556: MATHEMATICAL STATISTICS I

CONVERGENCE IN DISTRIBUTION: EXAMPLES

EXAMPLE 1: Continuous random variable X_n with range $\mathbb{X} \equiv (0, n]$ for $n > 0$ and cdf

$$F_{X_n}(x) = 1 - \left(1 - \frac{x}{n}\right)^n \qquad 0 < x \leq n$$

and standard cdf behaviour outside of this range. Then as $n \to \infty$, $\mathbb{X} \equiv (0, \infty)$, and for all $x > 0$

$$F_{X_n}(x) \to 1 - e^{-x} \qquad \therefore \qquad F_{X_n}(x) \to F_X(x) = 1 - e^{-x}$$

and hence

$$X_n \xrightarrow{d} X \qquad X \sim Exponential(1)$$

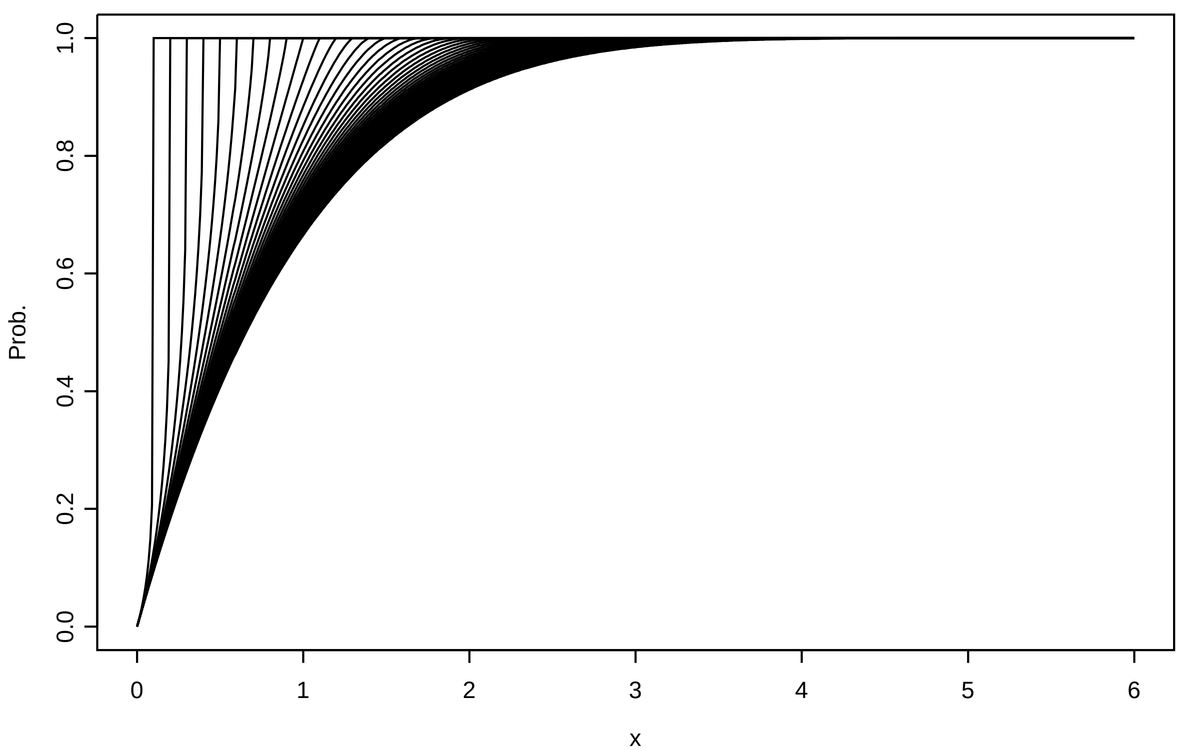

Figure 1: $F_{X_n}(x) = 1 - \left(1 - \frac{x}{n}\right)^n$ for $0 \leq x \leq n$, $n = 0, 1, 2, \ldots$

