Master's Thesis

Rankin L-functions and the Birch and Swinnerton-Dyer Conjecture

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Abstract

In this thesis we use Rankin's method to evaluate the central critical value of the L-series attached to an elliptic curve E over $\mathbb{Q}$ and certain odd irreducible 2-dimensional Artin representation $\tau : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$. The motivation for this study is the twisted Birch and Swinnerton-Dyer conjecture.

Let K be a number field and $E(K)$ be the abelian group of K-rational points on E. Consider the natural representation of $\mathrm{Gal}(K/\mathbb{Q})$ on $\mathbb{C} \otimes_{\mathbb{Z}} E(K)$, i.e. for any point $P = (x, y) \in E(\overline{\mathbb{Q}})$, an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts as

$$\begin{array}{rcl} \mathrm{Gal}(K/\mathbb{Q}) \times E(K) & \to & E(K) \\ (\sigma, P) & \mapsto & P^{\sigma} = (\sigma(x), \sigma(y)). \end{array}$$

Let ρ_E be the 2-dimensional ℓ-adic Galois representation attached to the elliptic curve E, namely, the p-adic Tate module of E.

Twisted form of the Birch and Swinnerton-Dyer conjecture: Let τ be a continuous and irreducible finite dimensional complex representation of $\mathrm{Gal}(K/\mathbb{Q})$. Then

$$\mathrm{ord}_{s=1} L(\tau \otimes \rho_E, s) = \langle \tau, \mathbb{C} \otimes_{\mathbb{Z}} E(K) \rangle = \text{multiplicity of } \tau \text{ in } \mathbb{C} \otimes_{\mathbb{Z}} E(K).$$

This conjecture is a natural strengthening of the Birch and Swinnerton-Dyer conjecture. In fact, if we replace τ by the trivial representation, then we recover this conjecture. We explain it briefly. Let K be a number field and consider $E(K)$ the abelian group of K-rational points on E. The Mordell-Weil theorem tells us that $E(K)$ has the form

$$E(K) \cong E(K)_{\mathrm{tors}} \oplus \mathbb{Z}^r$$

where the torsion subgroup $E(K)_{\mathrm{tors}}$ is finite and the *rank* r of $E(K)$ is a nonnegative integer. It is relatively easy to compute the torsion subgroup but there is no known procedure that is guaranteed to yield the rank $r_K(E)$.

The *Hasse-Weil L-function of E* is:

$$L(E, s) = \prod_p \frac{1}{1 - a_p p^s + \mathbf{1}_E(p) p^{1-2s}}$$

where $\mathbf{1}_E$ is the trivial character modulo the conductor N_E and

$$a_p(E) = p - (\text{the number of solutions (x,y) of equation E working modulo p}).$$

More generally, for any number field K, one can associate to E an L-series $L(E/K, s)$. The product defining $L(E/K, s)$ converges and gives an analytic function for all $\mathcal{R}(s) > \frac{3}{2}$. Its analytic continuation is conjectured as follows:

Conjecture: The L-series $L(E/K, s)$ has an analytic continuation to the entire complex plane and satisfies a functional equation relating its values at s and $2 - s$.

The original half plane of convergence of $L(E/K, s)$ is the half plane $\mathcal{R}(s) > \frac{3}{2}$ and the functional equation then determines $L(E/K, s)$ for $\mathcal{R}(s) < \frac{1}{2}$, but the behaviour of $L(E/K, s)$ at the center of the remaining strip $\{\frac{1}{2} < \mathcal{R}(s) < \frac{3}{2}\}$ is what conjecturally determines the rank of $E(K)$. Birch and Swinnerton-Dyer conjectures

that the rank of E over K is equal to the order of vanishing of $L(E/K,s)$ at $s=1$:

Birch and Swinnerton-Dyer Conjecture: The order of vanishing of $L(E/K,s)$ at $s=1$ is the rank of $E(K)$. That is, if $E(K)$ has rank r then: [1]

$$L(E/K,s) = (s-1)^r g(s) \quad , \ g(1) \neq 0, \infty.$$

This conjecture relates the algebraic group of an elliptic curve E to analytic properties of $L(E,s)$.

Let τ be a continuous and irreducible finite-dimensional complex representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. One can associate to τ an L-series $L(\tau,s)$. On the other hand, there is a representation ρ_E of the elliptic curve E, namely, the p-adic Tate module of E, such that its associated L-series $L(\rho_E,s)$ is equal to $L(E,s)$. One can construct an L-series $L(\rho_E\otimes\tau,s)$ which corresponds to the tensor product of two representations ρ_E and τ. By Rankin method, one can show that if τ is arising from a modular form, then the L-series $L(\rho_E\otimes\tau,s)$ admits analytic continuation to $\mathbb{C}$. (see chapter 2.1)

In this thesis, we provide some numerical evidence for the twisted form of the Birch and Swinnerton-Dyer conjecture using the Deligne-Serre theorem and Rankin's method. Deligne and Serre proved a correspondence between the modular forms of weight 1 and certain 2-dimensional Galois representations. Given a cusp form $g=\sum_{n=0}^{\infty} b_n q^n \in \mathcal{S}_1(\Gamma_0(N),\chi)$, one gets an irreducible 2-dimensional complex representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$:

$$\rho_g : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$$

with the property that

$$\mathrm{char}(\rho_g(\mathrm{Frob}_p)) = X^2 - b_p X + \chi(p) \quad \text{for all } p \nmid N.$$

The image of ρ_g in projective space $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^*$ is a dihedral group D_n or one of the groups A_4, S_4 or A_5.

For a given cusp form $g \in \mathcal{S}_1(\Gamma_0(N),\chi)$, we apply the twisted form of Birch and Swinnerton-Dyer conjecture to $\tau=\rho_g$:

$$\mathrm{ord}_{s=1} L(\rho_g\otimes\rho_E,s) = \text{multiplicity of } \rho_g \text{ in } \mathbb{C}\otimes_{\mathbb{Z}} E(K).$$

Then for some elliptic curve E of conductor M with $N|M$, we computed $L(\rho_E\otimes\rho_g,1)$ and verified if it vanishes. Assuming the BSD conjecture, one can say

$$L(\rho_g\otimes\rho_E,1)=0 \quad \Leftrightarrow \quad \rho_g \text{ occurs in the representation } \mathbb{C}\otimes_{\mathbb{Z}} E(K).$$

where K denotes the finite extension of Q which is fixed by the kernel ρ_g. As a consequence, if $L(\rho_g\otimes\rho_E,1)=0$, then the rank $r_K(E)$ of elliptic curve E over K is $\geqslant 2$.

In chapter one, we introduce some background about modular forms and Eisenstein series of weight 1. Chapter two introduces the Rankin convolution L-series $L(f\otimes g,s)$ attached to two modular forms f and g. Then its analytic continuation

[1] In 2000, this Conjecture was declared a million dollar millennium prize problem by the Clay Mathematics Institute.

is discussed via the Rankin-Selberg method and some applications of this method is presented. In chapter three, we state and prove the Deligne-Serre theorem. Finally in chapter four, we discuss the conjecture of Birch and Swinnerton-Dyer. Then we present a twisted form of it. One interesting case is when we twist an elliptic curve with a cusp form of weight 1. We develop techniques to compute the constant term of the twisted L-series $L(\rho_E \otimes \rho_g, s)$ at $s = 1$. We perform some computations for certain elliptic curves with small conductor N using the database Sage. We presented these computations in tables 1 to 14.

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Contents

1 Modular forms of weight 1

In this chapter, we introduce the non-holomorphic Eisenstein series of weight 1 and characters ψ and χ with parameter s. Since it can be analytically extended to a meromorphic function such that it is holomorphic on $\mathcal{R}(s) > \frac{-1}{2}$, one obtain a modular form of weight 1 at $s = 0$. Then we present theta series and give some examples of cusp forms of weight 1 arising from theta series.

1.1 Eisenstein series of weight 1

Let ψ and χ be Dirichlet characters mod M and mod N, respectively. For any integer $k \geqslant 1$ and $z \in \mathcal{H}$, we put:

$$\widetilde{E}_k(z;\psi,\chi) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}' \frac{\psi(m)\chi(n)}{(mz+n)^k}.$$

Here the summation $\sum'$ is over all integers $(m,n) \neq (0,0)$. This series is absolutely convergent for $k \geqslant 3$. Therefore we need some modification to discuss the case $k = 1$.

We define a new kind of Eisenstein series in two variables z and s such that it is holomorphic as a function of s but it is not holomorphic as a function of z.

Definition 1. *For $z = x + iy \in \mathcal{H}$, $s \in \mathbb{C}$ and $k \in \mathbb{Z}$, define*

$$\widetilde{E}_k(z,s;\psi,\chi) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}' \frac{\psi(m)\chi(n)}{(mz+n)^k}\frac{y^s}{|mz+n|^{2s}}. \tag{1}$$

This function is called non-holomorphic Eisenstein series or Epstein zeta function of weight k and characters ψ and χ.

The right hand side of the formula is uniformly and absolutely convergent for $k + 2\mathcal{R}(s) \geqslant 2 + \varepsilon$ for any $\varepsilon > 0$. Therefore it is holomorphic on $\mathcal{R}(s) > \frac{2-k}{2}$. However, it is not holomorphic as a function of z. Put:

$$\Gamma_0(M,N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) |\ c \equiv 0 \bmod \mathrm{N}, \quad b \equiv 0 \bmod \mathrm{M} \right\}.$$

Then $\Gamma_0(M,N)$ is a modular group. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M,N)$, we have:

$$E_k(\gamma z,s;\psi,\chi) = \psi(d)\overline{\chi(d)}(cz+d)^k|cz+d|^{2s}\widetilde{E}_1(z,s;\psi,\chi).$$

We easily see that if $\psi(-1)\chi(-1) \neq (-1)^k$, then $\widetilde{E}_1(z,s;\psi,\chi) = 0$. So we assume:

$$\psi(-1)\chi(-1) = (-1)^k \tag{2}$$

throughout this section.

We wish to study the case $k = 1$. The series $\widetilde{E}_1(z,s;\psi,\chi)$ is not convergent at $s = 0$. But if $\widetilde{E}_1(z,s;\psi,\chi)$ is continued analytically to $s = 0$ and holomorphic at $s = 0$, then we will obtain a modular form of weight 1.

Theorem 2. *The Eisenstein series $\widetilde{E}_1(z,s;\psi,\chi)$ is analytically continued to a meromorphic function on the whole s-plane. If χ is non-trivial, then $\widetilde{E}_1(z,s;\psi,\chi)$ is an entire function of s. If χ is trivial, then it is holomorphic for $\mathcal{R}(s) > \frac{-1}{2}$. At $s=0$, we get a modular form of weight 1 for the modular group $\Gamma_0(M,N)$. Its Fourier expansion is given by:*

$$\widetilde{E}_1(z,s;\psi,\chi) = C + D + A\sum_{n=0}^{\infty} a_n q_N^n.$$

Here

$$C = \begin{cases} 0 & \textit{if} \quad \psi \textit{ is the principal character} \\ 2L(\chi,1) & \textit{if} \quad \psi \textit{ is not the principal character} \end{cases},$$

$$D = \begin{cases} 0 & \textit{if} \quad \chi \textit{ is non-trivial} \\ -2\pi i L(\psi,0)\prod_{p|M}(1-p^{-1}) & \textit{if} \quad \chi \textit{ is trivial} \end{cases},$$

$$A = \frac{-4\pi i\tau(\chi')}{N},$$

$$a_n = \sum_{0<c|n}\psi(\tfrac{n}{c})\sum_{0<d|gcd(l,c)} d\mu(\tfrac{l}{d})\chi'(\tfrac{l}{d})\overline{\chi'}(\tfrac{c}{d}),$$

$$q_N = e^{2\pi i/N}$$

where χ' is the primitive character of mod N' associated with χ,

$$\tau(\chi') = \sum_{a=1}^{N'}\chi'(a)e^{2\pi i a/N'}$$

is the Gauss sum attached to χ'; μ is the Mobius function and $l = \frac{N}{N'}$.

Proof: See [11] Theorem 7.29, Corollary 7.2.10 and Theorem 7.2.13. ■

In the next sections, we will need a more general kind of Eisenstein series.

Definition 3. *Let χ be a character mod N with $\chi(-1) = -1$ and **1** the trivial character mod N. Assume M is an integer with $N|M$. Then:*

$$\widetilde{E}_1(z,s;\chi;M) := \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi(n)}{Mmz+n}\frac{y^s}{|Mmz+n|^{2s}} \tag{3}$$

is called the non-holomorphic Eisenstein series of weight 1, character χ and level M.

*When $M=1$, $\widetilde{E}_1(z,s;\chi;M)$ is the non-holomorphic Eisenstein series with characters **1** and χ:*

$$\widetilde{E}_1(z,s;\chi;1) = \widetilde{E}_1(z,s;\mathbf{1},\chi).$$

We will need the q-expansion of $\widetilde{E}_1(z,\chi,M)$ for the next sections. The case $M = N$ is easier to handle:

$$\begin{aligned}
\widetilde{E}_1(z,s;\chi;N) &= \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi(n)}{Nmz+n}\frac{y^s}{|Nmz+n|^{2s}} \\
&= \sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi(n)}{mz+n}\frac{y^s}{|mz+n|^{2s}} \\
&= \frac{1}{N^s}\widetilde{E}_1(Nz,s;\mathbf{1},\chi).
\end{aligned}$$

Define:

$$\widetilde{E}_1(z;\chi;N) := \widetilde{E}_{1,}(z;0,\chi;N). \tag{4}$$

This is an Eisenstein series of weight 1 and character χ. It belongs to $\mathcal{M}_1(\Gamma_0(N),\chi)$. Its q-expansion is given by:

$$\widetilde{E}_1(z;\chi;N) = 2L_N(\chi,1) - \frac{4\pi i\tau(\chi)}{N}\sum_{n=0}^{\infty}\sigma_{\overline{\chi}}(n)q^n \tag{5}$$

where $\sigma_{\overline{\chi}}(n) = \sum_{d|n}\overline{\chi}(d)$. Since χ is a character mod N, for any $p|N$, the factor $1 - \frac{\chi(p)}{p^s}$ is one. Therefore

$$L(\chi,s) = L_N(\chi,s).$$

Assume that χ is a primitive character. The functional equation satisfied by $L(\chi,s)$ is:

$$\pi^{-(s+1)/2}\Gamma(\tfrac{s+1}{2})N^sL(\chi,s) = -i\pi^{-(2-s)/2}\Gamma(\tfrac{2-s}{2})\tau(\chi)L(\overline{\chi},1-s).$$

Specializing the above equation at $s = 1$ and by using $\Gamma(1/2) = \sqrt{\pi}$ we get:

$$\begin{aligned}
L(\overline{\chi},0) &= \frac{-i\Gamma(1)N}{\sqrt{\pi}\Gamma(1/2)\tau(\chi)}L(\chi,1) \\
&= \frac{-iN}{\pi\tau(\chi)}L(\chi,1).
\end{aligned}$$

There is a nice formula (see [20]) for computing $L(\chi,0)$ where χ is any character of mod N:

$$L(\chi,0) = \frac{-1}{N}\sum_{i=1}^{N} i\chi(i).$$

We can rewrite the q-expansion of $\widetilde{E}_1(z;\chi;N)$:

$$\widetilde{E}_1(z;\chi;N) = \frac{-2\pi i\tau(\chi)}{N}L(\overline{\chi},0) - \frac{4\pi i\tau(\chi)}{N}\sum_{n=0}^{\infty}\sigma_{\overline{\chi}}(n)q^n.$$

