_{k_{2n}} - S_{k_{1n}})$, and $(S_{n+1} - S_{k_{2n}})$.

RESULT 5: If $U_{(1)}, \ldots, U_{(n)}$ are the order statistics from a random sample of size n from a Uniform $(0,1)$ distribution, and if $n \longrightarrow \infty$, $k_{1n} \longrightarrow \infty$ and $k_{2n} \longrightarrow \infty$ in such a way that

$$\sqrt{n}\left(\frac{k_{1n}}{n} - p_1\right) \longrightarrow 0 \qquad \text{and} \qquad \sqrt{n}\left(\frac{k_{2n}}{n} - p_2\right) \longrightarrow 0$$

for $0 < p_1 < p_2 < 1$, then

$$\sqrt{n}\left(\begin{pmatrix} U_{(k_{1n})} \\ U_{(k_{2n})} \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \begin{bmatrix} p_1(1-p_1) & p_1(1-p_2) \\ p_1(1-p_2) & p_2(1-p_2) \end{bmatrix}\right).$$

Proof: Define

$$\mathbf{g}(x_1, x_2, x_3) = \frac{1}{x_1 + x_2 + x_3}\begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} \qquad \dot{\mathbf{g}}(x_1, x_2, x_3) = \frac{1}{(x_1 + x_2 + x_3)^2}\begin{bmatrix} x_2 + x_3 & -x_1 & -x_1 \\ x_3 & x_3 & -(x_1 + x_2) \end{bmatrix}.$$

$$\therefore \mathbf{g}\left(\frac{S_{k_{1n}}}{n+1}, \frac{S_{k_{2n}} - S_{k_{1n}}}{n+1}, \frac{S_{n+1} - S_{k_{2n}}}{n+1}\right) = \frac{1}{S_{n+1}}\begin{bmatrix} S_{k_{1n}} \\ S_{k_{2n}} \end{bmatrix}$$

which has the same distribution as $\left(U_{(k_{1n})}, U_{(k_{2n})}\right)^\top$, by Result 1. By the Delta Method

$$\sqrt{n}\left(\begin{pmatrix} U_{(k_{1n})} \\ U_{(k_{2n})} \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \dot{\mathbf{g}}(\mu)\Sigma\dot{\mathbf{g}}(\mu)^\top\right)$$

where Σ is as defined in the Result 4, where here $\mu = (p_1, p_2 - p_1, 1 - p_2)^T$. It can be easily verified that

$$\dot{\mathbf{g}}(\mu)\Sigma\dot{\mathbf{g}}(\mu)^T = \begin{bmatrix} p_1(1-p_1) & p_1(1-p_2) \\ p_1(1-p_2) & p_2(1-p_2) \end{bmatrix}.$$

RESULT 6: If $X_{(1)}, \ldots, X_{(n)}$ are the order statistics from a random sample of size n from a distribution with continuous distribution function F_X and density f_X which is continuous and non-zero in a neighbourhood of quantiles x_{p_1} and x_{p_2} corresponding to probabilities $p_1 < p_2$, then if $k_{1n} = \lceil np_1 \rceil$ and $k_{2n} = \lceil np_2 \rceil$

$$\sqrt{n}\left(\left(\begin{array}{c} X_{(k_{1n})} \\ X_{(k_{2n})} \end{array}\right) - \left(\begin{array}{c} x_{p_1} \\ x_{p_2} \end{array}\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \left[\begin{array}{cc} \dfrac{p_1(1-p_1)}{\{f_X(x_{p_1})\}^2} & \dfrac{p_1(1-p_2)}{f_X(x_{p_1}) f_X(x_{p_2})} \\ & \\ \dfrac{p_1(1-p_2)}{f_X(x_{p_1}) f_X(x_{p_2})} & \dfrac{p_2(1-p_2)}{\{f_X(x_{p_2})\}^2} \end{array}\right]\right)$$

Proof: We use the Delta Method on the result from Result 5, with the transformation

$$\mathbf{g}(y_1, y_2) = \left[\begin{array}{c} F_X^{-1}(y_1) \\ F_X^{-1}(y_2) \end{array}\right]$$

so that

$$\dot{\mathbf{g}}(y_1, y_2) = \left[\begin{array}{cc} \dfrac{1}{f_X\left(F_X^{-1}(y_1)\right)} & 0 \\ 0 & \dfrac{1}{f_X\left(F_X^{-1}(y_2)\right)} \end{array}\right]$$

with $y_1 = p_1$ and $y_2 = p_2$.

By properties of the multivariate normal distribution, we have that the marginal distribution of $X_{(k_{1n})}$ can be approximated for large n by using the relationship

$$\sqrt{n}(X_{(k_{1n})} - x_{p_1}) \xrightarrow{d} \mathcal{N}\left(0, \frac{p_1(1-p_1)}{\{f_X(x_{p_1})\}^2}\right)$$

For example, if $p_1 = 1/2$, x_{p_1} is the **median** $x_{F_X}(0.5)$ of the distribution, and $X_{(k_{1n})}$ is the **sample median** $\widetilde{X}_n(0.5)$, and we have that

$$\sqrt{n}(\widetilde{X}_n(0.5) - x_{F_X}(0.5)) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{4\{f_X(x(0.5))\}^2}\right)$$

If F_X is the $\mathcal{N}(\mu, \sigma^2)$ distribution, then $x_{F_X}(0.5) = \mu$ and

$$f_X(x(0.5)) = f_X(\mu) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2}$$

so this result says that

$$\sqrt{n}(\widetilde{X}_n(0.5) - \mu) \xrightarrow{d} \mathcal{N}\left(0, \frac{\pi\sigma^2}{2}\right) \mathrel{\dot{\sim}} \mathcal{N}\left(0, 1.57\sigma^2\right)$$

which contrasts with the exact result for the sample mean

$$\sqrt{n}(\overline{X}_n - \mu) \sim \mathcal{N}\left(0, \sigma^2\right).$$