EXAMPLE 2: Continuous random variable X_n with range $\mathbb{X} \equiv (0, \infty)$ and cdf

$$F_{X_n}(x) = \left(1 - \frac{1}{1+nx}\right)^n \qquad 0 < x < \infty$$

and zero otherwise. Then as $n \to \infty$, for all $x > 0$

$$F_{X_n}(x) \to e^{-1/x} \qquad \therefore \qquad F_{X_n}(x) \to F_X(x) = e^{-1/x}$$

as

$$\lim_{n\to\infty} \left(1 - \frac{1}{1+nx}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{nx}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1/x}{n}\right)^n$$

and for any z

$$\lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = e^z$$

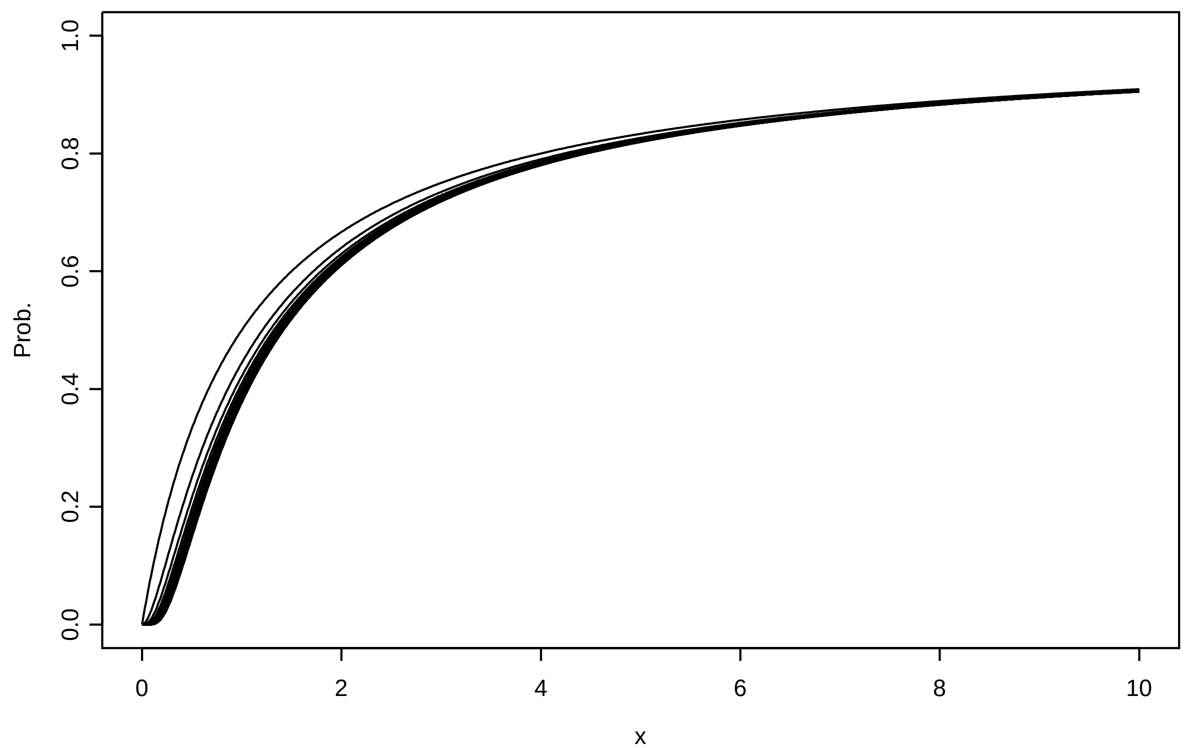

Figure 2: $F_{X_n}(x) = \left(1 - \frac{1}{1+nx}\right)^n$, $0 < x < \infty$, $n = 0, 1, 2, \ldots, 20$