We introduce the *normalised Eisenstein series* $E_1(z;\chi;N)$, related to $\widetilde{E}_1(z;\chi;N)$ by the equation

$$\widetilde{E}_1(z,s;\chi;N) = \frac{-4\pi i\tau(\chi)}{N}E_1(z,s;\chi;N). \tag{6}$$

Thus the q-expansion of $E_1(z;\chi;N)$ is given by

$$E_1(z;\chi;N) = \frac{1}{2}L(\overline{\chi},0) + \sum_{n=0}^{\infty} \sigma_{\overline{\chi}}(n) q^n. \tag{7}$$

Define:

$$\begin{aligned} \widetilde{E}_1'(z,s;\chi;N) &= \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m,n)=1}} \frac{\chi(n)}{Nmz+n} \frac{y^s}{|Nm'z+n'|^{2s}} \\ &= \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(Nm,n)=1}} \frac{\chi(n)}{Nmz+n} \frac{y^s}{|Nm'z+n'|^{2s}}. \end{aligned} \tag{8}$$

(the last equality holds since $\chi(n) = 0$ for any $p|\gcd(n,N)$) We want to find the q-expansion of the following Eisenstein series:

$$\widetilde{E}_1'(z;\chi;N) := \widetilde{E}_1'(z,0;\chi;N). \tag{9}$$

If $\mathcal{R}(s) > \frac{1}{2}$, the Eisenstein series $\widetilde{E}_1(z,s;\chi;N)$ is absolutely convergent and thus we can rearrange it and write

$$\begin{aligned} \widetilde{E}_1(z,s;\chi;N) &= \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi(n)}{Nmz+n} \frac{y^s}{|Nmz+n|^{2s}} \\ &= \sum_{k=1}^{\infty} \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m,n)=k}} \frac{\chi(n)}{Nmz+n} \frac{y^s}{|Nmz+n|^{2s}} \\ &= \sum_{k=1}^{\infty} \frac{\chi(k)}{k^{1+2s}} \sum_{\substack{(m',n')\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m',n')=1}} \frac{\chi(n')}{Nm'z+n'} \frac{y^s}{|Nm'z+n'|^{2s}} \\ &= L(\chi,1+2s)\widetilde{E}_1'(z,s;\chi;N). \end{aligned}$$

Both $\widetilde{E}_1(z,s;\chi;N)$ and $\widetilde{E}_1'(z,s;\chi;N)$ are holomorphic on $\mathcal{R}(s) > \frac{-1}{2}$. So we can set $s=0$ in the above equation and get

$$\widetilde{E}_1(z;\chi;N) = L(\chi,1)\widetilde{E}_1'(z;\chi;N). \tag{10}$$

Now we consider a more general situation where $N|M$. As before, one can show that:

$$\widetilde{E}_1(z,s;\chi;M) = L(\chi,1+2s)\widetilde{E}_1'(z,s;\chi;M) \tag{11}$$

where $\widetilde{E}_1'(z,s,\chi,M) = \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m,n)=1}} \frac{\chi(n)}{Mmz+n} \frac{y^s}{|Mmz+n|^{2s}}$.

Define:

$$\widetilde{E}_1(z;\chi;M) \quad := \quad \widetilde{E}_1(z,0;\chi;M). \tag{12}$$

Then:

$$\widetilde{E}_1(z;\chi;M) = L(\chi,1)\widetilde{E}_1'(z;\chi;M)$$

where $\widetilde{E}_1'(z;\chi;M) = \widetilde{E}_1'(z,0;\chi;M)$. By an example, we show how to compute Fourier coefficients of $\widetilde{E}_1(z,\chi,M)$.

Example 4. *Let χ be a primitive character mod N and let $M = pN$ where p is relatively prime to N. We want to compute the q-expansion of $\widetilde{E}_1(z;\chi;M)$. It is enough to do it for $\widetilde{E}_1'(z;\chi;M)$. For $\mathcal{R}(s) > \frac{-1}{2}$, the series $\widetilde{E}_1'(z,s;\chi,M)$ is holomorphic and one can write*

$$\begin{aligned}
\frac{1}{p^s}\widetilde{E}_1'(pz,s;\chi;N) &= \frac{1}{p^s}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1}} \frac{\chi(n)}{Npmz+n}\frac{(py)^s}{|Npmz+n|^{2s}}\\
&= \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1\\ p\nmid n}} \frac{\chi(n)}{Npmz+n}\frac{y^s}{|Npmz+n|^{2s}} + \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1\\ p|n}} \frac{\chi(n)}{Npmz+n}\frac{y^s}{|Npmz+n|^{2s}}\\
&= \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(pm,n)=1}} \frac{\chi(n)}{Npmz+n}\frac{y^s}{|Npmz+n|^{2s}} + \frac{\chi(p)}{p^{1+2s}}\sum_{\substack{(m,n')\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n')=1}} \frac{\chi(n')}{Nmz+n'}\frac{y^s}{|Nmz+n'|^{2s}}\\
&= \widetilde{E}_1'(z,s;\chi;Np) + \frac{\chi(p)}{p^{1+2s}}\widetilde{E}_1'(z,s;\chi,N).
\end{aligned}$$

Setting $s=0$ in the above equation gives:

$$\begin{aligned}
\widetilde{E}_1'(z;\chi;Np) &= \widetilde{E}_1'(pz;\chi;N) - \frac{\chi(p)}{p}\widetilde{E}_1'(z;\chi;N) &(13)\\
&= \frac{1}{2}(1-\tfrac{\chi(p)}{p})L(\overline{\chi},0) + \sum_{n=0}^{\infty}\left(\sigma_{\overline{\chi}}(\tfrac{n}{p}) - \tfrac{\chi(p)}{p}\sigma_{\overline{\chi}}(n)\right)q^n &(14)
\end{aligned}$$

where $\sigma_{\overline{\chi}}(\frac{n}{p}) = 0$ if $p\nmid n$.

By similar computation, we can get a more general formula. Assume $N = \prod_{i=1}^{v} p_i^{\alpha_i}$ and $M = \prod_{i=1}^{v} p_i^{\beta_i}\prod_{j=1}^{w} q_j^{\gamma_j}$ where p_i's and q_i's are distinct prime numbers and $\beta_i \geqslant \alpha_i$ for any $i = 1,...,v$ (thus $N|M$). Set $Q = \{q_1,q_2,...,q_w\}$. We have

$$\begin{aligned}
\widetilde{E}_1'(z;\chi;M) &= \widetilde{E}_1'(Mz;\chi;N)\\
&\quad -\sum_{q_i\in Q}\frac{\chi(q_i)}{q_i}\widetilde{E}_1'(\tfrac{M}{q_i}z;\chi;N)\\
&\quad +\sum_{\substack{q_i,q_j\in Q\\ q_i\neq q_j}}\frac{\chi(q_iq_j)}{q_iq_j}\widetilde{E}_1'(\tfrac{M}{q_iq_j}z;\chi;N)\\
&\quad \pm\cdots\\
&\quad +(-1)^w\frac{\chi(q_1q_2...q_w)}{q_1q_2...q_w}\widetilde{E}_1'(\tfrac{M}{q_1q_2...q_w}z;\chi;N).
\end{aligned}$$

Since we already computed the q-expansion of $\widetilde{E}_1'(z;\chi;N)$, we obtain the q-expansion of $\widetilde{E}_1'(z,\chi,M)$ using the above equation.

1.2 Theta series

We give a brief description of theta series and provide some examples. For more details, see [19]. Let $Q:\mathbb{Z}^r\to\mathbb{Z}$ be any positive definite integer-valued quadratic form in r variables, r even. Define the *theta series* associated to Q as

$$\Theta_Q(\tau)=\sum_{x\in\mathbb{Z}^r} q^{Q(x)}.$$

Θ_Q belongs to $\mathcal{M}_{\frac{r}{2}}(\Gamma_0(N),\chi)$. The level N is determined as follows: write $Q(x)=\frac{1}{2}x^tAx$ where A is an even symmetric $r\times r$ matrix, i.e. $A=(a_{ij}),\ a_{ii}\in 2\mathbb{Z}$; then N is the smallest positive integer such that NA^{-1} is again even. The character χ is the Kronecker symbol $\chi=\left(\frac{D}{\cdot}\right)$ with $D=(-1)^{\frac{r}{2}}\det A$. With some examples, we explain how to get cusp forms of weight 1 arising from theta series.

Example 5. *The following quadratic forms* $Q_1(x_1,x_2)=x_1^2+x_1x_2+6x_2^2$ *and* $Q_2(x_1,x_2)=2x_1^2+x_1x_2+3x_2^2$ *have level* $N=23$ *and character* $\chi(d)=\left(\frac{-23}{d}\right)=\left(\frac{d}{23}\right)$. *Put*

$$\begin{aligned} f &= \tfrac{1}{2}(\Theta_{Q_1}-\Theta_{Q_2})\\ &= q-q^2-q^3+q^6+q^8-q^{13}+\cdots \quad . \end{aligned}$$

Then f is a cusp form of weight 1 and character χ. Further:

$$f=\eta(z)\eta(23z)=q\prod_{n=1}^{\infty}(1-q^n)(1-q^{23n}),$$

where η is the Dedekind's η-function.
More generally, for any prime number p with $p\equiv -1$ (mod 24) set $Q_1(x_1,x_2)=6x_1^2+x_1x_2+\frac{p+1}{24}x_2^2$ *and* $Q_2(x_1,x_2)=6x_1^2+5x_1x_2+\frac{p+25}{24}x_2^2$. *Put*

$$\begin{aligned} g &= \frac{1}{2}(\Theta_{Q_1}-\Theta_{Q_2})\\ &= q^{\frac{p+1}{24}}(1-q-q^2+q^5+q^7-q^{12}+\cdots) \end{aligned}$$

One has

$$g(z)=\eta(z)\eta(pz)=q^{\frac{p+1}{24}}\prod_{n=1}^{\infty}(1-q^n)(1-q^{pn}),$$

g is a cusp form of weight 1 and level p. For more details, see [16]

Example 6. *We can define in a similar way a cusp form of weight 1 and level 31. Let* $Q_1(x_1,x_2)=x_1^2+x_1x_2+8x_2^2$ *and* $Q_2(x_1,x_2)=2x_1^2+x_1x_2+4x_2^2$. *Set*

$$\begin{aligned} f &= \frac{1}{2}(\Theta_{Q_1}-\Theta_{Q_2})\\ &= q-q^2-q^5-q^7+q^8+q^9+q^{10}+\cdots \quad . \end{aligned}$$

Then $f\in\mathcal{S}_1(\Gamma_0(31),\left(\frac{-31}{\cdot}\right))$.

2 The Rankin-Selberg method and applications

The 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which are geometric are all expected to arise from modular forms, i.e. we expect that if ρ is an odd 2-dimensional compatible system of ℓ-adic representations, then there is a modular form f and integer j such that:

$$L(\rho, s) = L(f, s + j).$$

We can construct representations of higher dimension built up from those arising from modular forms. For example, given 2 representations V_1 and V_2, we have:

$$L(V_1 \oplus V_2) = L(V_1, s).L(V_2, s).$$

This L-function inherits its analytic properties from $L(V_1, s)$ and $L(V_2, s)$ so it is not interesting. However, one can try to construct an L-series corresponding to the representation $V_1 \otimes V_2$. In this chapter, we study the representation associated to $V_1 \otimes V_2$ where V_1 and V_2 are modular, i.e. they arise from modular forms.

2.1 Rankin convolution L-series

Let

$$f = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(N), \chi_f)$$

and

$$g = \sum_{n=1}^{\infty} b_n q^n \in \mathcal{S}_\ell(\Gamma_0(N), \chi_g)$$

be normalized eigenforms of level N (we assume also the case $N = 1$). We do not assume that they are new of this level, but we do assume that they are simultaneous eigenvectors for the Hecke operators T_r with $\gcd(r, N) = 1$ as well as the operators U_r attached to the primes r dividing N. Then their associated L-functions have an Euler product expansion:

$$\begin{aligned} L(f, s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \\ &= \prod_{p|N} \left(1 - a_p p^{-s}\right)^{-1} \prod_{p\nmid N} (1 - a_p p^{-s} + \chi_f(p) p^{k-1-2s})^{-1}. \end{aligned}$$

We define:

$$L_N(f, s) := \prod_{p\nmid N} \left(1 - a_p p^{-s} + \chi_f(p) p^{k-1-2s}\right)^{-1}.$$

For each prime p, let α_p and $\alpha_p^{'}$ be the roots of the Hecke polynomials $x^2 - a_p x + \chi_f(p) p^{k-1}$, choosing $(\alpha_p, \alpha_p^{'}) = (a_p, 0)$ when $p|N$. It follows:

$$L_N(f, s) = \prod_{p\nmid N} \left(1 - \alpha_p p^{-s}\right)^{-1} (1 - \alpha_p^{'} p^{-s})^{-1}.$$

For each prime p, let α_p and $\alpha_p^{'}$ be the roots of the Hecke polynomials $x^2 - a_p x + \chi_f(p)p^{k-1}$, choosing $(\alpha_p, \alpha^{'}) = (a_p, 0)$ when $p|N$. Hence for each prime $p|N$, we have $L_{(p)}(f,s) = (1 - a_p p^{-s})^{-1}$. Therefore we can simply write:

$$L(f,s) = \prod_{p\in\mathcal{P}} L_{(p)}(f,s).$$

We do the same for g. For each prime p, let β_p and $\beta_p^{'}$ be the roots of the Hecke polynomials $x^2 - b_p x + \chi_g(p)p^{\ell-1}$, choosing $(\beta_p, \beta_p^{'}) = (b_p, 0)$ when $p|N$. We write

$$L(g,s) = \prod_{p\in\mathcal{P}} L_{(p)}(g,s),$$

where

$$L_{(p)}(g,s) = \left(1 - \beta_p p^{-s}\right)^{-1}(1 - \beta_p^{'} p^{-s})^{-1}.$$

We want to define an L-series attached to both modular forms f and g. For it, we can use their product expansion:

Definition 7. *The Rankin L-series or Rankin convolution L-series attached to* (f,g) *is defined as:*

$$L(f\otimes g,s) = \prod_{p\in\mathcal{P}} L_p(f\otimes g,s) \tag{15}$$

where:

$$L_{(p)}(f\otimes g,s) := (1-\alpha_p\beta_p p^{-s})^{-1}(1-\alpha_p\beta_p^{'}p^{-s})^{-1}(1-\alpha_p^{'}\beta_p p^{-s})^{-1}(1-\alpha_p^{'}\beta_p^{'}p^{-s})^{-1}.$$

$V_f \otimes V_g$ *is the tensor product of the two representations which is 4-dimensional so that* $L(f\otimes g,s)$ *is defined by an Euler product with factors of degree 4.*

We study the analytic continuation of $L(f\otimes g,s)$ and try to find a functional equation for it. First, we would like to write $L(f\otimes g,s) = \displaystyle\sum_{n=1}^{\infty}\frac{A_n}{n^s}$ and compute the coefficients A_n. We have:

$$\begin{aligned} L_{(p)}(f\otimes g,s) &= (1-\alpha_p\beta_p p^{-s})^{-1}(1-\alpha_p\beta_p^{'}p^{-s})^{-1}(1-\alpha_p^{'}\beta_p p^{-s})^{-1}(1-\alpha_p^{'}\beta_p^{'}p^{-s})^{-1} \\ &= (1+\alpha_p\beta_p p^{-s} + (\alpha_p\beta_p)^2 p^{-2s} + ...)(1+\alpha_p\beta_p^{'}p^{-s} + (\alpha_p\beta_p^{'})^2p^{-2s} + ...) \\ &\quad (1+\alpha_p^{'}\beta_p p^{-s} + (\alpha_p^{'}\beta_p)^2p^{-2s} + ...)(1+\alpha_p^{'}\beta_p^{'}p^{-s} + (\alpha_p^{'}\beta_p^{'})^2p^{-2s} + ...). \end{aligned}$$

