MATH 556 - ASSIGNMENT 4SOLUTIONS

1. The cdf is
$$F_X(x) = 1 - \exp\{-\lambda(x-\eta)\} \qquad x > \eta$$
and zero otherwise. Therefore $Y_{n,1}$ has cdf
$$F_{Y_{n,1}}(y) = 1 - \exp\{-n\lambda(y-\eta)\} \quad y > \eta$$
and as $n \longrightarrow \infty$ we have
$$F_{Y_{n,1}}(y) \longrightarrow \begin{cases} 0 & y \leq \eta \\ 1 & y > \eta \end{cases}$$
Therefore the limiting distribution is degenerate at η; the limit function is not right continuous, but clearly for any $\epsilon > 0$,P$[Y_{n,1} < \epsilon] \longrightarrow 1$.

2 Marks

2. We have
$$F_X(x) = 1 - \frac{1}{(1+x)} \qquad x > 0$$
and zero otherwise. Then $Y_{n,n}$ has cdf
$$F_{Y_{n,n}}(y) = \left(1 - \frac{1}{1+y}\right)^n \qquad y > 0$$
Let $V_n = Y_{n,n}/n$. Then for $v > 0$
$$F_{V_n}(v) = \left(1 - \frac{1}{1+nv}\right)^n$$
and in the limit we have that
$$F_{V_n}(v) = \longrightarrow \begin{cases} 0 & v \leq 0 \\ \exp\{-1/v\} & v > 0 \end{cases}$$

4 Marks

3. By Markov's inequality, for $r = 2$ and $\epsilon > 0$,
$$\mathrm{P}_X\left[X^2 \geq \epsilon\right] \leq \frac{\mathbb{E}_X\left[X^2\right]}{\epsilon}.$$
Consider the sequence of random variables $\{X_n - X, n \geq 1\}$. Then this inequality becomes
$$\mathrm{P}_{X_n,X}\left[(X_n - X)^2 \geq \epsilon\right] \leq \frac{\mathbb{E}_{X_n,X}\left[(X_n - X)^2\right]}{\epsilon}.$$
But the right hand side goes to zero as $n \longrightarrow \infty$. Therefore, for any $\epsilon > 0$,
$$\mathrm{P}_{X_n,X}\left[(X_n - X)^2 \geq \epsilon\right] \longrightarrow 0$$
which confirms that $X_n \xrightarrow{p} X$.

4 Marks

4. (a) We have from the Weak Law of Large Numbers and Central Limit Theorem that

$$\overline{X}_n \xrightarrow{p} \phi \qquad\qquad \sqrt{n}(\overline{X}_n - \phi) \xrightarrow{d} \mathcal{N}(0, \phi(1-\phi))$$

so using the Delta Method for transform $g(t) = t^2$, so that $\dot{g}(t) = 2t$, we have

$$S_n = \{\overline{X}_n\}^2 \xrightarrow{p} \phi^2$$

and

$$\sqrt{n}(S_n - \phi^2) \xrightarrow{d} \mathcal{N}(0, 4\phi^3(1-\phi))$$

so that for large n

$$S_n \stackrel{\cdot}{\sim} \mathcal{N}(\phi^2, 4\phi^3(1-\phi)/n).$$

5 Marks

(b) We have that

$$T_n = \log M_n = \frac{1}{n}\sum_{i=1}^n \log U_i = \frac{1}{n}\sum_{i=1}^n Y_i$$

say. Note that, by parts

$$\mathbb{E}_{Y_i}[Y_i] = \int_0^1 \log u \ \ \mathrm{d}u = [-u(1-\log u)]_0^1 = -1$$

and also by parts

$$\mathbb{E}_{Y_i}[Y_i^2] = \int_0^1 (\log u)^2 \ \ \mathrm{d}u = \left[u(\log u)^2\right]_0^1 - 2\int_0^1 \log u \ \ \mathrm{d}u = 2$$

so that $\mathrm{Var}_{Y_i}[Y_i] = 1$. Therefore by the CLT

$$\sqrt{n}(T_n + 1) \xrightarrow{d} \mathcal{N}(0,1).$$

Using the Delta Method for transform $g(t) = e^t$, so that $\dot{g}(t) = e^t$, we have