EXAMPLE 3: Continuous random variable X_n with range $\mathbb{X} \equiv [0,1]$ and cdf

$$F_{X_n}(x) = x - \sin(2n\pi x)/(2n\pi) \qquad 0 \le x \le 1$$

and standard cdf behaviour outside of this range. Then as $n \to \infty$, and for all $0 \le x \le 1$

$$F_{X_n}(x) \to x \qquad \therefore \qquad F_{X_n}(x) \to F_X(x) = x$$

and hence

$$X_n \xrightarrow{d} X \qquad \text{where } X \sim Uniform(0,1)$$

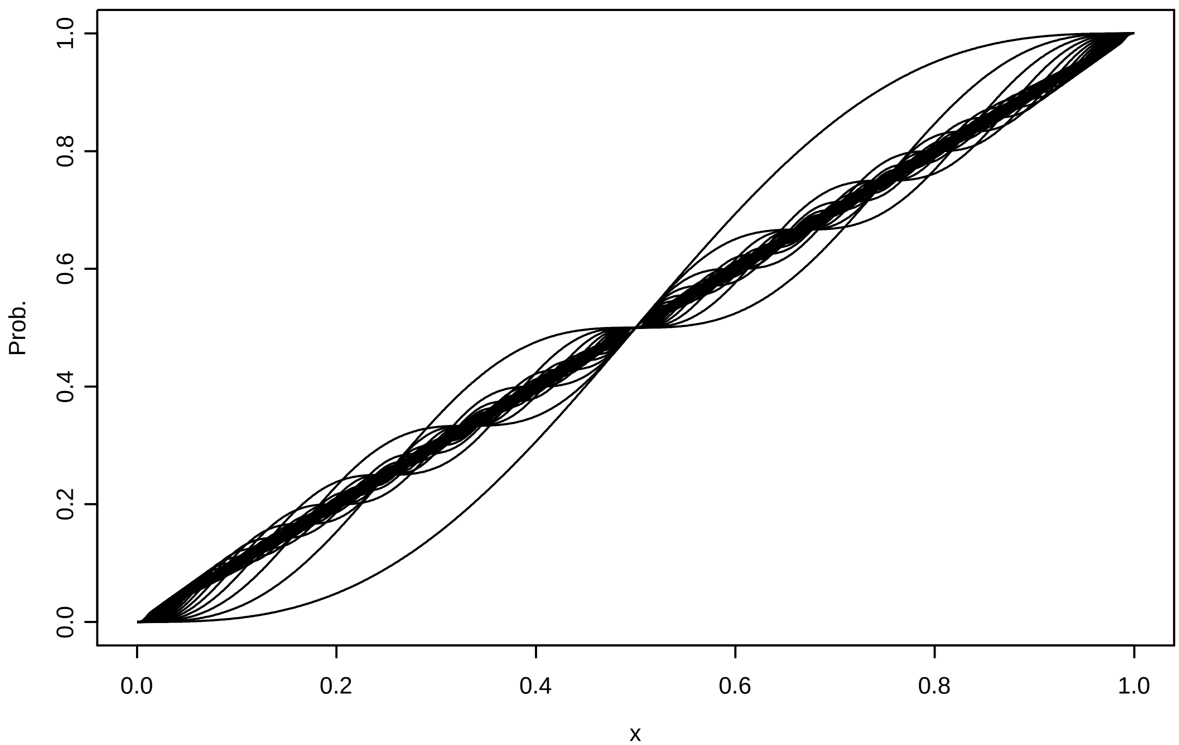

Figure 3: $F_{X_n}(x) = x - \sin(2n\pi x)/(2n\pi),\ 0 \le x \le 1$, $n = 0, 1, 2, \dots, 10$

NOTE: for the pdf

$$f_{X_n}(x) = 1 - \cos(2n\pi x) \qquad 0 \le x \le 1$$

and there **is no limit as** $n \to \infty$.