It follows that:

$$\begin{aligned} A_p &= \alpha_p\beta_p + \alpha_p\beta_p^{'} + \alpha_p^{'}\beta_p + \alpha_p^{'}\beta_p^{'} \\ &= (\alpha_p + \alpha_p^{'})(\beta_p + \beta_p^{'}) \\ &= a_p b_p. \end{aligned}$$

$$\begin{aligned}
A_{p^2} &= (\alpha_p\beta_p)^2 + (\alpha_p\beta_p^{'})^2 + (\alpha_p^{'}\beta_p)^2 + (\alpha_p^{'}\beta_p^{'})^2 \\
&\quad +(\alpha_p\beta_p)(\alpha_p\beta_p^{'}) + (\alpha_p\beta_p)(\alpha_p^{'}\beta_p) + (\alpha_p\beta_p)(\alpha_p^{'}\beta_p^{'}) \\
&\quad +(\alpha_p\beta_p^{'})(\alpha_p^{'}\beta_p) + (\alpha_p\beta_p^{'})(\alpha_p^{'}\beta_p^{'}) + (\alpha_p^{'}\beta_p)(\alpha_p^{'}\beta_p^{'}) \\
&= \frac{1}{2}\Big((\alpha_p^2+\alpha_p^{'2})(\beta_p^2+\beta_p^{'2}) + (\alpha_p\beta_p+\alpha_p\beta_p^{'}+\alpha_p^{'}\beta_p+\alpha_p^{'}\beta_p^{'})(\alpha_p\beta_p+\alpha_p\beta_p^{'}+\alpha_p^{'}\beta_p+\alpha_p^{'}\beta_p^{'})\Big) \\
&= \frac{1}{2}\Big((a_p^2-2\chi_f(p)p^{k-1})(b_p^2-2\chi_g(p)p^{\ell-1}) + A_p^2\Big) \\
&= a_p^2b_p^2 - a_p^2\chi_g(p)p^{\ell-1} - b_p^2\chi_f(p)p^{k-1} + 2\chi_f(p)p^{k-1}\chi_g(p)p^{\ell-1} \\
&= (a_p^2-\chi_f(p)p^{k-1})(b_p^2-\chi_g(p)p^{\ell-1}) + \chi_f(p)p^{k-1}\chi_g(p)p^{\ell-1} \\
&= a_{p^2}b_{p^2} + \chi_f(p)\chi_g(p)p^{k+\ell-2}.
\end{aligned}$$

We see that $A_{p^2} \neq a_{p^2}b_{p^2}$. This motivates us to consider the following "modified Rankin L-series" as an approximation of the Rankin convolution L-series.

Definition 8. *The modified Rankin function attached to f and g is defined by:*

$$\mathcal{D}(f,g,s) = \sum_{n=1}^{\infty}\frac{a_nb_n}{n^s}.$$

The function a_nb_n is weakly multiplicative and therefore we can write:

$$\mathcal{D}(f,g,s) = \prod_p (1 + a_pb_pp^{-s} + a_{p^2}b_{p^2}p^{-2s} + \ldots).$$

Define

$$\mathcal{D}_{(p)}(f,g,s) = \sum_{n=0}^{\infty} a_{p^n}b_{p^n}p^{-ns}.$$

The series of $\mathcal{D}(f,g,s)$ is absolutely convergent if $\mathcal{R}(s) > \frac{k+\ell}{2}$. Hence we can rearrange it and write:

$$\mathcal{D}(f,g,s) = \prod_p \mathcal{D}_{(p)}(f,g,s). \tag{16}$$

We study $\mathcal{D}(f,g,s)$ locally, i.e. prime by prime. We try to find a formula relating $\mathcal{D}_p(f,g,s)$ to $L_{(p)}(f\otimes g,s)$. We give two preliminary lemmas.

Lemma 9. *Let $\left(B_{p^j}\right)_{j=1,2,\ldots}$ be a sequence of complex numbers satisfying an r-term linear recurrence of the form*

$$\begin{aligned}
B_{p^0} &= 1 \\
B_{p^{j+r}} &= \lambda_1B_{p^{j+r-1}} + \lambda_2B_{p^{j+r-2}} + \ldots + \lambda_rB_{p^j}
\end{aligned}$$

for all $j \geqslant 0$. Then:

$$\sum_{n=0}^{\infty} B_{p^n}x^n = \frac{Q(x)}{1-\lambda_1x-\lambda_2x^2-\ldots-\lambda_rx^r}$$

for some $Q(x) \in \mathbb{C}[x]$ of degree strictly less than r.

Proof: If we compute the product $(1+B_px+B_{p^2}x^2+...)(1-\lambda_1x-\lambda_2x^2-...-\lambda_rx^r)=Q(x)$, we see that the term of degree $t\geqslant r$ is $B_{p^t}-\lambda_1B_{p^{t-1}}-\lambda_2B_{p^{t-2}}-...-\lambda_{t-r}B_{p^r}=0$ by the recurrence formula. Hence it has no terms of degree $\geqslant r$. ■

In the above lemma, put $B_{p^i}=a_{p^i}b_{p^i}$. We try to find a recurrence formula for B_{p^i}.

Lemma 10. *The sequence $B_{p^i}=a_{p^i}b_{p^i}$ satisfies a recurrence formula of the form*

$$B_{p^{j+4}}=\lambda_1B_{p^{j+3}}+\lambda_2B_{p^{j+2}}+\lambda_3B_{p^{j+1}}+\lambda_4B_{p^j},$$

where

$$(1-\lambda_1x-\lambda_2x^2-\lambda_3x^3-\lambda_4x^4)=(1-\alpha_p\beta_px)(1-\alpha_p\beta_p^{'}x)(1-\alpha_p^{'}\beta_px)(1-\alpha_p^{'}\beta_p^{'}x).$$

Proof: a_{p^i} satisfies a two term recurrence formula:

$$a_{p^{i+2}}=a_pa_{p^{i+1}}-\chi_f(p)p^{k-1}a_{p^i}\quad\forall i\geqslant 0.$$

Let

$$W=\left\{(x_i)_{i\geqslant 0}:\ x_{i+2}=a_px_{i+1}-\chi_f(p)p^{k-1}x_i\right\}.$$

We claim that $\dim(W)=2$ and a basis for this vector space is given by $(\alpha_p^j)_{j=1,2,...}$ and $(\alpha_p^{'j})_{j=1,2,...}$. To prove it, consider the following linear transformation on W:

$$\begin{aligned}\varphi:\quad & W && \to W\\ & (x_0,x_1,...) && \mapsto (x_1,x_2,...).\end{aligned}$$

φ is an invertible map, since x_0 can be determined by x_1 and x_2. The eigenvalues of this transformations are geometric progressions. Suppose $\varphi((x_0,x_1,...))=\lambda(x_0,x_1,...)$. Then $x_2=\lambda x_1=\lambda^2x_0$. From $x_2=a_px_1-\chi_f(p)p^{k-1}x_0$ we get $\lambda^2x_0=a_p\lambda x_0-\chi_f(p)p^{k-1}x_0$ therefore $\lambda^2=a_p\lambda-\chi_f(p)p^{k-1}$, i.e. λ is a root of $x^2=a_px-\chi_f(p)p^{k-1}$ so is equal to α_p or $\alpha_p^{'}$. Hence $\left(a_{p^i}\right)_i$ is a linear combination of α_p^i and $\alpha_p^{'i}$. Likewise, $\left(b_{p^i}\right)_i$ is a linear combination of β_p^i and $\beta_p^{'i}$. It follows that $\left(B_{p^i}\right)=a_{p^i}b_{p^i}$ is a linear combinations of the four geometric progressions $(\alpha_p\beta_p)^i$, $(\alpha_p\beta_p^{'})^i$, $(\alpha_p^{'}\beta_p)^i$ and $(\alpha_p^{'}\beta_p^{'})^i$ and they satisfy the desired recurrence formula.

Corollary 1. *We have:*

$$\sum_{n=0}^{\infty}a_{p^n}b_{p^n}p^{-ns}=\frac{1-\chi(p)p^{k+\ell-2}}{1-\alpha_p\beta_pp^{-s}-\alpha_p^{'}\beta_pp^{-2s}-\alpha_p\beta_p^{'}p^{-s}-\alpha_p^{'}\beta_p^{'}p^{-s}}$$

where $\chi=\chi_f\chi_g$. In other words:

$$\mathcal{D}_{(p)}(f,g,s)=(1-\chi(p)p^{k+\ell-2}p^{-2s})L_{(p)}(f\otimes g,s).\tag{17}$$

Proof: By the lemma 9:

$$\sum_{n=0}^{\infty}a_{p^n}b_{p^n}p^{-ns}=\frac{Q(p^{-s})}{1-\alpha_p\beta_pp^{-s}-\alpha_p^{'}\beta_pp^{-2s}-\alpha_p\beta_p^{'}p^{-s}-\alpha_p^{'}\beta_p^{'}p^{-s}}$$

for $Q(x)$ a polynomial of degree $\leqslant 3$. We have:

$$\begin{aligned}
1 &- a_p\beta_p p^{-s} - a_p^{'}\beta_p p^{-2s} - \alpha_p\beta_p^{'} p^{-s} - \alpha_p^{'}\beta_p^{'} p^{-s} \\
&= (1-\alpha_p\beta_p p^{-s} + \beta_p^2\chi_f(p)p^{k-1-2s})(1-\alpha_p\beta_p^{'}p^{-s} + \beta_p^{'2}\chi_f(p)p^{k-1-2s}) \\
&= 1 - a_pb_pp^{-s} + [\beta_p^{'2}\chi_f(p)p^{k-1} + a_p^2\chi_g(p)p^{\ell-1} + \beta_p^2\chi_f(p)p^{k-1}]p^{-2s} \\
&\quad -[a_pb_p\chi(p)p^{k+\ell-2}]p^{-3s} + \chi(p)p^{2(k+\ell-2)}p^{-4s}. \qquad (18)
\end{aligned}$$

Using $\beta_p^2 + \beta_p^{'2} = (\beta_p + \beta_p^{'})^2 - 2\beta_p\beta_p^{'} = b_p^2 - 2\chi_g(p)p^{\ell-1}$ we get:

$$\begin{aligned}
(18) \quad &= \quad 1 - a_pb_pp^{-s} + [b_p^2\chi_f(p)p^{k-1} + a_p^2\chi_g(p)p^{\ell-1} - 2\chi(p)p^{k+\ell-2}]p^{-2s} \qquad (19) \\
&\quad -[a_pb_p\chi(p)p^{k+\ell-2}]p^{-3s} + \chi(p)p^{2(k+\ell-2)}p^{-4s}. \qquad (20)
\end{aligned}$$

One can show:

$$\begin{aligned}
Q(p^{-s}) \quad &= \quad (1 + a_pb_pp^{-s} + a_{p^2}b_{p^2}p^{-2s} + ...) \\
&\quad \times(1 - a_pb_pp^{-s} + [\beta_p^{'2}\chi_f(p)p^{k-1} + a_p^2\chi_g(p)p^{\ell-1} + \beta_p^2\chi(p)p^{k-1}]p^{-2s} \\
&\quad -[a_pb_p\chi(p)p^{k+\ell-2}]p^{-3s} + \chi(p)p^{2(k+\ell-2)}p^{-4s}) \\
&= \quad 1 - \chi(p)p^{k+\ell-2}p^{-2s}.
\end{aligned}$$

Hence the corollary holds. ■

Theorem 11. *Let $f \in \mathcal{S}_k(\Gamma_0(N), \chi_f)$ and $g \in \mathcal{S}_\ell(\Gamma_0(N), \chi_g)$. Then*

$$L(f \otimes g, s) = L(\chi, 2s - k - \ell + 2)\mathcal{D}(f, g, s). \qquad (21)$$

In particular, if $f \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ and $g \in \mathcal{S}_\ell(SL_2(\mathbb{Z}))$ ($N = 1$ and χ_f, χ_g are trivial characters) we have:

$$L(f \otimes g, s) = \zeta(2s - k - \ell + 2)\mathcal{D}(f, g, s). \qquad (22)$$

Proof: Using (17) , (15), (16), we get (21). For the case $N = 1$, as $L(\mathbf{1}, 2s-k-\ell+2) = \zeta(2s - k - \ell + 2)$, we get (22). ■

We study the analytic properties of $\mathcal{D}(f, g, s)$ for $N = 1$, then we can get a formula for $\mathcal{D}(f, g, k - 1)$ when $k > \ell + 2$.

As for the case of weight 1, we can define an Eisenstein series of weight k and level 1:

$$\widetilde{E}_k(z) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}^{'} \frac{1}{|mz+n|^k}$$

and

$$\widetilde{E}_k^{'}(z) := \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m,n)=1}} \frac{1}{(mz+n)^k}.$$

Similarly, we define the non-holomorphic Eisenstein series of weight k:

$$\widetilde{E}_k(z, s) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}^{'} \frac{1}{(mz+n)^k}\frac{y^s}{|mz+n|^{2s}}.$$

- $\widetilde{E}_k(z,s)$ is convergent for $\mathcal{R}(s) \gg 0$ and for any k. It is holomorphic as a function of s.
- $\widetilde{E}_k(\frac{az+b}{cz+d}, s) = (cz+d)^k \widetilde{E}_k(z,s)$ so it behaves like a modular form as a function of z however it is not holomorphic as a function of z.

Define analogously:

$$\widetilde{E}'_k(z,s) := \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m,n)=1}} \frac{1}{(mz+n)^k} \frac{y^s}{(mz+n)^{2s}}.$$

Then

$$\widetilde{E}_k(z,s) = \zeta(2z)\widetilde{E}'_k(z,s).$$

We need the following preliminary lemma.

Lemma 12. *Put* $(\mathbb{Z}\times\mathbb{N})' := \{(a,b)\in\mathbb{Z}\times\mathbb{N} |\ gcd(a,b)=1\}$. *Define* $\Gamma_\infty := \left\{\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} \mid n\in\mathbb{Z}\right\}$ *which is a subgroup of* $SL_2(\mathbb{Z})$. *Then the map*

$$\begin{array}{rcl} \Gamma_\infty\backslash SL_2(\mathbb{Z}) & \to & (\mathbb{Z}\times\mathbb{N})' \\ \begin{pmatrix}a & b\\ c & d\end{pmatrix} & \mapsto & (c,d) \end{array}$$

is a bijection.

