MATH 556 - ASSIGNMENT 4

To be handed in not later than 10pm, 3rd December 2014.
Please submit your solutions as pdf via myCourses

1. Suppose that $\{X_n, n = 1, 2, \ldots\}$ are independent and identically distributed continuous random variables with pdf
$$f_X(x) = \lambda \exp\{-\lambda(x - \eta)\} \qquad x > \eta$$
and zero otherwise, for parameters $\lambda > 0$ and $\eta \in \mathbb{R}$. Find the limiting distribution, as $n \longrightarrow \infty$, of the sequence of random variables $\{Y_{n,1}, n = 1, 2, \ldots\}$ defined by
$$Y_{n,1} = \min\{X_1, \ldots, X_n\}.$$
The double subscript notation $Y_{n,j}$ for the jth order statistic arising from a sample of size n more readily lends itself to the study of asymptotic properties, and is commonly used.

2 Marks

2. Suppose that $\{X_n, n = 1, 2, \ldots\}$ are independent and identically distributed continuous random variables with pdf
$$f_X(x) = \frac{1}{(1+x)^2} \qquad x > 0$$
and zero otherwise. Let
$$Y_{n,n} = \max\{X_1, \ldots, X_n\} \qquad n = 1, 2, \ldots.$$
Show that the sequence of random variables $\{Y_{n,n}/n, n = 1, 2, \ldots\}$ converges in distribution, and find the limiting distribution.

4 Marks

3. Show that, for sequence of random variables $\{X_n, n = 1, 2, \ldots\}$ and random variable X (defined on a common probability space) if
$$\mathbb{E}_{X_n,X}[(X_n - X)^2] \longrightarrow 0$$
as $n \longrightarrow \infty$, then $X_n \xrightarrow{p} X$.

4 Marks

4. A simple version of the *Central Limit Theorem* states that if $\{X_n, n = 1, 2, \ldots\}$ are independent and identically distributed with finite expectation μ and variance σ^2, then if
$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$
is the sample mean random variable, we have that
$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} \text{Normal}(0, \sigma^2)$$
In addition, a version of the *first order Delta method* states that if $\{Y_n, n = 1, 2, \ldots\}$ is a sequence of rvs such that
$$\sqrt{n}(Y_n - \theta) \xrightarrow{d} \text{Normal}(0, V(\theta))$$
for some constant θ, where $V(.)$ is a positive function, and if $g(t)$ is a continuous function such that the first derivative of g, $\dot{g}$, is non-zero at θ, then
$$\sqrt{n}(g(Y_n) - g(\theta)) \xrightarrow{d} \text{Normal}(0, V(\theta)\{\dot{g}(\theta)\}^2)$$

(a) Suppose $\{X_n, n = 1, 2, \ldots\}$ are independent and identically distributed Bernoulli(ϕ) random variables, where $0 < \phi < 1$. Describe the limiting behaviour of the sequence $\{S_n, n = 1, 2, \ldots\}$, where

$$S_n = \{\overline{X}_n\}^2$$

and construct a large sample normal approximation to the distribution of S_n.

5 Marks

(b) Suppose that $U_1, U_2, \ldots$ are independent and identically distributed Uniform$(0, 1)$ random variables. Let M_n be the geometric mean of the $U_1, \ldots, U_n$, that is

$$M_n = \left\{ \prod_{i=1}^{n} U_i \right\}^{1/n} \qquad n = 1, 2, \ldots.$$

Describe the limiting behaviour of the sequence of random variables $\{M_n, n = 1, 2, \ldots\}$, and construct a large sample normal approximation to the distribution of M_n.

5 Marks