EXAMPLE 4: Continuous random variable X_n with range $\mathbb{X} \equiv [0,1]$ and cdf

$$F_{X_n}(x) = 1 - (1-x)^n \qquad 0 \leq x \leq 1$$

and standard cdf behaviour outside of this range. Then as $n \to \infty$, and for $x \in \mathbb{R}$

$$F_{X_n}(x) \to \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} .$$

This limiting form is **not** continuous at $x = 0$, as $x = 0$ is not a point of continuity, and the **ordinary definition of convergence in distribution cannot be applied**. However, it is clear that for $\epsilon > 0$,

$$P\left[|X_n| < \epsilon\right] = 1 - (1-\epsilon)^n \to 1 \text{ as } n \to \infty$$

so it is still correct to say

$$X_n \xrightarrow{d} X \qquad \text{where } P\left[X = 0\right] = 1$$

so the limiting distribution is **degenerate at** $x = 0$.

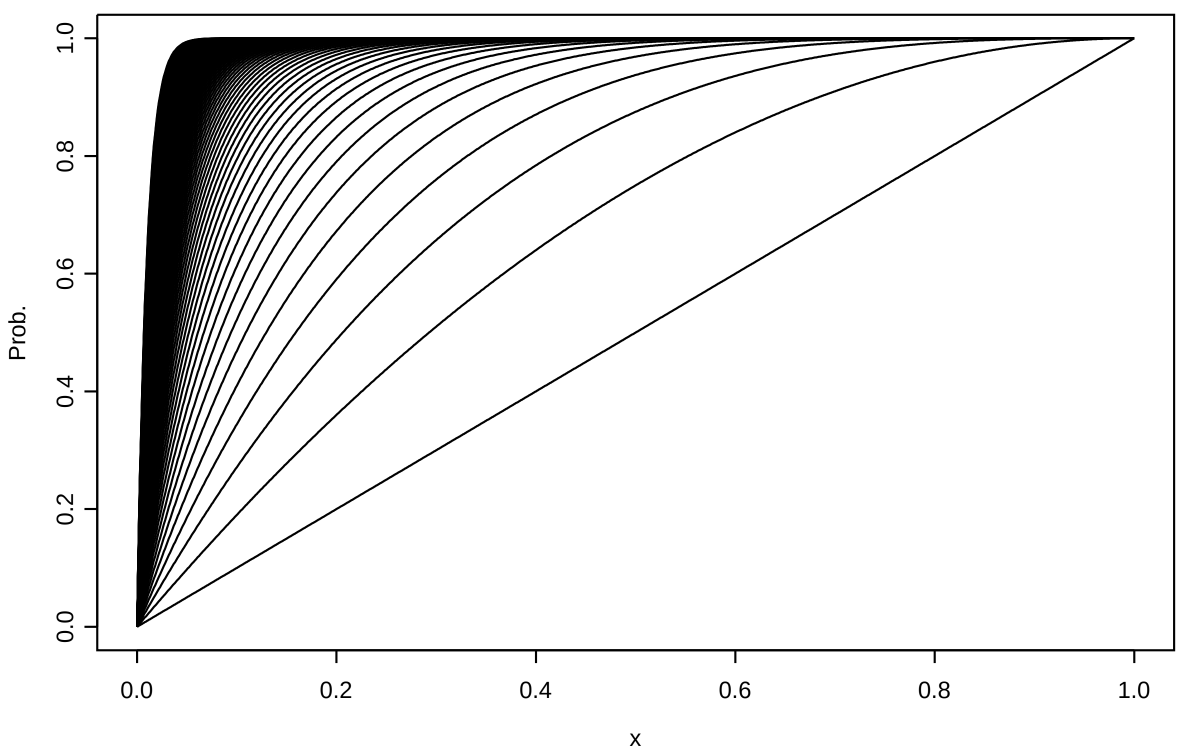

Figure 4: $F_{X_n}(x) = 1 - (1-x)^n$, $0 < x < 1$, $n = 0, 1, 2, \ldots, 100$

EXAMPLE 5: Continuous random variable X_n with range $\mathbb{X} \equiv (0, \infty)$ and cdf

$$F_{X_n}(x) = \left(\frac{x}{1+x}\right)^n \qquad x > 0$$

and zero otherwise. Then as $n \to \infty$, and for $x > 0$

$$F_{X_n}(x) \to 0$$

Thus there is **no limiting distribution**.