Proof: The map obviously surjects. Moreover, it is left unchanged by multiplying by any matrix in Γ_∞, so Γ_∞ is inside the kernel. We can easily see that Γ_∞ is all the kernel hence we have the injectivity. ■

Proposition 13. *Let* $f\in\mathcal{S}_k(SL_2(\mathbb{Z}))$ *and* $g\in\mathcal{S}_\ell(SL_2(\mathbb{Z}))$. *For* $\mathcal{R}(s) > 2-\frac{k-\ell}{2}$ *we have:*

$$\left\langle \widetilde{E}'_{k-\ell}(z,s)g(z), f^*(z)\right\rangle_k = \frac{2\Gamma(k+s-1)}{(4\pi)^{k+s-1}}\mathcal{D}(f,g,k+s-1) \tag{23}$$

where $f^*\in\mathcal{S}_k(SL_2(\mathbb{Z}))$ *is the modular form satisfying* $a_n(f^*) = \overline{a_n}$. *In particular, if* $s=0$, *then*

$$\left\langle \widetilde{E}'_{k-\ell}(z)g(z), f^*(z)\right\rangle_k = \frac{2\Gamma(k-1)}{(4\pi)^{k-1}}\mathcal{D}(f,g,k-1). \tag{24}$$

Proof: We can easily see that:

$$\begin{aligned} y(\gamma z)^{k+s}g(\gamma z)f(\gamma(-\bar{z})) &= \frac{y^{k+s}}{|mz+n|^{2(k+s)}}(mz+n)^\ell g(z)\overline{(mz+n)^k}f(-\bar{z}) \\ &= \frac{y^{k+s}}{(mz+n)^{k-\ell}|mz+n|^{2s}}g(z)f(-\bar{z}) \end{aligned}$$

for any $\gamma = \begin{pmatrix} * & * \\ m & n \end{pmatrix}$. Remark that $f^*(z) = \overline{f(-\bar{z})}$. Then we compute:

$$\begin{aligned}
\left\langle \widetilde{E}'_{k-\ell}(z,s)g(z), f^*(z) \right\rangle_k &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} y^k \widetilde{E}'_{k-\ell}(z,s)g(z)f(-\bar{z})\frac{dx\ dy}{y^2} \\
&= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} y^k \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m,n)=1}} \frac{y^s}{(mz+n)} . \frac{g(z)f(-\bar{z})}{|mz+n|^{2s}} \frac{dx\ dy}{y^2} \\
&= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{N} \\ gcd(m,n)=1}} \frac{y^{k+s}}{(mz+n)} . \frac{g(z)f(-\bar{z})}{|mz+n|^{2s}} \frac{dx\ dy}{y^2} \\
&= 2\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} y(\gamma z)^{k+s} g(\gamma z)f(-\gamma\bar{z})\frac{dx(\gamma z)\ dy(\gamma z)}{y^2(\gamma z)} \\
&= 2\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \int_{\gamma(\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H})} y(z)^{k+s} g(z)f(-\bar{z})\frac{dx\ dy}{y^2}. \qquad (25)
\end{aligned}$$

The different translates $\gamma(\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H})$ of the original fundamental domain $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ are disjoint and they fit together exactly to form a fundamental domain for the action of Γ_∞ on $\mathcal{H}$. This is called Rankin's unfolding trick. Hence:

$$\begin{aligned}
(25) &= 2\int_{\Gamma_\infty\backslash\mathcal{H}} y^{k+s} g(z)f(-\bar{z})\frac{dx\ dy}{y^2} \\
&= 2\int_{y=0}^{\infty}\int_{x=0}^{1} y^{k+s}\left(\sum_{n\geqslant 1} b_n e^{2\pi i n z}\right)\left(\sum_{m\geqslant 1} a_m e^{-2\pi i m\bar{z}}\right)\frac{dxdy}{y^2} \\
&= 2\int_{y=0}^{\infty}\int_{x=0}^{1} y^{k+s} \sum_{n,m\geqslant 1} b_n a_m e^{2\pi i n(x+iy)} e^{-2\pi i m(x-iy)} \frac{xdy}{y^2} \\
&= 2\int_{y=0}^{\infty} y^{k+s} \sum_{n,m\geqslant 1} b_n a_m e^{-2\pi(n+m)y} \left(\int_{x=0}^{1} e^{2\pi i(n-m)x}dx\right)\frac{dy}{y^2}.
\end{aligned}$$

The integral in the parenthesis is equal to the Kronecker delta $\delta_{(n,m)}$. So the last line is equal to:

$$\begin{aligned}
&= 2\int_{y=0}^{\infty} y^{k+s}\sum_{n\geqslant 1} a_n b_n e^{-2\pi(n+n)y}\frac{dy}{y^2} \\
&= 2\sum_{n\geqslant 1} a_n b_n \int_{y=0}^{\infty} y^{k-1+s} e^{-4\pi n y}\frac{dy}{y} \\
&= 2\left(\sum_{n\geqslant 1}\frac{a_n b_n}{(4\pi n)^{k-1+s}}\right)\int_0^{\infty} u^{k-1+s}e^{-u}\frac{du}{u} \\
&= \frac{2\Gamma(k-1+s)}{(4\pi)^{k-1+s}}\mathcal{D}(f,g,k-1+s)
\end{aligned}$$

where we have made the change of variable $u = 4\pi n y$ for each integral in the sum. ■

The formula (23) makes sense even when $k \not> \ell + 2$. In particular, it makes sense when $k = \ell$. Consider the following Eisenstein series

$$\begin{aligned} \widetilde{E}(z,s) &= \widetilde{E}_0(z,s) \\ &= \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}' \frac{y^s}{|mz+n|^{2s}} \end{aligned}$$

which converges for $\mathcal{R}(s) > 1$. Define

$$\begin{aligned} \widetilde{E}'(z,s) &:= \widetilde{E}_0'(z,s) \\ &= \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m,n)=1}} \frac{y^s}{|mz+n|^{2s}}. \end{aligned}$$

Then

$$\widetilde{E}(z,s) = \zeta(2s)\widetilde{E}'(z,s).$$

So $\widetilde{E}(z,s)$ is a nonholomorphic Eisenstein series of weight 0. As a function of s, with z fixed, we write:

$$\widetilde{E}'(z,s) = \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(m,n)=1}} \frac{1}{Q_z^s(m,n)}$$

where $Q_z^s(m,n) = \dfrac{|mz+n|^2}{y}$ is a quadratic form in two variables with $\mathrm{disc}(Q_z) = -4$.

$\widetilde{E}'(z,s)$ is a non-holomorphic Eisenstein series of weight zero attached to Q_z when considered as a function of s.

Lemma 14. *Let $f, g \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ be two modular forms of the same weight. Then:*

$$\left\langle \widetilde{E}(z,s)g(z), f(z) \right\rangle_k = \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}} L(f\otimes g, s+k-1).$$

Proof: We have:

$$\begin{aligned} \left\langle \widetilde{E}_k(z,s)g(z), f(z) \right\rangle_k &= \zeta(2s) \left\langle \widetilde{E}_k'(z,s)g(z), f(z) \right\rangle_k \\ &= \zeta(2s)\frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \mathcal{D}(f,g,s+k-1) \\ &= \zeta(2s)\frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}} L(f\otimes g, s+k-1)\zeta(2(s+k-1)+2-2k)^{-1} \\ &= \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}} L(f\otimes g, s+k-1). \quad \blacksquare \end{aligned}$$

Theorem 15. *Let $z \in \mathcal{H}$ be fixed. Then*
1) The function $\widetilde{E}(z,s)$ has a meromorphic continuation to $s \in \mathbb{C}$ and is entire except for a simple pole with residue π at $s = 1$.

2) The function $G(z,s) := \dfrac{\Gamma(s)}{\pi^s}\widetilde{E}(z,s)$ *is holomorphic except for simple poles at* $s=1$ *and* $s=0$ *with residue 1 and -1 repectively. Moreover*

$$G(z,s) = G(z,1-s).$$

Proof: Consider

$$\theta_z(t) = \sum_{(m,n)\in\mathbb{Z}^2} e^{-\pi Q_z(m,n)t}.$$

We compute its Mellin transform and get:

$$\begin{aligned} G(z,s) &= \Gamma(s) \sideset{}{'}\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} [\pi Q_z(m,n)]^{-s} \\ &= \int_0^\infty (\theta_z(t)-1)t^s \frac{dt}{t}. \end{aligned}$$

The Poisson summation formula implies that

$$\theta_z(\tfrac{1}{t}) = t\theta_z(t).$$

Then we can write for $\mathcal{R}(s) > 1$:

$$\begin{aligned} G(z,1-s) &= \int_0^\infty (\theta_z(t)-1)t^{1-s}\frac{dt}{t} \\ &= \int_0^\infty (\frac{1}{t}\theta_z(\frac{1}{t})-1)t^{1-s}\frac{dt}{t} \\ &= \int_0^\infty \left(\frac{1}{t}(\theta_z(\frac{1}{t})-1)-1+\frac{1}{t}\right)t^{1-s}\frac{dt}{t} \\ &= \int_0^\infty (t(\theta_z(t)-1)-1+t)\, t^{s-1}\frac{dt}{t} \\ &= \int_0^\infty (\theta_z(t)-1)t^s\frac{dt}{t} + \int_0^\infty (-1+t)\, t^{s-1}\frac{dt}{t} \\ &= G(z,s) + \frac{1}{1-s} + \frac{1}{s}. \end{aligned}$$

(in the fourth line, we make the change of variable $s = \frac{1}{u}$) Hence $G(z,s)$ is invariant under the change $s \to 1-s$. It is entire except for simple poles at $s=1$ and $s=0$ with residue 1 and -1, respectively.

We compute the residue of $\widetilde{E}(z,s)$ at $s=1$:

$$\begin{aligned} \mathrm{Res}_{s=1}\widetilde{E}(z,s) &= \mathrm{Res}_{s=1}\frac{\pi^s}{\Gamma(s)}G(z,s) \\ &= \frac{\pi}{\Gamma(1)}\mathrm{Res}_{s=1}G(z,s) \\ &= \pi. \end{aligned}$$

Since the Gamma function $\Gamma(s)$ has a simple pole at $s=0$, then $\widetilde{E}(z,s)$ it holomorphic at this point. ■

In the lemma 14, we obtained an integral representation for $L(f \otimes g, s+k-1)$. Now define

$$\begin{aligned} \Lambda(f \otimes g, s) &:= \langle G(z, s-k+1)g, f\rangle_k \\ &= \frac{2\Gamma(s-k+1)\Gamma(s)}{4^s \pi^{s-k+1}} L(f \otimes g, s). \end{aligned} \tag{26}$$

This function has some nice properties.

Proposition 16. *Let $f, g \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ be two modular forms of the same weight. The function $\Lambda(f \otimes g, s)$ extends to a meromorphic function of s. It is holomorphic except at $s = k-1$ and $s = k$ where it has simple poles with residues $-\langle g, f\rangle$ and $\langle g, f\rangle$ respectively.*

Proof: We have seen that $G(z, s)$ is an entire function except at $s = 0$ and $s = 1$ where it has simple poles with residue 1 and -1 respectively. So $\Lambda(f \otimes g, s)$ extends to a meromorphic function with two poles at points $s = k-1$ and $s = k$. We compute the residue of $\Lambda(f \otimes g, s)$ at $s = 0$:

$$\begin{aligned} \mathrm{Res}_{s=k-1}\Lambda(f \otimes g, s) &= \mathrm{Res}_{s=k-1} \langle G(z, s-k+1)g, f\rangle_k \\ &= \langle \mathrm{Res}_{s=0} G(z, s)g, f\rangle_k \\ &= -\langle g, f\rangle . \end{aligned}$$

Similarly, $\mathrm{Res}_{s=k}\Lambda(f \otimes g, s) = \langle g, f\rangle$. ■

Corollary 2. *Let $f, g \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ be two modular forms of the same weight. $L(f \otimes g, s)$ extends to a meromorphic function of $s \in \mathbb{C}$. It has a simple pole at $s = k$ if and only if $\langle f, g\rangle \neq 0$.*

Proof: Write $L(f \otimes g, s) = \dfrac{4^s \pi^{2(s-k+1)}}{2\Gamma(s)\Gamma(s-k+1)} \Lambda(f \otimes g, s)$. The function $\Gamma(s)$ has simple pole at all points $s = 0, -1, -2, ...$ with residue $\mathrm{Res}_{s=n}\Gamma(s) = \frac{(-1)^n}{n}$ for $n = 0, -1, -2, ...$. So $\dfrac{1}{\Gamma(s)}$ has zeros at points $s = 0, -1, -2, ...$. Hence $L(f \otimes g, s)$ cannot have a pole at $s = k-1$ and it has a pole at $s = k$ if and only if $\langle f, g\rangle = 0$. ■

We can also find similar formulas for modular forms of level $N > 1$. Let $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}^*$ be a character modulo N. Put:

$$\begin{aligned} \widetilde{E}(z, s; \chi; N) &:= \widetilde{E}_0(z, s; \chi; N) \\ &= \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}{}' \frac{\chi(n) y^s}{(Nmz + n)^{2s}} \end{aligned} \tag{27}$$

and

$$\begin{aligned} \widetilde{E}'(z, s; \chi; N) &:= \widetilde{E}'_0(z, s; \chi; N) \\ &= \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z} \\ gcd(Nm,n)=1}} \frac{\chi(n) y^s}{(Nmz + n)^{2s}}. \end{aligned} \tag{28}$$

It follows that:

$$\widetilde{E}(z,s;\chi;N) = L(\chi,2s)\widetilde{E}'(z,s;\chi;N). \tag{29}$$

As before, set:

$$\begin{aligned}\widetilde{E}(z;\chi;N) &:= \widetilde{E}(z,0;\chi;N) \quad (30)\\ \widetilde{E}'(z;\chi;N) &:= \widetilde{E}'(z,0;\chi;N). \quad (31)\end{aligned}$$

Proposition 17. *Let $f \in \mathcal{S}_k(\Gamma_0(N),\chi_f)$ and $g \in \mathcal{S}_\ell(\Gamma_0(N),\chi_g)$. For $\mathcal{R}(s) > 2 - \frac{k-\ell}{2}$ we have:*

$$\left\langle \widetilde{E}'_{k-\ell}(z,s;\chi^{-1};N)g(z), f^*(z)\right\rangle_{k,N} = \frac{2\Gamma(k+s-1)}{(4\pi)^{k+s-1}}\mathcal{D}(f,g,k+s-1) \tag{32}$$

where $\chi = \chi_f\chi_g$ and $f^* \in \mathcal{S}_k(\Gamma_0(N),\overline{\chi_f})$ is the modular form satisfying $a_n(f^*) = \overline{a_n}$. In particular, if $s=0$, then

$$\left\langle \widetilde{E}'_{k-\ell}(z;\chi^{-1};N)g(z), f^*(z)\right\rangle_{k,N} = \frac{2\Gamma(k-1)}{(4\pi)^{k-1}}\mathcal{D}(f,g,k-1). \tag{33}$$

Proof: The proof is similar to the proposition 13. ■

Lemma 18. *Let $f \in \mathcal{S}_k(\Gamma_0(N),\chi_f)$ and $g \in \mathcal{S}_\ell(\Gamma_0(N),\chi_g)$ be two modular forms. Then:*

$$\left\langle \widetilde{E}_{k-\ell}(z,s;\chi^{-1};N)g(z), f^*(z)\right\rangle_{k,N} = \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}}\frac{L(\chi^{-1},2s+k-\ell)}{L(\chi,2s+k-\ell)}L(f\otimes g,s+k-1) \tag{34}$$

where $\chi = \chi_f\chi_g$. In particular, if $\chi_f = \chi_g^{-1}$, then

$$\left\langle \widetilde{E}_{k-\ell}(z,s;\mathbf{1};N)g(z), f^*(z)\right\rangle_{k,N} = \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}}L(f\otimes g,s+k-1). \tag{35}$$

Proof: We have:

$$\begin{aligned}\left\langle \widetilde{E}_{k-\ell}(z,s;\chi^{-1};N)g(z), f^*(z)\right\rangle_{k,N} &= L(\chi^{-1},2s+k-\ell)\left\langle \widetilde{E}'_{k-\ell}(z,s;\chi^{-1};N)g(z), f^*(z)\right\rangle_{k,N}\\ &= L(\chi^{-1},2s+k-\ell)\frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}}\mathcal{D}(f,g,s+k-1)\\ &= L(\chi^{-1},2s+k-\ell)\frac{2\Gamma(s+k-1)L(f\otimes g,s+k-1)}{(4\pi)^{s+k-1}L(\chi,2(s+k-1)+2-k-\ell)}\\ &= \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}}\frac{L(\chi^{-1},2s+k-\ell)}{L(\chi,2s+k-\ell)}L(f\otimes g,s+k-1).\end{aligned}$$

In particular, if $\chi_f = \chi_g^{-1}$ or equivalently if $\chi = \mathbf{1}$, then $L(\chi^{-1},2s+k-\ell) = L(\chi,2s+k-\ell) = \zeta_N(2s+k-\ell)$, so (35) holds. ■

For two arbitrary modular forms f and g, one has the following more general result:

Theorem 19. *Let $f \in \mathcal{S}_k(\Gamma_0(N_f), \chi_f)$ and $g \in \mathcal{S}_\ell(\Gamma_0(N_g), \chi_g)$. Assume that χ_f and χ_f are primitive characters modulo N_f and N_g, respectively. Moreover, $\chi_f\chi_g^{-1}$ is primitive modulo $M = gcd(N_f, N_g)$. As before, set*

$$\Lambda(f \otimes g, s) = \frac{2\Gamma(s-k+1)\Gamma(s)}{4^s\pi^{s-k+1}} L(f \otimes g, s).$$

Then $\Lambda(f \otimes g, s)$ is an entire function of s unless $f = g^*$. If $f = g^*$, then $\Lambda(f \otimes f^*, s)$ has two simple poles at $s = k$ and $s = k-1$.