$$\sqrt{n}(M_n - e^{-1}) \xrightarrow{d} \mathcal{N}(0, e^{-2})$$

so that for large n

$$M_n \stackrel{\cdot}{\sim} \mathcal{N}(e^{-1}, e^{-2}/n).$$

5 Marks

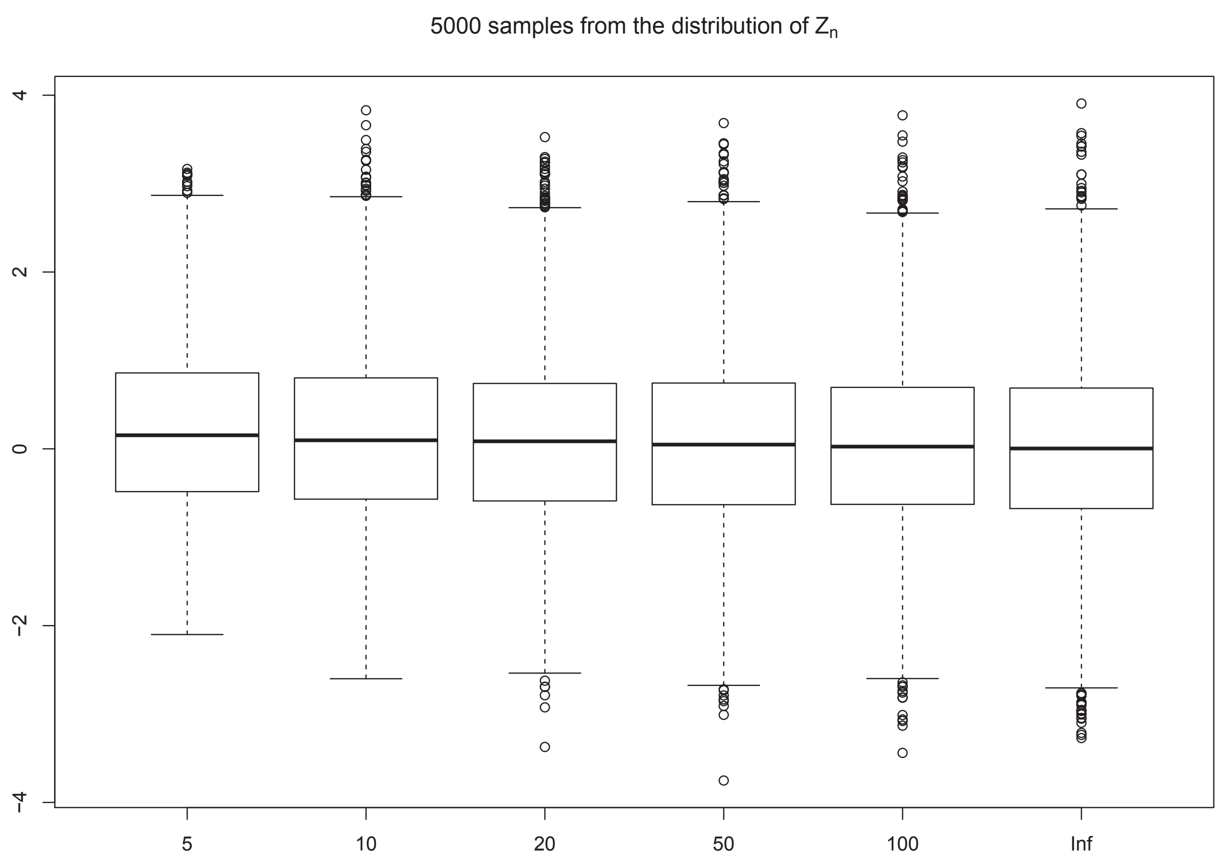
Here is a Monte Carlo study: the code simulates the distribution of the standardized version of M_n, that is

$$Z_n = \sqrt{n}\frac{M_n - e^{-1}}{e^{-1}}$$

for $n = 5, 10, 20, 50, 100$.

```
set.seed(2323)
nvec<-c(5,10,20,50,100)
nreps<-5000
Xsamp<-matrix(0,nrow=nreps,ncol=length(nvec))
for(i in 1:length(nvec)){
        Xsamp[,i]<-replicate(nreps,exp(mean(log(runif(nvec[i])))))
        Xsamp[,i]<-sqrt(nvec[i])*(Xsamp[,i]-exp(-1))/exp(-1)
}
Xsamp<-cbind(Xsamp,rnorm(nreps))
boxplot(Xsamp,names=c(nvec,Inf))
```

The boxplots for increasing n seem to match the asymptotic limit for $n \geq 20$.


5000 samples from the distribution of Z_n
4
2
0
−2
−4
5
10
20
50
100
Inf