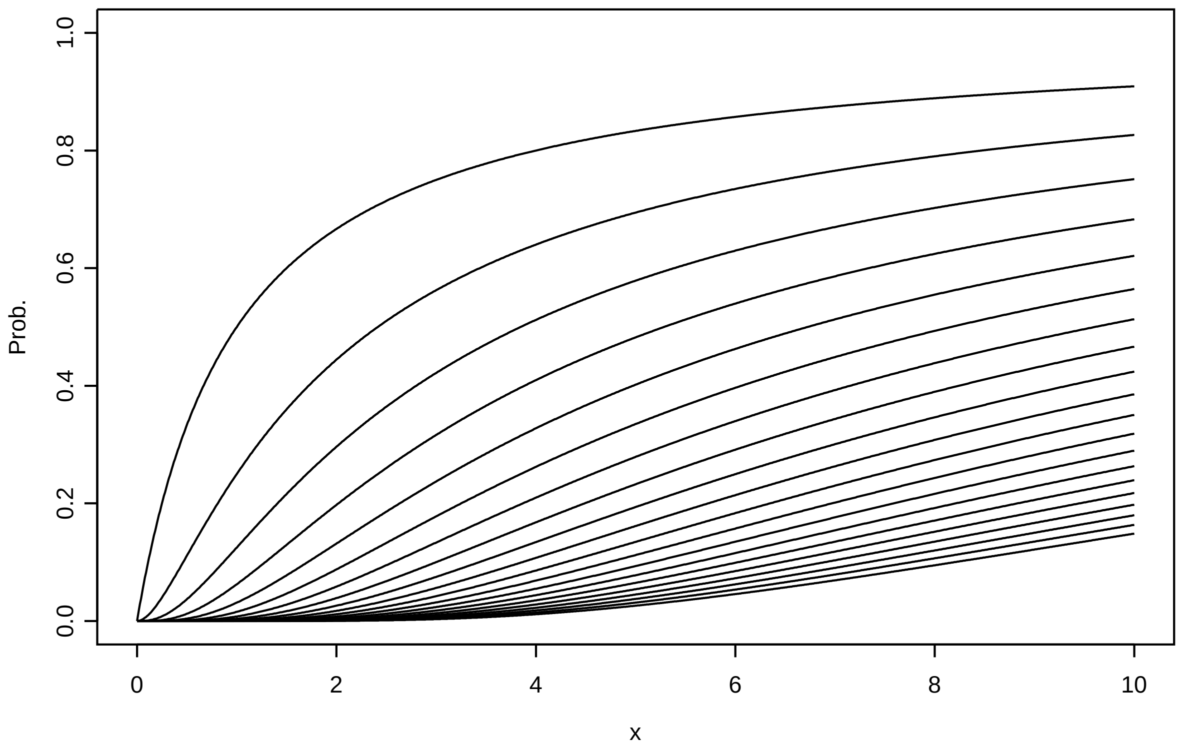

Figure 5: $F_{X_n}(x) = \left(\frac{x}{1+x}\right)^n$, $x > 0$, $n = 0, 1, 2, \ldots, 20$

Now let $V_n = X_n/n$. Then $\mathbb{V} \equiv (0, \infty)$ and the cdf of V_n is

$$F_{V_n}(v) = P[V_n \le v] = P[X_n/n \le v] = P[X_n \le nv] = F_{X_n}(nv) = \left(\frac{nv}{1+nv}\right)^n \qquad v > 0$$

and as $n \to \infty$, for all $v > 0$

$$F_{V_n}(v) \to e^{-1/v} \qquad \therefore \qquad F_{V_n}(v) \to F_V(v) = e^{-1/v}$$

and the limiting distribution of V_n does exist.