Proof: See [10]. ■

2.2 Some applications of the Rankin-Selberg method

In this section, we apply the Rankin-Selberg method to compute the norm of a modular form whose fourier coefficients are real:

$$||f|| = \sqrt{\langle f, f\rangle}.$$

As a second application of the Rankin-Selberg method, we state and prove an estimate for $\sum_{p \nmid N} |a_p|^2 p^{-s}$.

We need the following lemma:

Lemma 20. ***(Landau)*** *Let $f(s) = \sum_{n=0}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series with real coefficients $a_n \geqslant 0$. Suppose that the series defining f(s) converges for $\mathcal{R}(s) > \sigma_0$. Suppose further that the function f extends to a function holomorphic in a neighborhood of $s = \sigma_0$. Then, in fact, the series defining $f(s)$ converges for $\mathcal{R}(s) > \sigma_0 - \varepsilon$ for some $\varepsilon > 0$.*

Proof: See [15]. ■

Proposition 21. *Let $f = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ be a normalised Hecke eigenform. Then:*

(a)

$$\sum_{n=0}^{\infty} a_n^2 n^{-s} = \zeta(s-k+1) \sum_{n=0}^{\infty} a_{n^2} n^{-s}. \tag{36}$$

(b)

$$\langle f, f\rangle_k = \frac{\pi(k-1)!}{3(4\pi)^k} \sum_{n=0}^{\infty} \frac{a_{n^2}}{n^k}. \tag{37}$$

Proof: Being f a normalised Hecke eigenform, its fourier coefficients satisfy:

1. $a_1 = 1$,

2. $a_{p^n} = a_p a_{p^{n-1}} - p^{k-1} a_{p^{n-2}}$ for all p prime and $n \geqslant 2$ (38)

3. $a_{mn} = a_m a_n$ when $\gcd(m, n) = 1$.

We compute the product expansion of two series $\sum_{n=0}^{\infty} a_n^2 n^{-s}$ and $\sum_{n=0}^{\infty} a_{n^2} n^{-s}$.
Define $b_n := a_n^2$. Then from (38), we have:

$$\begin{aligned}
b_{p^n} &= (a_p a_{p^{n-1}} - p^{k-1} a_{p^{n-2}})^2 \\
&= a_p^2 b_{p^{n-1}} + p^{2(k-1)} b_{n-2} - 2a_p p^{k-1} a_{p^{n-1}} a_{p^{n-2}} \\
&= a_p^2 b_{p^{n-1}} + p^{2(k-1)} b_{p^{n-2}} - 2a_p p^{k-1} (a_p a_{p^{n-2}} - p^{k-1} a_{p^{n-3}}) a_{p^{n-2}} \\
&= a_p^2 b_{p^{n-1}} + p^{k-1}(p^{k-1} - 2a_p^2) b_{p^{n-2}} + 2a_p p^{2(k-1)} a_{p^{n-2}} a_{p^{n-3}}.
\end{aligned}$$

Now, we can replace n by $n-1$ in the equation (38) and then replace $a_{p^{n-2}} a_{p^{n-3}}$ by its equivalent in the above equation. So we get

$$b_{p^n} = (a_p^2 - p^{k-1}) b_{p^{n-1}} + p^{k-1}(p^{k-1} - a_p^2) b_{p^{n-2}} + p^{3(k-1)} b_{p^{n-3}}.$$

Using the lemma 9, we have the following product expansion:

$$\sum_{i=0}^{\infty} b_{p^i} p^{-is} = \frac{1 + p^{k-1} p^{-s}}{1 - (a_p^2 - p^{k-1}) p^{-s} - p^{k-1}(p^{k-1} - a_p^2) p^{-2s} - p^{3(k-1)} p^{-3s}}.$$

The series $\sum_{n=0}^{\infty} a_n^2 n^{-s}$ is absolutely convergent for $\mathcal{R}(s) > k$. On the other hand, $b_1 = 1$ and $b_{mn} = b_m b_n$ when $\gcd(m,n) = 1$. So we can rearrange the series and write:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n^2 n^{-s} &= \prod_p \sum_{i=0}^{\infty} b_{p^i} p^{-is} \\
&= \prod_p \frac{1 + p^{k-1} p^{-s}}{1 - (a_p^2 - p^{k-1}) p^{-s} - p^{k-1}(p^{k-1} - a_p^2) p^{-2s} - p^{3(k-1)} p^{-3s}} \quad , \mathcal{R}(s) > k.
\end{aligned}$$

Define $c_n = a_{n^2}$. For a prime number p, we have:

$$\begin{aligned}
c_{p^n} &= a_p a_{p^{2n-1}} - p^{k-1} a_{p^{2n-2}} \\
&= a_p (a_p a_{p^{2n-2}} - p^{k-1} a_{p^{2n-3}}) - p^{k-1} a_{p^{2n-2}} \\
&= (a_p^2 - p^{k-1}) c_{p^{n-1}} - p^{k-1} a_p a_{p^{2n-3}} \\
&= (a_p^2 - p^{k-1}) c_{p^{n-1}} - p^{k-1} (a_{p^{2n-2}} + p^{k-1} a_{p^{2n-4}}) \\
&= (a_p^2 - 2p^{k-1}) c_{p^{n-1}} - p^{2(k-1)} c_{p^{n-2}}.
\end{aligned}$$

Clearly $c_1 = 1$ and $c_{mn} = c_m c_n$ when $\gcd(m,n) = 1$. The series $\sum_{n=0}^{\infty} a_{n^2} n^{-s}$ is absolutely convergent when $\mathcal{R}(s) > k$. By the same argument as for b_n, we can write:

$$\sum_{n=0}^{\infty} a_{n^2} n^{-s} = \prod_p \frac{1 + p^{k-1} p^{-s}}{1 - (a_p^2 - 2p^{k-1}) p^{-s} + p^{2(k-1)} p^{-2s}}.$$

We can easily see that:

$$1 - (a_p^2 - p^{k-1}) p^{-s} - p^{k-1}(p^{k-1} - a_p^2) p^{-2s} - p^{3(k-1)} p^{-3s} = (1 - p^{k-1} p^{-s})(1 - (a_p^2 - 2p^{k-1}) p^{-s} + p^{2(k-1)} p^{-2s}).$$

Hence the part (a) is proved. For the second part, consider

$$\begin{aligned}\left\langle \widetilde{E}(z,s)f(z), f^*(z)\right\rangle_k &= \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}L(f\otimes f, s+k-1)\\ &= \frac{\Gamma(s+k-1)\zeta(2s)}{(4\pi)^{s+k-1}}D(f,f,s+k-1).\end{aligned}$$

Since f is a Hecke eigenform, its fourier coefficients a_n are real. Thus $f^*(z) = f(z)$ and

$$D(f,f,s+k-1) = \sum_{n=0}^{\infty}\frac{a_n^2}{n^{s+k-1}} = \zeta(s)\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^{s+k-1}}.$$

The series $f(s) := \sum_{n=0}^{\infty}\frac{a_{n^2}}{n^{s+k-1}}$ is absolutely convergent for $\mathcal{R}(s) > 1$. Since $\zeta(s)$ and $D(f,f,s+k-1)$ are meromorphic functions over $\mathbb{C}$ and have simple pole at $s=1$, by Landau's lemma, then $f(s)$ is holomorphic in a neighborhood of $s=1$ and the series $f(s) = \sum_{n=0}^{\infty}\frac{a_{n^2}}{n^{s+k-1}}$ is convergent at $s=1$.

We compute:

$$\begin{aligned}\mathrm{Res}_{s=1}\left\langle G(z,s)f(z), f(z)\right\rangle_k &= \mathrm{Res}_{s=1}\left\langle \frac{\Gamma(s)}{\pi^s}\widetilde{E}(z,s)f(z), f(z)\right\rangle_k\\ &= \mathrm{Res}_{s=1}\frac{2\Gamma(s)\Gamma(s+k-1)\zeta(2s)}{\pi^s(4\pi)^{s+k-1}}D(f,f,s+k-1)\\ &= \mathrm{Res}_{s=1}\frac{2\Gamma(s)\Gamma(s+k-1)\zeta(2s)}{\pi^s(4\pi)^{s+k-1}}\zeta(s)\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^{s+k-1}}\\ &= \frac{2\Gamma(1)\Gamma(k)\zeta(2)}{\pi(4\pi)^k}\left(\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^k}\right)\mathrm{Res}_{s=1}\zeta(s)\\ &= \frac{\pi(k-1)!}{3(4\pi)^k}\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^k}.\end{aligned}$$

On the other hand:

$$\begin{aligned}\mathrm{Res}_{s=1}\left\langle G(z,s)f(z), f(z)\right\rangle_k &= \left\langle \mathrm{Res}_{s=1}G(z,s)f(z), f(z)\right\rangle_k\\ &= \left\langle f(z), f(z)\right\rangle_k.\end{aligned}$$

Thus we have the required result. ■

We can also find similar formula for a modular form of level $N > 1$ for the case when the fourier coefficients of f are real.

Proposition 22. *Let $f = \sum_{n=0}^{\infty} a_n q^n \in \mathcal{S}_k(\Gamma_0(N))$ be a normalised Hecke eigenform with real fourier coefficients where $N = \prod_{i=1}^{r} p_i^{\alpha_i}$ (p_i's are distinct prime numbers and*

$\alpha_i \geqslant 1$). Then:
(a)

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{a_n^2}{n^s} &= L(\mathbf{1}, s-k+1)\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^s}\\ &= \zeta(s-k+1)\prod_{p|N}(1-\frac{1}{p})\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^s}.\end{aligned}$$

where $\mathbf{1}$ is the trivial character of mod N.
(b)

$$\langle f, f\rangle_{k,N} = \frac{N\pi(k-1)!}{3(4\pi)^k}\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^k}. \tag{39}$$

Proof: In the proposition (17), we put $g = f$ and so $\chi = \mathbf{1}$. We then have

$$\begin{aligned}\left\langle \widetilde{E}'(z,s;\mathbf{1};N)f(z), f^*(z)\right\rangle_{k,N} &= \left\langle \widetilde{E}'(z,s;\mathbf{1};N)f(z), f(z)\right\rangle_{k,N}\\ &= \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}}\mathcal{D}(f,f,s+k-1).\end{aligned}$$

Let $N = \prod_{i=1}^{v} p_i^{\alpha_i}$ where p_i's are distinct prime numbers and $\alpha_i \geqslant 1$ and set $P = \{p_1, p_2, ..., p_r\}$. Then:

$$\begin{aligned}\frac{1}{N^s}\widetilde{E}_1'(Nz,s) &= \frac{1}{N^s}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1}}\frac{(Ny)^s}{(Nmz+n)^{2s}}\\ &= \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1}}\frac{y^s}{(Nmz+n)^{2s}}\\ &= \sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(Nm,n)=1}}\frac{y^s}{(Nmz+n)^{2s}} + \sum_{p_i\in P}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1\\ p_i|n}}\frac{y^s}{(Nmz+n)^{2s}}\\ &\quad - \sum_{\substack{p_i,p_j\in P\\ p_i\neq p_j}}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1\\ p_ip_j|n}}\frac{y^s}{(Nmz+n)^{2s}} + \cdots - (-1)^v\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1\\ p_1p_2...p_v|n}}\frac{y^s}{(Nmz+n)^{2s}}\\ &= \widetilde{E}'(z,s;\mathbf{1};N) + \sum_{p_i\in P}\frac{1}{p_i^{2s}}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1}}\frac{y^s}{(\frac{N}{p_i}mz+n)^{2s}}\\ &\quad - \sum_{\substack{p_i,p_j\in P\\ p_i\neq p_j}}\frac{1}{(p_ip_j)^{2s}}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1}}\frac{y^s}{(\frac{N}{p_ip_j}mz+n)^{2s}} + \cdots\\ &\quad -(-1)^v\frac{1}{(p_1...p_v)^{2s}}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1}}\frac{y^s}{(\frac{N}{p_1...p_v}mz+n)^{2s}}.\end{aligned}$$

Since for any M:

$$\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1}} \frac{y^s}{(Mmz+n)^{2s}} = \frac{1}{M^s}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\ gcd(m,n)=1}} \frac{(My)^s}{(Mmz+n)^{2s}} = \frac{1}{M^s}E^{'}(Mz,s),$$

it follows

$$\begin{aligned}
\widetilde{E}^{'}(z,s;\mathbf{1};N) &= \frac{1}{N^s}\widetilde{E}^{'}_1(Nz,s) - \sum_{p_i\in P}\frac{1}{N^s p_i^s}E^{'}(\tfrac{N}{p_i}z,s)\\
&\quad + \sum_{\substack{p_i,p_j\in P\\ p_i\neq p_j}}\frac{1}{N^s(p_ip_j)^s}E^{'}(\tfrac{N}{p_ip_j}z,s)\\
&\quad - \cdots\\
&\quad + (-1)^v\frac{1}{N^s(p_1...p_v)^s}E^{'}(\tfrac{N}{p_1p_2...p_v}z,s).
\end{aligned}$$

One can compute the residue of $E^{'}(Mz,s)$ at $s=1$ for any M. In fact:

$$\mathrm{Res}_{s=1}E^{'}(Mz,s) = \mathrm{Res}_{s=1}\frac{E(Mz,s)}{\zeta(2s)} = \mathrm{Res}_{s=1}\frac{\pi^sG(Mz,s)}{\Gamma(s)\zeta(2s)} = \frac{\pi}{\Gamma(1)\zeta(2)}\mathrm{Res}_{s=1}G(Mz,s) = \frac{6}{\pi}.$$

Therefore

$$\begin{aligned}
\mathrm{Res}_{s=1}\widetilde{E}^{'}(z,s;\mathbf{1};N) &= \frac{6}{N\pi}\left(1 - \sum_{p_i\in P}\frac{1}{p_i} + \sum_{\substack{p_i,p_j\in P\\ p_i\neq p_j}}\frac{1}{p_ip_j} + \cdots + (-1)^v\frac{1}{p_1...p_v}\right)\\
&= \frac{6}{N\pi}\prod_{p|N}(1-\tfrac{1}{p}).
\end{aligned}$$

We compute then the residue of both sides of the formula (2.2) at $s=1$:

$$\mathrm{Res}_{s=1}\left\langle \widetilde{E}^{'}_{k-\ell}(z,s;\mathbf{1};N)f(z),f(z)\right\rangle_{k,N} = \frac{6}{N\pi}\left(\prod_{p|N}(1-\tfrac{1}{p})\right)\langle f(z),f(z)\rangle_{k,N}$$

and

$$\begin{aligned}
\mathrm{Res}_{s=1}\left(\frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}}\mathcal{D}(f,f,s+k-1)\right) &= \frac{2\Gamma(k)}{(4\pi)^k}\mathrm{Res}_{s=1}\mathcal{D}(f,f,s+k-1)\\
&= \frac{2\Gamma(k)}{(4\pi)^k}\prod_{p|N}(1-\frac{1}{p})\left(\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^k}\right)\mathrm{Res}_{s=1}\zeta(s)\\
&= \frac{2(k-1)!}{(4\pi)^k}\prod_{p|N}(1-\frac{1}{p})\left(\sum_{n=0}^{\infty}\frac{a_{n^2}}{n^k}\right).
\end{aligned}$$

So we have obtained the formula (39). ■

Remark 23. *The series* $\sum_{n=0}^{\infty} \frac{a_{n^2}}{n^k}$ *in the formulas (37) and (39) does not converge fast. However, there are faster methods to compute numerically the norm of a modular form. For example, let* $f \in \mathcal{S}_k(SL_2(\mathbb{Z}))$ *be a normalised Hecke eigenform. Then*

$$\langle f, f \rangle_k = \frac{2}{\pi} \frac{(k-1)!}{(4\pi)^k} \sum_{n \geqslant 1} \frac{A(n)}{n^k}$$

where $A(n) = \sum_{m|n} (-1)^{\Omega(m)} m^{k-1} (a_{n/m})^2$ *while* $\Omega(m)$ *is the number of prime divisors of* m *counted with multiplicity. For more details, see [4].*

As a second application of the Rankin-Selberg method, we give an estimate for $\sum_{p \nmid N} \frac{|a_p|^2}{p^s}$.

