MATH 556 - ASSIGNMENT 2

1. (a) We have by definition that

$$\varphi_k(t) = \int e^{itx} \, \mathrm{d}F_k(x)$$

say, for some cdf F_k. Clearly $\varphi(t)$ is finite as the c_k are summable, and the individual cf integrals are finite. We have, because of this, by exchanging the order of summation and integration,

$$\varphi(t) = \sum_{k=1}^{n} c_k \varphi_k(t) = \sum_{k=1}^{n} \left\{ c_k \int e^{itx} \, \mathrm{d}F_k(x) \right\} = \int e^{itx} \left\{ \sum_{k=1}^{n} c_k \, \mathrm{d}F_k(x) \right\} = \int e^{itx} \, \mathrm{d} \left\{ \sum_{k=1}^{n} c_k F_k(x) \right\}.$$

The function

$$F_X(x) = \sum_{k=1}^{n} c_k F_k(x)$$

is a valid cdf; this is easily verified by checking the standard properties, as the c_ks sum to one. Therefore $\varphi(t)$ is the cf corresponding to F_X.

4 Marks

The distribution characterized by F_X and $\varphi(t)$ is termed a finite mixture distribution.

(b) Consider the limiting case function $F_X(x)$

$$F_X(x) = \sum_{k=1}^{\infty} c_k F_k(x)$$

for any fixed x. The right hand side can be considered as the probability of a disjoint union of the events

$$(X \leq x \cap Z = k) \qquad k = 1, 2, \ldots$$

where Z is a discrete random variable on the positive integers, with probabilities $c_1, c_2, \ldots$ attached to $1, 2, \ldots$. Now by standard limit results for event sequences

$$\bigcup_{k=1}^{\infty} (X \leq x \cap Z = k) \equiv (X \leq x)$$

and thus $F_X(x)$ is a well-defined cdf. Hence the limiting case as $n \longrightarrow \infty$ provides no difficulty.

2 Marks

(c) By inspection of the formula sheet, and the realization that if the mgf $M(t)$ exists for $|t| < \delta$, then $\varphi(t) = M(it)$, we may deduce that $\varphi_1(t)$ is the cf of the normal density with mean zero and variance 8.

For $\varphi_2(t)$, we note that

$$\limsup_{t \longrightarrow \pm\infty} |\varphi_2(t)| = 1$$

so the distribution is discrete; we must find mass function $f_2(x)$ with support $\mathbb{X}$ such that

$$\sum_{x \in \mathbb{X}} e^{ixt} f_2(x) = (3 + \cos(t) + \cos(2t))/5$$

Now, note that $e^{itx} = \cos(tx) + i\sin(tx)$, so we can deduce that $x = 0, 1, 2$ must be in $\mathbb{X}$. Note also that the cf is entirely real, and

$$\cos(tx) = \frac{e^{itx} + e^{-itx}}{2}$$

and so

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \cos(2t) = \frac{e^{it2} + e^{-it2}}{2}$$

Hence $f_2(x)$ can be deduced to be of the form

$$f_2(x) = \begin{cases} 1/10 & x = -2 \\ 1/10 & x = -1 \\ 6/10 & x = 0 \\ 1/10 & x = 1 \\ 1/10 & x = 2 \\ 0 & \text{otherwise} \end{cases}$$

Hence, using part(a), the distribution is an equal mixture of the two components, that is,

$$F(x) = \frac{1}{2}\Phi(x/\sqrt{8}) + \frac{1}{2}F_2(x).$$

4 Marks

2. As $X_2 = X_1 + (X_2 - X_1)$, we have by the independence statements

$$\varphi_{X_2}(t) = \varphi_{X_1}(t)\varphi_{X_2-X_1}(t) = \varphi_{X_1}(t)\varphi_{X_2}(t)\varphi_{X_1}(-t) = \varphi_{X_2}(t)|\varphi_{X_1}(t)|^2$$

with the final step being an elementary property of complex numbers. Now $\varphi_{X_2}(0) = 1$ and thus by continuity $\varphi_{X_2}(t) \neq 0$ for t at least in a neighbourhood of zero. Thus, equating the two sides, we must have $|\varphi_{X_1}(t)|^2 = 1$ for all t. Thus as t varies, $\varphi_{X_1}(t)$ is always a complex valued quantity that lies on the unit circle in the complex plane. Hence we must have

$$\varphi_{X_1}(t) = e^{itc}$$

for some c, and X_1 is degenerate at c.

