MATH 556 - EXERCISES 1

Characteristic Functions

1. Suppose that X is a continuous rv with pdf f_X and characteristic function (cf) φ_X. Find $\varphi_X(t)$ if

   (a)
   $$f_X(x) = \frac{1}{2}|x| \exp\{-|x|\} \qquad x \in \mathbb{R}.$$

   (b)
   $$f_X(x) = \exp\{-x - e^{-x}\} \qquad x \in \mathbb{R}.$$

   (c)
   $$f_X(x) = \frac{1}{\cosh(\pi x)} = \frac{2}{e^{-\pi x} + e^{\pi x}} = \sum_{k=0}^{\infty} (-1)^k \exp\{-(2k+1)\pi|x|\} \qquad x \in \mathbb{R}.$$

   Leave your answer as an infinite sum if necessary.

2. Find $f_X(x)$ if the cf is given by
   $$\varphi_X(t) = 1 - |t| \qquad -1 < t < 1$$
   and zero otherwise.

3. Suppose that random variable Y has cf φ_Y. Find the distribution of Y if

   (a)
   $$\varphi_Y(t) = \frac{2(1-\cos t)}{t^2} \qquad t \in \mathbb{R}.$$

   (b)
   $$\varphi_Y(t) = \cos(\theta t) \qquad t \in \mathbb{R}.$$

   for some parameter $\theta > 0$.

4. By considering derivatives at $t = 0$, and the implied moments, assess whether the function
   $$\varphi(t) = \frac{1}{1+t^4}$$
   is a valid cf for a mass function (pmf) or density function (pdf).

5. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed Cauchy rvs each with
   $$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2} \qquad x \in \mathbb{R} \qquad\qquad \varphi_X(t) = \exp\{-|t|\} \qquad t \in \mathbb{R}.$$
   Let continuous random variable Z_n be defined by
   $$Z_n = \frac{1}{\overline{X}} = \frac{n}{\sum\limits_{j=1}^{n} X_j}.$$
   Find an expression for $\mathrm{P}_{Z_n}[|Z_n| \leq c]$ for constant $c > 0$.

6. A probability distribution for rv X is termed *infinitely divisible* if, for all positive integers n, there exists a sequence of independent and identically distributed rvs $Z_{n1}, \ldots, Z_{nn}$ such that X and

$$Z_n = \sum_{j=1}^{n} Z_{nj}$$

have the same distribution, that is, the characteristic function of X can be written

$$\varphi_X(t) = \{\varphi_Z(t)\}^n$$

for some characteristic function φ_Z. Show that the $Gamma(\alpha, \beta)$ distribution is infinitely divisible.

7. Prove that if f_X is pdf for a continuous random variable, then

$$|\varphi_X(t)| \longrightarrow 0 \qquad \text{as} \qquad |t| \longrightarrow \infty.$$

Use the fact that f_X can be approximated to arbitrary accuracy by a step-function; for each $\epsilon > 0$, there exists a step-function $g_\epsilon(x)$ defined (for some K) as

$$g_\epsilon(x) = \sum_{k=1}^{K} c_k \mathbb{1}_{A_k}(x)$$

where $c_k, k = 1, \ldots, K$ are real constants, and $A_1, \ldots, A_K$ form a partition of $\mathbb{R}$, such that

$$\int_{-\infty}^{\infty} |f_X(x) - g_\epsilon(x)| \, dx < \epsilon.$$

The function $\mathbb{1}_A(x)$ is an ***indicator function*** *for set A*

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$