Theorem 24. *Let* $f \in \mathcal{S}_k(\Gamma_0(N), \chi)$. *Suppose* f *is an normalized eigenform for the* T_p *operator with* $p \nmid N$. *Then the series* $\sum_{p \nmid N} \frac{|a_p|^2}{p^s}$ *converges for all* $\mathcal{R}(s) > k$ *and we have:*

$$\sum_{p \nmid N} \frac{|a_p|^2}{p^s} \leqslant log \left(\frac{1}{s-k} \right) + O(1) \quad \text{as } s \to k^+. \tag{40}$$

To prove this theorem, we need two preliminary results:

Theorem 25.

$$\sum_p p^{-s} \leqslant log \left(\frac{1}{s-1} \right) + O(1) \quad \text{as } s \to 1^+. \tag{41}$$

Proof: Recall that the zeta function $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ has a simple pole at $s = 1$.

Taking logarithms of both sides and using the Taylor expansion for the logarithms, we obtain:

$$\begin{aligned}
\log\zeta(s) &= \sum_p -\log(1 - p^{-s}) \\
&= \sum_p \sum_{m=1}^{\infty} \frac{p^{-ms}}{m} \\
&= \sum_{m=1}^{\infty} \sum_p \frac{p^{-ms}}{m} \\
&= \sum_{m=1}^{\infty} g_m(s)
\end{aligned}$$

where $g_m(s) = \sum_p \frac{p^{-ms}}{m}$. Notice that $g_1(s) = \sum_p p^{-s}$. We know that the residue of $\zeta(s)$ at s=1 is equal to 1:

$$\mathrm{Res}_{s=1}\zeta(s) = 1$$

or equivalently

$$\lim_{s\to 1} (s-1)\zeta(s) = 1\ .$$

Taking logarithm gives:

$$\lim_{s\to 1^+} [\log(s-1) + \log\zeta(s)] = 0.$$

We claim that the series $\sum_{m=2}^{\infty} g_m(s)$ converges for $s = 1$. Indeed,

$$\begin{aligned}
\sum_{m=2}^{\infty} g_m(s) &= \sum_p \sum_{m=2}^{\infty} \frac{p^{-ms}}{m} \\
&= \sum_p \sum_{n=1}^{\infty} \left(\frac{p^{-2ns}}{2n} + \frac{p^{-(2n+1)s}}{2n+1} \right) \\
&\leqslant \sum_p \sum_{n=1}^{\infty} \left(\frac{p^{-2ns}}{2n} + \frac{p^{-2ns}}{2n} \right) \\
&\leqslant \sum_p \sum_{n=1}^{\infty} \frac{p^{-2ns}}{n} \\
&= \log\zeta(2s).
\end{aligned}$$

Then

$$\sum_p p^{-s} + \sum_{m=2}^{\infty} g_m(s) = \log\zeta(s) = \log\frac{1}{s-1}.$$

This completes the proof. ∎

Proof of the theorem [24]: Using Deligne's inequality, i.e. $|a_n| \leqslant Cn^{\frac{k-1}{2}}$. We see that the series $\sum_{n=1}^{\infty} \frac{|a_n|^2}{n^s}$ converges for $\mathcal{R}(s) > \frac{k+1}{2}$. Denote $\alpha_{p,1}$ and $\alpha_{p,2}$ for the roots of $x^2 - a_p x - \chi(p)p^{k-1}$. Consider

$$L_N(f \otimes f^*, s) = \prod_{p\nmid N} L_p(f \otimes f^*, s) \tag{42}$$

where

$$L_p(f \otimes f^*, s) = \prod_{i,j=1}^{2} (1 - \alpha_{p,i}\overline{\alpha_{p,j}}p^{-s})^{-1}.$$

Recall that $D_N(f, f^*, s) = \sum_{(n,N)=1} \frac{|a_n|^2}{n^s}$. Then

$$L_N(f \otimes f^*, s) = D_N(f, f^*, s)\zeta_N(2s + 2 - 2k)$$

where

$$\zeta_N(s) = \prod_{p\nmid N}(1-p^{-s})^{-1}.$$

Then

$$L_N(f\otimes f^*, s) = H(s)D(f, f^*, s)\zeta(2s+2-2k)$$

where $H(s) = \prod_{p\nmid N}\Big((1-p^{-2s+2k-2})(1-|a_p^2|p^{-s})\Big)$. We claim that $H(s)\neq 0$ in the half plane $\mathcal{R}(s)\geqslant k$:
For any $p|N$, one has $a_{p^m} = a_p^m$ for any $m\in\mathbb{N}$. By the Deligne's inequality, one has $|a_{p^m}| \leqslant C(p^m)^{\frac{k-1}{2}}$. Therefore $|a_p| \leqslant C^{\frac{1}{m}}p^{\frac{k-1}{2}}$. Taking limit $m\to\infty$, we get $|a_p| \leqslant p^{\frac{k-1}{2}} < p^k$. Then for $\mathcal{R}(s)\geqslant k$, $1-|a_p^2|p^{-s}\neq 0$, and so $H(s)\neq 0$ in the half plane $\mathcal{R}(s)\geqslant k$.

Since $\langle f, f^*\rangle \neq 0$, by the corollary [2], we see that $L(f\otimes f^*, s) = D(f, f^*, s)\zeta(2s+2-2k)$ extends to a meromorphic function with a unique simple pole at $s=k$. We conclude that $L_N(f\otimes f^*, s) = H(s)L(f\otimes f^*, s)$ is holomorphic for $\mathcal{R}(s)\geqslant k$ and has a unique simple pole at $s=k$.

We have $\lim_{s\to k}(s-k)L_N(f\otimes f^*, s) = O(1)$. Hence:

$$\lim_{s\to k^+}\log(s-k) + \lim_{s\to k^+}\log L_N(f\otimes f^*, s) = O(1).$$

Taking logarithm of both sides of the formula (42) gives:

$$\begin{aligned}
\log L_N(f\otimes f^*, s) &= -\sum_{p\nmid N}\left[\sum_{i,j=1}^{2}\log(1-\alpha_{p,i}\overline{\alpha_{p,j}}p^{-s})\right]\\
&= \sum_{p\nmid N}\sum_{m=1}^{\infty}\left(\frac{(\alpha_{p,1}\overline{\alpha_{p,1}})^m}{mp^{ms}} + \frac{(\alpha_{p,1}\overline{\alpha_{p,2}})^m}{mp^{ms}} + \frac{(\alpha_{p,2}\overline{\alpha_{p,1}})^m}{mp^{ms}} + \frac{(\alpha_{p,2}\overline{\alpha_{p,2}})^m}{mp^{ms}}\right)\\
&= \sum_{p\nmid N}\left(\sum_{m=1}^{\infty}\frac{(\alpha_{p,1}^m+\alpha_{p,2}^m)(\overline{\alpha_{p,1}}^m+\overline{\alpha_{p,2}}^m)}{mp^{ms}}\right)\\
&= \sum_{m=1}^{\infty}\left(\sum_{p\nmid N}\frac{(\alpha_{p,1}^m+\alpha_{p,2}^m)(\overline{\alpha_{p,1}}^m+\overline{\alpha_{p,2}}^m)}{mp^{ms}}\right)\\
&= \sum_{m=1}^{\infty} g_m(s)
\end{aligned}$$

where $g_m(s) = \sum_{p\nmid N}\frac{(\alpha_{p,1}^m+\alpha_{p,2}^m)(\overline{\alpha_{p,1}}^m+\overline{\alpha_{p,2}}^m)}{mp^{ms}}$. Remark that

$$g_1(s) = \sum_{p\nmid N}\frac{(\alpha_{p,1}+\alpha_{p,2})(\overline{\alpha_{p,1}}+\overline{\alpha_{p,2}})}{p^s} = \sum_{p\nmid N}\frac{|\alpha_{p,1}+\alpha_{p,2}|^2}{p^s} = \sum_{p\nmid N}\frac{|a_p|^2}{p^s}\ .$$

It follows that

$$g_1(s) \leqslant \sum_{m=1}^{\infty} g_m(s) = L_N(f \otimes f^*, s).$$

Taking limit $s \to k^+$ gets:

$$g_1(s) = \lim_{s \to k^+} \sum_{p \nmid N} \frac{|a_p|^2}{p^s} \leqslant \log\left(\frac{1}{s-k}\right).$$

This completes the proof. ■

The theorem above suggests us to give the following definition:

Definition 26. *Let $\mathcal{P}$ be the set of natural primes and let $X \subseteq \mathcal{P}$. Define the* ***superior density of*** *X to be*

$$dens.sup(X) = \limsup_{s \to 1^+} \frac{\sum_{p \in X} p^{-s}}{log \frac{1}{s-1}}.$$

Remark 27. *Let $X \subset \mathcal{P}$ be a subset of natural primes such that $dens.sup(X)$ exists. Since $\sum_{p \in \mathcal{P}} p^{-s} \leqslant log\left(\frac{1}{s-1}\right) + O(1)$ as $s \to 1^+$, we have $dens.sup(X) \in [0, 1]$.*

Remark 28. *Let $X \subset \mathcal{P}$ be a finite subset of primes. Then $dens.sup(X) = 0$. But the converse is not true. For example, assume $X = \{p_1, p_2, p_3, ...\}$ is any ordered subset of primes such that $2p_i < p_{i+1}$ for any $i \geqslant 1$. Then*

$$\sum_{i=1}^{\infty} p_i^{-s} < p_1^{-s} \sum_{i=1}^{\infty} 2^{-(i-1)s} = 2p_1^{-s}.$$

Therefore $dens.sup(X)$=0.

Remark 29. ***(Dirichlet's Theorem on Primes in Arithmetic Progressions)*** *Let m be a positive integer and a be an integer for which $gcd(m, a) = 1$. If X=$\{p \in \mathcal{P} : p \equiv a \bmod m\}$, then $dens.sup(X) = \frac{1}{\varphi(m)}$ where φ is the Euler totient function, i.e. $\varphi(m)$ is the number of integers k in the range $1 \leqslant k \leqslant n$ for which $gcd(n, k) = 1$.*

In particular, there are infinitely many primes p satisfying $p \equiv a \bmod m$.

Proposition 30. *Let $f \in \mathcal{S}_1(\Gamma_0(N), \chi)$ be a normalized newform. Then for each $\eta > 0$, there exists sets $X_\eta, Y_\eta \in \mathbb{C}$ such that:*

- *$|Y_\eta| < \infty$*
- *$dens.sup(X) \leqslant \eta$*
- *$a_p \in Y_\eta$ if $p \notin X_\eta$.*

We need the following result in order to prove the proposition above.

Proposition 31. *Let $f = \sum_{n \geqslant 1} a_n q^n \in \mathcal{M}_k(\Gamma_0(N), \chi)$ be a normalised newform. Then*

1. The field $K = \mathbb{Q}(a_n : n \in \mathbb{N})$ is a finite extension of $\mathbb{Q}$ and each a_n is an algebraic integer.

2. For any embedding $\sigma : K_f \hookrightarrow \mathbb{C}$, we have

$$\sigma(f) := \sum \sigma(a_n) q^n \in \mathcal{M}_k(\Gamma_0(N), \chi \circ \sigma).$$

Proof: See [16]. ■

Proof of the proposition 30: Let $K = \mathbb{Q}(a_1, a_2, ...)$ be the number field containing all fourier coefficients of f. Fix $c > 0$ and let

$$Y(c) := \left\{ \alpha \in \mathcal{O}_K : |\sigma(\alpha)|^2 < c \text{ for all } \sigma \in \mathrm{Hom}(K, \mathbb{C}) \right\}.$$

We claim that the set $Y(c)$ is finite: Let $x \in \mathcal{O}_K$ with minimal polynomial of degree m over $\mathbb{Z}$:

$$X^m + b_{m-1} X^{m-1} + ... + b_0 \quad , b_i \in \mathbb{Z}.$$

The j-th coefficient b_j can be given by

$$b_j = \sum_{\substack{i_1, ..., i_{m-j} \\ i_k \neq i_l \text{ for } k \neq l}} \sigma_{i_1}(a) ... \sigma_{i_{m-j}}(a)$$

and therefore one has:

$$|b_j| \leqslant \sum_{\substack{i_1, ..., i_{m-j} \\ i_k \neq i_l \text{ for } k \neq l}} |\sigma_{i_1}(a)| ... |\sigma_{i_{m-j}}(a)| \leqslant \binom{m}{m-j} \sqrt{c}.$$

Since the coefficients b_j are integers, this means that the minimal polynomials of the elements of $Y(c)$ are just a finite number. Therefore $Y(c)$ must be finite.
Now set

$$X(c) = \{ p \in \mathcal{P} : a_p \notin Y(c) \}.$$

By proposition (31), for each embedding $\sigma : K \to \mathbb{C}$, $\sigma(f) = \sum_{n=1}^{\infty} \sigma(a_n) q^n$ is a normalised newform which lies in $\mathcal{S}_k(\Gamma_0(N), \chi \circ \sigma)$. Applying the theorem 24 to $\sigma(f)$ gives:

$$\sum_{p \nmid N} \frac{|\sigma(a_p)|^2}{p^s} \leqslant \log\left(\frac{1}{s-1}\right) + O(1).$$

Doing summation over all embeddings $\sigma : K \to \mathbb{C}$ gives:

$$\sum_{\sigma \in \mathrm{Hom}(K, \mathbb{C})} \left(\sum_{p \nmid N} \frac{|\sigma(a_p)|^2}{p^s} \right) \leqslant [K : \mathbb{Q}] \log\left(\frac{1}{s-1}\right) + O(1).$$

For any $p \in X(c)$, there is an embedding σ such that $|\sigma(a_p)|^2 \geqslant c$, so $\sum\limits_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} |\sigma(a_p)|^2 \geqslant c$. It follows

$$\begin{aligned} c \sum_{p \in X(c)} p^{-s} &\leqslant \sum_{p \in X(c)} \left(\sum_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} |\sigma(a_p)|^2 \right) p^{-s} \\ &\leqslant [K:\mathbb{Q}] \log \left(\frac{1}{s-1} \right) + O(1) \end{aligned}$$

and so $\mathrm{dens.sup}(X(c)) \leqslant \dfrac{[K:\mathbb{Q}]}{c}$. It follows that if $\eta \geqslant \dfrac{[K:\mathbb{Q}]}{c}$, then $X_\eta = X(c)$ and $Y_\eta = Y(c)$ satisfy the necessary conditions of the proposition and we are done. ■

3 Artin representations attached to modular forms of weight 1

3.1 The Deligne-Serre theorem

There is a correspondence between the modular forms of weight 1 and certain representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{GL}_2(\mathbb{C})$. The existence of this correspondence is conjectured by Langlands and then constructed by Serre and Deligne.