4 Marks

3. First, note that

$$M_X(t) = e^{-t}M_Z(t)$$

where

$$M_Z(t) = \frac{9}{(3+2t)^2} = \frac{1}{(1+(2/3)t)^2}$$

and, by linear transformation results for mgfs, $X \stackrel{d}{=} Z - 1$. Hence

$$\mathbb{E}_X[X^r] = \mathbb{E}_Z[(Z-1)^r] = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j}\mathbb{E}_Z[Z^j].$$

We have that

$$M_Z^{(j)}(t) = (-2)(-3)...(-2-j+1)\frac{(2/3)^j}{(1+(2/3)t)^{2+j}} = (-1)^j(j+1)!\frac{(2/3)^j}{(1+(2/3)t)^{2+j}}$$

so that

$$\mathbb{E}_Z[Z^j] = M_Z^{(j)}(0) = (-1)^j(j+1)!(2/3)^j$$

which we may substitute in above to get

$$\mathbb{E}_X[X^r] = \mathbb{E}_Z[(Z-1)^r] = (-1)^r \sum_{j=0}^{r} \binom{r}{j}(j+1)!(2/3)^j.$$

3 Marks

4. From the formula sheet we have

$$M_X(t) = (1-\theta+\theta e^t)^n$$

and thus

$$K_X(t) = n\log(1-\theta+\theta e^t).$$

Using the linear transformation result,

$$M_{Z_n}(t) = e^{b_n t} M_X(a_n t)$$

where

$$a_n = \frac{1}{\sqrt{n\theta(1-\theta)}} \qquad b_n = -\frac{n\theta}{\sqrt{n\theta(1-\theta)}} = -n\theta a_n$$

we have

$$K_{Z_n}(t) = -n\theta a_n t + K_X(a_n t) \quad = -n\theta a_n t + n\log(1-\theta+\theta e^{a_n t})$$

Now if

$$g_n(t) = e^{a_n t} - 1 = a_n t + \frac{1}{2}a_n^2 t^2 + \frac{1}{6}a_n^3 t^3 + \cdots$$

we have up to terms in t^3

$$\begin{aligned} n\log\{(1+\theta g_n(t))\} &= n\theta g_n(t) - n\theta^2\{g_n(t)\}^2/2 + n\theta^3\{g_n(t)\}^3/6\cdots \\ &= n\theta(a_n t + \frac{1}{2}a_n^2 t^2 + \frac{1}{6}a_n^3 t^3 + \cdots) \\ &\quad -\frac{n\theta^2}{2}(a_n^2 t^2 + a_n^3 t^3 + \cdots) \\ &\quad +\frac{n\theta^3}{3}(a_n^3 t^3 + \cdots) + \cdots \end{aligned}$$

Therefore, from the earlier expression, the term in t cancels, and we are left with

$$\begin{aligned} K_{Z_n}(t) &= \frac{n\theta(1-\theta)}{2}a_n^2 t^2 + n\theta\left(\frac{1}{6} - \frac{\theta}{2} + \frac{\theta^2}{3}\right)a_n^3 t^3 + \cdots \\ &= \frac{t^2}{2} + \frac{1}{n^{1/2}}\frac{\theta(1-3\theta+2\theta^2)}{6(\theta(1-\theta))^{3/2}}t^3 + \cdots \end{aligned}$$

The truncation after the second term leads to an approximation which has order na_n^4; this is a constant times $nn^{-2} = n^{-1}$. Hence we may equivalently write

$$K_{Z_n}(t) = \frac{t^2}{2} + \frac{1}{n^{1/2}} \frac{\theta(1 - 3\theta + 2\theta^2)}{6(\theta(1-\theta))^{3/2}} t^3 + \mathrm{O}(n^{-1})$$

or

$$K_{Z_n}(t) = \frac{t^2}{2} + \frac{1}{n^{1/2}} \frac{\theta(1 - 3\theta + 2\theta^2)}{6(\theta(1-\theta))^{3/2}} t^3 + \mathrm{o}(n^{-1/2})$$

as $n \longrightarrow \infty$.

3 Marks