EXAMPLE 6: Continuous random variable X_n with range $\mathbb{X} \equiv (0, \infty)$ and cdf

$$F_{X_n}(x) = \frac{\exp(nx)}{1+\exp(nx)} \qquad x \in \mathbb{R}$$

and zero otherwise. Then as $n \to \infty$

$$F_{X_n}(x) \to \begin{cases} 0 & x < 0 \\ \dfrac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \qquad x \in \mathbb{R}$$

This limiting form is **not** a cdf, as it is not right continuous at $x = 0$. However, as $x = 0$ is not a point of continuity, and the ordinary definition of convergence in distribution does not apply. However, it is clear that for $\epsilon > 0$,

$$P\left[|X| < \epsilon\right] = \frac{\exp(n\epsilon)}{1+\exp(n\epsilon)} - \frac{\exp(-n\epsilon)}{1+\exp(-n\epsilon)} \to 1 \text{ as } n \to \infty$$

so it is still correct to say

$$X_n \xrightarrow{d} X \qquad \text{where} \qquad \mathrm{P}\left[X = 0\right] = 1$$

and the limiting distribution is **degenerate at** $x = 0$.

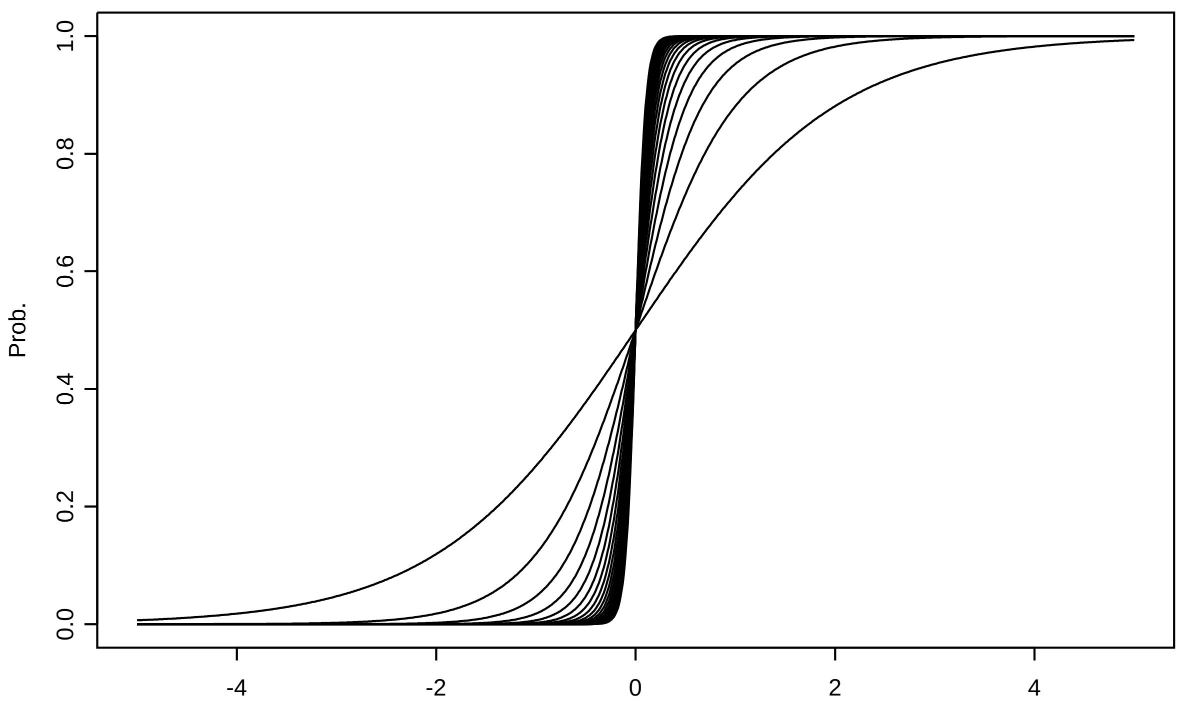

Figure 6: $F_{X_n}(x) = \dfrac{\exp(nx)}{1+\exp(nx)}$, $x \in \mathbb{R}$, $n = 0, 1, 2, \ldots, 20$