Theorem 32. ***(Deligne-Serre)*** *Given a normalised eigenform* $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{S}_1(\Gamma_0(N), \chi)$ *with* χ *an odd character mod* N*, there exists a continuous Galois representation* $\rho_f : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$ *with the property that*

$$char(\rho_f(Frob_p)) = X^2 - a_p X + \chi(p) \quad \text{for all } p \nmid N \quad .$$

In addition, ρ_f *is irreducible if and only if* f *is a cusp form.*

Assuming the theorem, we can restrict the image of ρ_f by conjugating it.

Lemma 33. *Let* $K_f = \mathbb{Q}(a_1, a_2, ...)$ *be the number field generated by the Fourier coefficients of* f*. If the Deligne-Serre representation* ρ_f *exists, then it is realisable over* K_f*, i.e. one can conjugate it in such a way that it takes values on* $GL_2(K_f)$*.*

Proof: Let $C \in G_{\mathbb{Q}}$ be the complex conjugation. As the order of C is two, i.e $C \circ C = 1$, the eigenvalues of $\rho_f(C)$ are 1 or/and -1. Let φ_N be the mod N cyclotomic character. By the isomorphism $\mathrm{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q})$ with $(\mathbb{Z}/N\mathbb{Z})^*$, we can consider $\varphi_N : G_{\mathbb{Q}} \to (\mathbb{Z}/N\mathbb{Z})^* \cong \mathrm{Gal}(\mathbb{Q}(\xi_N)/\mathbb{Q})$ which takes $g \in G_{\mathbb{Q}}$ to the automorphism induced by g on the Nth cyclotomic extension $\mathbb{Q}(\xi_N)$ of $\mathbb{Q}$. We can compose two maps φ_N and χ:

$$G_{\mathbb{Q}} \xrightarrow{\varphi_N} (\mathbb{Z}/N\mathbb{Z})^* \xrightarrow{\chi} \mathbb{C}^*.$$

We abuse the language and use χ in place of $\varphi_N \circ \chi$ and call it *Galois character*. Consider $\det\rho_f : G_{\mathbb{Q}} \to \mathbb{C}^*$. By the Deligne-Serre theorem, we have $\det(\rho_f) = \chi$. Hence

$$\det(\rho_f)(C) = \chi(-1) = -1$$

since χ is an odd character (in fact, if we had $\chi(1) = 1$, then $f = 0$). So the two eigenvalues of $\det(\rho_f)$ are 1 and -1. So we may assume, by conjugating if necessary, that

$$\rho_f(C) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Write

$$\rho_f(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \quad \text{for } \sigma \in G_{\mathbb{Q}}$$

where $a, b, c, d : G_{\mathbb{Q}} \to \mathbb{C}$ such that $a(C) = 1,\ b(C) = c(C) = 0,\ d(C) = -1$.

We claim that $a(\sigma), d(\sigma) \in K_f$ for all $\sigma \in G_{\mathbb{Q}}$. In fact by the theorem of Deligne-Serre we have $\mathrm{Tr}(\sigma) = a(\sigma) + d(\sigma) \in K_f$ for all $\sigma \in G_{\mathbb{Q}}$. On the other hand

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a(\sigma) & -b(\sigma) \\ c(\sigma) & -d(\sigma) \end{pmatrix} \in G_{\mathbb{Q}}.$$

Thus $a(\sigma) - d(\sigma) \in K_f$ for all $\sigma \in G_{\mathbb{Q}}$. Hence $a(\sigma), d(\sigma) \in K_f$.
Consider

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix} = \begin{pmatrix} a(\sigma)a(\tau) + b(\sigma)c(\tau) & \cdot \\ \cdot & \cdot \end{pmatrix}.$$

This implies $a(\sigma)a(\tau) + b(\sigma)c(\tau) \in K_f$. Therefore $b(\sigma)c(\tau) \in K_f$ for all $\sigma, \tau \in G_{\mathbb{Q}}$. There are two possible cases for $c : G_{\mathbb{Q}} \to \mathbb{C}$.

Case 1: c is identically zero: In this case, ρ_f is reducible hence by semi-simplicity $\rho_f = \chi_1 \bigoplus \chi_2$ where χ_1, χ_2 are one-dimensional representations. We can write:

$$\rho_f = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}.$$

Hence the image of ρ_f is in $\mathrm{GL}_2(K_f)$ in this case.

Case 2: $c \neq 0$: There exists σ_0 such that $\rho_f(\sigma_0) = \begin{pmatrix} a(\sigma_0) & b(\sigma_0) \\ c(\sigma_0) & d(\sigma_0) \end{pmatrix}$ with $c(\sigma_0) \neq 0$. Let $\lambda \in \mathbb{C}$ be such that $\lambda^2 = c(\sigma_0)$. We can conjugate $\rho_f(\sigma_0)$ by A=$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and get $A\rho_f(\sigma_0)A^{-1} = \begin{pmatrix} a(\sigma_0) & b(\sigma_0)c(\sigma_0) \\ 1 & d(\sigma_0) \end{pmatrix}$. Since $b(\sigma)c(\tau) \in K_f$, for $\tau = \sigma_0$ we get $b(\sigma) \in K_f$ for any $\sigma \in G_{\mathbb{Q}}$.

Consider the function $b : G_{\mathbb{Q}} \to \mathbb{C}$. If b is identically zero, then ρ_f will be reducible and like before, we have the result in this case. So assume $b \neq 0$ therefore there exists σ' such that $b(\sigma') \neq 0$, then $b(\sigma') \in K_f^*$. As $b(\sigma)c(\tau) \in K_f$, this implies $c(\tau) \in K_f$ for all $\tau \in G_{\mathbb{Q}}$. ■

Proposition 34. *Let $\rho : G_{\mathbb{Q}} \to GL_d(K)$ be a Galois representation where K is a number field. Then ρ is similar to a Galois representation $\rho' : G_{\mathbb{Q}} \to GL_d(\mathcal{O}_K)$, i.e. it can be conjugated in such a way that it takes values on $GL_d(\mathcal{O}_K)$.*

Proof: For a proof, See [6] Proposition 9.3.5 . ■

As a consequence, if we have a Deligne-Serre representation ρ_f, we can conjugate it so that

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_{K_f}).$$

3.2 The proof of the Deligne-Serre theorem

The steps of the proof of Deligne-Serre's theorem are as follows. Our purpose is to construct a representation with $\mathrm{Tr}(\mathrm{Frob}_p) = a_p$ and $\det(\mathrm{Frob}_p) = \chi(p)$.

Step 1: Starting with a modular form f of weight 1, we may multiply it with a certain Eisenstein series E of weight $\geqslant 1$, whose q-expansion is congruent to 1 modulo ℓ. So we obtain a modular form $E.f$ of weight $\geqslant 2$ whose q-expansion is congruent to that of f modulo ℓ. The representations attached to eigenforms of weight $\geqslant 2$ are pretty well understood and arise from the ℓ-adic representations. They may be reduced modulo ℓ to obtain representations in $\mathrm{Gl}_2(\mathbb{F}_\ell)$. We construct a representation $\overline{\rho}_\lambda : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_\ell)$ attached to $E.f$ (where λ is a prime above ℓ) satisfying $\mathrm{Tr}(\mathrm{Frob}_p) = a_p \bmod \ell$ and $\det(\mathrm{Frob}_p) = \chi(p) \bmod \ell$. We can do this step for almost all primes ℓ.

Step 2: We use an analytic result of Rankin to show that the a_p's are finite in number if we exclude a set of primes p of small density. Then we can get a uniform bound (independent of ℓ on the image of $\overline{\rho}_\ell$.)

Step 3: We glue the $\overline{\rho}_\lambda$'s to obtain a preresentation ρ into $\mathrm{GL}_2(\mathcal{O}_L)$ for some ring of integers $\mathcal{O}_L$ which would reduce to $\overline{\rho}_\lambda$ for infinitely many ℓ. This is possible thanks to the bound on the image of $\overline{\rho}_\lambda$ obtained in the third step. The representation ρ has the desired properties.

3.2.1 Step 1: Construction ℓ-adic representations $\overline{\rho}_\lambda : G_{\mathbb{Q}} \to \mathbf{GL}_2(\mathbb{F}_\ell)$

Theorem 35. *Let $0 \neq f \in M_k(\Gamma_0(N), \chi)$ with $k \geqslant 2$. Suppose that f is a normalised eigenform for all T_p with $p \nmid N$. Let K be a number field which contains all the a_p and all the $\chi(p)$. Let λ be a finite place of K of residual characteristic ℓ and let K_λ be the completion of K with respect to it. Then there exists a semi- simple Galois representation*

$$\rho_\lambda : G_{\mathbb{Q}} \to GL_2(K_\lambda)$$

which is unramified at all primes that don't divide Nl and such that:

$$\begin{aligned} Tr(Frob_p) &= a_p \\ det(Frob_p) &= \chi(p)p^{k-1} \quad \text{if } p \nmid Nl. \end{aligned}$$

After the lemma below, such a representation is unique up to isomorphism.

Lemma 36. *Let ρ, $\rho^{'} : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$ be two Galois representations and X be a subset of rational prime numbers with density 1. Assume that for all $p \in X$, we have $char(\rho(Frob_p)) = char(\rho^{'}(Frob_p))$. Then $\rho = \rho^{'}$.*

Proof: See [7].■

Remark 37. *If f is an Eisenstein series, the attached representation to it is the direct sum of two 1-dimensional representations and is therefore reducible. We show how one can construct this representation. Let $N \in \mathbb{N}$ and let ψ and φ be primitive characters modulo u and v respectively such that $(\psi\varphi)(-1) = -1$ and $uv|N$. Set*

$$E_1^{\psi,\varphi}(z) := \delta(\varphi)L(\psi, 0) + \delta(\psi)L(\varphi, 0) + 2\sum_{n=1}^{\infty} \sigma_0^{\psi,\varphi}(n)q^n$$

where $q = e^{2\pi iz}$, $\delta(\varphi) = 1$ *iff* $\varphi = \mathbf{1}$ *and* 0 *otherwise, while* $\sigma_0^{\psi,\varphi} = \sum_{\substack{m|n \\ m>0}} \psi(\frac{n}{m})\varphi(m)$ *and* $L(\varphi, s)$ *(resp.* $L(\psi, s)$*) is the function associated to* φ *(resp.* ψ*). Set* $\chi = \psi\varphi$ *which is a character modulo* N*. Consider* ψ *and* φ *as characters of* $G_{\mathbb{Q}}$*. Then the representation*

$$\begin{array}{rcl} \rho: & G_{\mathbb{Q}} \to & GL_2(\mathbb{C}) \\ & \sigma \mapsto & \begin{pmatrix} \psi(\sigma) & 0 \\ 0 & \varphi(\sigma) \end{pmatrix} \end{array}$$

is reducible with the desired properties.(see [6] and [7])

In the theorem above, the weight of modular form f is assumed to be $\geqslant 2$, so for the case weight 1, we will need a different construction.

From here to the end of the theorem 39, assume that $K \subseteq \mathbb{C}$ is the number field containing all the coefficients of f, λ is a finite place of K, $\mathcal{O}_\lambda$ is its valuation ring and m_λ its maximal ideal. Furthermore, $k_\lambda = \mathcal{O}_\lambda / m_\lambda$ is the residue field and l its characteristic.

Definition 38. *Let* $K \subset \mathbb{C}$ *be a number field,* λ *a finite place of* K*,* $\mathcal{O}_\lambda$ *is the valuation ring and* m_λ *its maximal ideal. Furthermore,* $k_\lambda = \mathcal{O}_\lambda / m_\lambda$ *is the residue field and* ℓ *is its characteristic.*

Let $f \in M_k(N, \chi)$*,* $k \geqslant 1$*. We say that* f *is* λ*-integral (resp. that* $f \equiv 0 \mod m_\lambda$*) if all the coefficients of* f *lie in* $\mathcal{O}_\lambda$ *(resp. in* m_λ*).*

If f *is* λ*-integral, we say that* f *is an eigenform mod* m_λ *of the Hecke operator* T_p *with eigenvalue* $a_p \in k_f$ *if*

$$T_p f \equiv a_p f \mod m_\lambda.$$

Theorem 39. *Let* $0 \neq f \in M_k(\Gamma_0(N), \chi)$ *with* $k \geqslant 1$ *with Fourier coefficients in* K*. Suppose that* f *is* λ*-integral but* $f \not\equiv 0 \mod m_\lambda$ *and that* f *is an eigenform of* T_p *modulo* m_λ *for* $p \nmid N$*:*

$$T_p(f) \equiv a_p f \mod m_\lambda \quad \text{for all } p \nmid Nl.$$

Let k_f *be the subextension of* k_λ *generated by the* a_p *and the* $\chi(p)$ *mod* m_λ*. Then there is a semi-simple representation*

$$\rho : G_{\mathbb{Q}} \to GL_2(k_f)$$

unramified outside Nl *such that for all primes* $p \nmid Nl$ *one has:*

$$\begin{array}{l} Tr(Frob_p) = a_p \\ det(Frob_p) \equiv \chi(p)p^{k-1} \mod \lambda; \end{array} \tag{43}$$

Proof: First, we do three preliminary reductions:

1. Suppose that $(K', \lambda', f', k', \chi', (a'_p))$ is as in the hypothesis of the theorem with $K \subseteq$

K' and $\lambda'|\lambda$. We can reduce to the case where $f \equiv f' \bmod \lambda'$, $\chi = \chi'$ and $k \equiv k' \bmod (l-1)$: In fact, if $a_p \equiv a'_p \bmod m_{\lambda'}$ and $\chi(p)p^{k-1} \equiv \chi'(p)p^{k'-1} \bmod m_{\lambda'}$ for all $p \nmid Nl$, then the theorem holds for f if and only if it holds for f'.

2. Reduction to the case $k \geqslant 2$:
Fix a prime $\lambda \lhd \mathcal{O}_K$ and let ℓ be the prime dividing $\mathrm{Norm}_{\mathbb{Q}}^K(\lambda)$ (in fact, $\mathrm{Norm}_{\mathbb{Q}}^K(\lambda) = \ell^{f(K|\mathbb{Q})}$). Consider the Eisenstein series $E_{\ell-1}$ of weight $\ell - 1$ where $\ell \geqslant 5$. Its Fourier expansion is given by

$$E_{\ell-1} = 1 - \frac{2(\ell-1)}{B_{\ell-1}} \sum_{n=1}^{\infty} \sigma_{\ell-2}(n) q^n$$

where B_n is the nth Bernoulli number, is given by

$$\frac{x}{e^x - 1} = \sum_{n \geqslant 1} B_n \frac{x^n}{n!}.$$

Proposition 40. ***(CLAUSEN-VON STAUDT)*** *The denominator of* $\frac{B_n}{2^n}$ *is* $\prod_{p-1|n} p^{1+v_p(n)}$.
In particular, if ℓ is a prime, then $\frac{B_k}{2k}$ is integral at ℓ if and only if $(\ell-1) \nmid k$.

