MATH 556 - EXERCISES 2

Moment generating functions and Laplace transforms

1. Suppose X is a random variable, with mgf $M_X(t)$ defined on a neighbourhood $(-\delta, \delta)$ of zero. Show that
$$\mathrm{P}_X[X \geq a] \leq e^{-at} M_X(t) \qquad \text{for } 0 < t < \delta$$

2. Suppose that X is a random variable with pmf/pdf f_X and mgf M_X, where for some $\delta > 0$,
$$M_X(t) = \mathbb{E}_X[e^{tX}] = \int e^{tx}\, dF_X(t) \qquad -\delta < t < \delta$$
The cumulant generating function of X, K_X, is defined by $K_X(t) = \log M_X(t)$. Verify that
$$\frac{d}{dt}\left\{K_X(t)\right\}_{t=0} = \mathbb{E}_X[X] \qquad \frac{d^2}{dt^2}\left\{K_X(t)\right\}_{t=0} = \mathrm{Var}_X[X]$$

3. The non-central chi-square distribution arises as the distribution of the square of a normal random variable. That is, if $X \sim \text{Normal}(\mu, 1)$, then $Y = X^2$ has the non-central chi-square distribution with one degree of freedom and non-centrality parameter λ, denoted $Y \sim \chi^2_\nu(\lambda)$, where $\nu = 1$ and $\lambda = \mu^2$.
In this setting,
   (a) Find the pdf of Y, and show that it can be expressed in the form
   $$f_Y(y) = e^{-\delta/2} \sum_{j=0}^{\infty} \frac{(\delta/2)^j}{j!} f_{Z_{2j+k}}(y) \qquad y > 0$$
   where f_{Z_m} is the pdf of a random variable Z_m which has a chi-square distribution with m degrees of freedom.
   (b) Find the characteristic function $\varphi_Y(t)$.
   (c) Find the Laplace transform $\mathcal{L}_Y(t)$.
   (d) Find the expectation and variance of Y.
   (e) Find the distribution of
   $$S = \sum_{i=1}^{n} Y_i$$
   where $Y_1, \ldots, Y_n$ are independent, with $Y_i \sim \chi^2_{\nu_i}(\lambda_i)$, $i = 1, \ldots, n$.

4. If $\mathcal{L}_X(t)$ is the Laplace transform of a nonnegative random variable X, show that
$$(-1)^r \frac{d^r}{dt^r}\{\mathcal{L}_{\mathcal{X}}(t)\} \geq 0 \qquad t \geq 0$$
for $r = 1, 2, \ldots$.

5. Let X be a nonnegative, real-valued random variable with distribution function F_X and Laplace transform $\mathcal{L}$. Show that
$$\mathcal{L}_X(t) = t \int_0^\infty \exp\{-tx\} F_X(x)\, \mathrm{d}x.$$

6. Suppose that $X_1 \sim Gamma(\alpha_1, \beta_1)$ and $X_2 \sim Gamma(\alpha_2, \beta_2)$ are independent random variables. Characterize the distribution of
$$Y = X_1 - X_2.$$