Proof: See [1] .■

The result above implies that the Eisenstein series $E_{\ell-1}(q)$ has Fourier coefficients in $\mathbb{Z}_{(\ell)}$(the localization of $\mathbb{Z}$ at ℓ) and

$$E_{\ell-1}(q) \equiv 1 \pmod{\ell} .$$

Since λ is a prime ideal of $\mathcal{O}_K$ above ℓ, we see that:

$$E_{\ell-1} \equiv 1 \mod \ell \Rightarrow E_{\ell-1} \equiv 1 \mod \lambda .$$

Hence $F_\lambda := fE_{\ell-1} \equiv f \bmod \lambda$. The modular form $F_\lambda = fE_{\ell-1}$ lies in $\mathcal{M}_{k+\ell-1}(\Gamma_0(N), \chi)$. Thus the theorem for f is equivalent to the theorem for F_λ which has weight $\geqslant 2$.

3. Reduction to the case where f is an eigenvector of T_p: It is enough to verify the theorem for f' eigenform of T_p's with $p \nmid Nl$ such that $(K', \lambda', f', k, \chi, (a'_p))$ is as in the theorem and $K \subseteq K'$, $\lambda'|\lambda$ and $a_p \equiv a'_p \bmod \lambda'$. We give a preliminary lemma.

Lemma 41. *Let M be a free module of finite rank over a discrete valuation ring $\mathcal{O}$. Let $m \subseteq \mathcal{O}$ be the maximal ideal, k the residue field and K the field of fractions of $\mathcal{O}$. Let $\mathcal{T} \subseteq End_{\mathcal{O}}(K)$ be a set of endomorphisms which commute two by two. Let $f \in M/mM$ be a nonzero common eigenvector for all the $T \in \mathcal{T}$, with eigenvalues a_T. Then there exists:*
1) a discrete valuation ring $\mathcal{O}' \supseteq \mathcal{O}$ with maximal ideal m' such that $m' \cap \mathcal{O} = m$ and with field of fractions K' such that $[K' : K] < \infty$;
2) an element $0 \neq f' \in M' = \mathcal{O}' \otimes_{\mathcal{O}} M$ which is an eigenvector for all the $T \in \mathcal{T}$ with eigenvalues a'_T with $a'_T \equiv a_T \bmod m'$.

Proof: See [7]. ■

We apply the above lemma to $M = \{f \in \mathcal{M}_k(\Gamma_0(N), \chi) |$ f has coefficients in $\mathcal{O}_\lambda\}$ and $\mathcal{T} = \{T_p\}_{p \nmid Nl}$.

Let

$$\rho_\lambda : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_\lambda)$$

be the representation associated to f by the theorem 35. We can assume that $\mathrm{im}(\rho_\lambda) \subseteq \mathrm{GL}_2(\widehat{\mathcal{O}}_\lambda)$ where $\widehat{\mathcal{O}}_\lambda$ is the ring of integers of K_λ or equivalently, the completion of $\mathcal{O}_\lambda$. By reduction mod λ, we get a representation

$$\widetilde{\rho}_\lambda : G_{\mathbb{Q}} \to \mathrm{GL}_2(k_\lambda).$$

Let φ be the semi-simplification of $\widetilde{\rho}_\lambda$; it is a semi-simple representation, unramified outside Nl which satisfies (43). The group $\varphi(G_{\mathbb{Q}})$ is finite. By Chebotarev density theorem, we deduce that every element in $\varphi(G_{\mathbb{Q}})$ is of the form $\varphi(\mathrm{Frob}_{\mathfrak{p}})$ with $\mathfrak{p} \cap \mathbb{Q} = p$ and $p \nmid Nl$. By the definition of k_f, we have:
- For all $g \in \varphi(G_{\mathbb{Q}})$, the coefficients of $\det(1 - gX)$ lie in k_f.

We can now apply the lemma below and conclude the result; *The end of the proof of the theorem 39* . ■

Lemma 42. *Let $\varphi : G \to GL_2(k')$ be a semi-simple representation of the group G over a finite field k'. Let k be a subfield of k' containing all the coefficients of polynomials* $\det(1 - \varphi(g)X)$ *for all $g \in G$. Then φ is realisable over k, i.e. it is isomorphic to a representation $\rho : G \to GL_2(k)$.*

Proof: The proof is essentially the same as the lemma 33. ■

3.2.2 Step 2: A uniform bound on the image of $\overline{\rho}_\lambda$'s

Let

$$\begin{aligned} \sum &= \{\lambda \lhd \mathcal{O}_K : \ \lambda | \ell \text{ and } \ell \text{ splits completely in } K/\mathbb{Q}\} \\ &= \{\lambda \lhd \mathcal{O}_K : \ \mathcal{O}_K/\lambda \cong \mathbb{F}_\ell\}\,. \end{aligned}$$

By Chebotarev density theorem, $\sum$ is infinite. Now ,for each $\lambda \in \sum$, let ℓ be the rational prime lying below it. Furthermore let

$$G_\lambda := \overline{\rho}_\lambda(G_{\mathbb{Q}}) \subseteq \mathrm{GL}_2(\mathbb{F}_\ell).$$

We wish to bound $|G_\lambda|$ independently of ℓ.

Definition 43. *Fix an integer $X > 0$. A subgroup $G \subseteq GL_2(\mathbb{F}_\ell)$ is X-sparse if there is a subset $H \subseteq G$ such that:*

1. *$|H| \geqslant \frac{3}{4}|G|$,*

2. *the elements of H have at most X distinct characteristic polynomials.*

Lemma 44. *There exists an* $X > 0$ *such that all the groups* G_λ $(\lambda \in \sum)$ *are X-sparse.*

Proof: By the proposition 30, for all $\eta > 0$, there exists finite set $X_\eta \subseteq \mathbb{C}$ such that $a_p \in X_\eta$ for all p outside a set Y_η of density η. Take $\eta < \frac{1}{4}$ and set $X = |X_\eta| \mathrm{ord}(\varepsilon)$. We show that G_λ is X-sparse. Let

$$H = \bigcup_{p \notin Y_\eta} \rho(\mathrm{Frob}_p).$$

By Chebotarev density theorem, the inequality $\mathrm{dens}(Y_\eta) > \frac{3}{4}$ implies $|H| \geqslant \frac{3}{4}|G|$. Moreover, the number of distinct characteristic polynomials of elements of H is less that $X = |X_\eta| \mathrm{ord}(\varepsilon)$. ■

Definition 45. *A subgroup* G *of* $GL_2(\mathbb{F}_\ell)$ *is semi-simple if the underlying 2 dimensional representation of* G *is semi-simple, i.e. either irreducible or a direct sum of 1 dimensional representations.*

Example 46. *The groups* $SL_2(\mathbb{F}_\ell)$ *and* $GL_2(\mathbb{F}_\ell)$ *are irreducible, hence semi-simple.*

Example 47. *Let* $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{F}_\ell^\times \right\}$ *the "split Cartan subgroup", then* G *is semi-simple and reducible.*

Example 48. *Let* $\mathbb{F}_{\ell^2}^\times$ *act by left multiplication on* $\mathbb{F}_{\ell^2}$ *viewed as a* $\mathbb{F}_\ell$*-vector space with any choice of basis. Let* G *be the image of* $\mathbb{F}_{\ell^2}^\times$ *in* $Aut_{\mathbb{F}_\ell}(\mathbb{F}_{\ell^2}) \cong GL_2(\mathbb{F}_\ell)$ *(a "non-split Cartaan subgroup"). Then* G *is semi-simple.*

Example 49. *If* T *is a Cartan subgroup, it has index 2 in its normalizer* G*. Then* G *is semi-simple.*

Example 50. *The group* $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{F}_\ell^\times,\ b \in \mathbb{F}_\ell \right\}$ *is reducible but not decomposable (the subspace* $\mathbb{F}_\ell \times O$ *is invariant under the action of* G *but its complement* $0 \times \mathbb{F}_\ell$ *is not invariant.) Hence* G *is not semi-simple.*

Theorem 51. *Fix* X*, there exists a constant* A_X *(depending on* X *but not* ℓ*) such that* $|G| < A_X$ *for all semi-simple X-sparse subgroups of* $GL_2(\mathbb{F}_\ell)$*.*

Remark 52. *The semi-simplicity assumption is crucial, for example if we consider*

$$G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} :\ a \in \mathbb{F}_\ell \right\},$$

then G *is 1-sparse but* $|G| = \ell$ *therefore one cannot bound* $|G|$ *independently of* ℓ*.*

We give a preliminary proposition before proving the theorem.

Proposition 53. *If* G *is a semi-simple subgroup of* $GL_2(\mathbb{F}_\ell)$ *then only the following four cases can arise:*
1. $G \supseteq SL_2(\mathbb{F}_\ell)$
2. G *is contained in a Cartan subgroup* T*, either split or non-split, which means that*

$T \simeq \mathbb{F}_\ell^\times \times \mathbb{F}_\ell^\times$ or $T \simeq \mathbb{F}_{\ell^2}^\times$.
3. $G \subset N_{GL_2(\mathbb{F}_\ell)}(T)$ where $N_{GL_2(\mathbb{F}_\ell)}(T)$ is the normaliser of a Cartan subgroup T. (therefore $[N_{GL_2(\mathbb{F}_\ell)}(T) : T] = 2$ and there exists a split exact sequence:
$1 \to T \to N(T) \to \pm 1 \to 1$.)
4. G is an "exceptional subgroup", namely its image in $PGL_2(\mathbb{F}_\ell)$ is A_4, S_4 or A_5.

Proof: See [14] section 2.5 or [21]. ∎

Theorem 54. *If G is a semi-simple X-sparse subgroup of $GL_2(\mathbb{F}_\ell)$, then there exists A independent of ℓ such that $|G| \leqslant A$.*

Proof: By the above proposition, we have to bound $|H|$ by bounding the number of elements in $\mathrm{GL}_2(\mathbb{F}_\ell)$ which have the same characteristic polynomial, i.e. by bounding the number of elements in a given conjugacy class.

We do this in the four cases of the proposition 53.

1. We have $|\mathrm{GL}_2(\mathbb{F}_\ell)| = (\ell^2-1)(\ell^2-\ell) = \ell(\ell+1)(\ell-1)^2$. Let $\sigma \in \mathrm{GL}_2(\mathbb{F}_\ell)$, then the cardinality of the set $C(\sigma) := \left\{\tau\sigma\tau^{-1} :\ \tau \in \mathrm{GL}_2(\mathbb{F}_\ell)\right\}$ is given by $|C(\sigma)| = \frac{|\mathrm{GL}_2(\mathbb{F}_\ell)|}{\mathrm{Z}(\sigma)}$ where $\mathrm{Z}(\sigma) = \{\tau :\ \tau\sigma = \sigma\tau\}$. There are 3 cases to consider for the characteristic polynomial of an element of $\sigma \in H$:

<u>Case 1: char(σ) has two roots in $\mathbb{F}_\ell$, i.e. char(σ) $= (T-a)(T-b)$ where $a \neq b$</u>: We have:

$$\left|\left\{\sigma :\ \mathrm{char}(\sigma) = (T-a)(T-b),\ a,b \in \mathbb{F}_\ell^\times,\ a \neq b\right\}\right| = \left|C(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix})\right| = \ell^2 + \ell.$$

<u>Case 2: char(σ) has one root in $\mathbb{F}_\ell$, i.e. char(σ) $= (T-a)^2$</u>: This means that

$$\sigma \in C(\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}) \cup C(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}).$$

Since

$$\left|Z(\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix})\right| = \left|\left\{\begin{pmatrix} u & v \\ 0 & u \end{pmatrix} :\ u \in \mathbb{F}_\ell^\times, v \in \mathbb{F}_\ell\right\}\right| = (\ell-1)\ell,$$

we have that $\sigma \in C(\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}) = \dfrac{\ell(\ell+1)(\ell-1)^2}{\ell^2-\ell} = \ell^2 - 1$. On the other hand, we have $C(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}) = 1$. So

$$\left|\left\{\sigma :\ \mathrm{char}(\sigma) = (T-a)^2,\ a \in \mathbb{F}_\ell^\times\right\}\right| = \ell^2.$$

<u>Case 3: char(σ) has no root in $\mathbb{F}_\ell$, i.e. char(σ) is irreducible over $\mathbb{F}_\ell$</u>: We have $|Z(\sigma)| = \ell^2 - 1$, so that:

$$|C(\sigma)| = (\ell-1)\ell.$$

Therefore we can deduce:

$$\frac{3}{4}|\mathrm{SL}_2(\mathbb{F}_\ell)| = \frac{3}{4}\ell(\ell+1)(\ell-1) \leqslant |H| \leqslant X.\mathrm{Max}\left\{\ell^2+\ell, \ell^2, \ell^2-\ell\right\} = X(\ell^2+\ell).$$

For the inequalities to hold, we must have $\ell - 1 \leqslant \frac{4}{3}X$. So:

$$|H| \leqslant X(\frac{4}{3}X+1)(\frac{4}{3}X+2).$$

Therefore, we found a bound on H independent of ℓ.

2. In T, there are at most two elements with a given characteristic polynomial, in fact σ and $\overline{\sigma}$ have the same characteristic polynomial. Hence $|H| \leqslant 2X$ and so

$$|G| \leqslant \frac{8}{3}X.$$

3. Let $G_0 = G \cap T$ which has index two in G, so $|G_0| = \frac{1}{2}|G|$ and let $H_0 = H \cap T$ so that $|H_0| \geqslant \frac{1}{2}|G_0|$. We can apply the case 2 and get $|H_0| \leqslant 2X$ so that $|G_0| \leqslant 4X$ which implies

$$|G| \leqslant 8X.$$

4. Consider the following homomorphism of groups:

$$\begin{aligned} \eta : \quad G &\to \mathrm{PGL}_2(\mathbb{F}_\ell) \times \mathbb{F}_\ell^\times \\ \sigma &\mapsto (\bar{\sigma}, \det(\sigma)). \end{aligned}$$

We know that the image of G in $\mathrm{PGL}_2(\mathbb{F}_\ell)$ is A_4, S_4 or A_5 and X is the number of different characteristic polynomials in H, therefore $|\eta(H)| \leqslant |A_5|X = 60X$. Since $\ker(\eta) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z}$. This implies that $|H| \leqslant 120X$ and then

$$|G| \leqslant 160X.$$

We have concluded the proof of the theorem. ■

3.2.3 Step 3: Gluing the ℓ-adic representations ρ_λ

Fix a constant A such that $|G_\ell| \leqslant A$. Let $K \subset \mathbb{C}$ be a Galois number field containing the a_p and the $\chi(p)$ for all primes p. As before, let

$$\sum = \{\lambda \lhd \mathcal{O}_K : \ \mathcal{O}_K/\lambda \cong \mathbb{F}_\ell\}.$$

For all $\ell \in \sum$, fix a place λ_ℓ of K extending ℓ. By theorem 39, there exists a semi-simple continuous representation

$$\rho_\ell : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F}_\ell)$$

unramified outside of Nl and such that

$$\begin{aligned} \mathrm{char}(\rho_\ell(\mathrm{Frob}_p)) &= \det(\mathrm{Id}_2 - \rho_\ell(\mathrm{Frob}_p)T) \\ &\equiv 1 - a_pT + \chi(p)T^2 \mod \lambda_\ell . \end{aligned}$$

for all primes $p \nmid Nl$.

Up to replacing K with a bigger number field (reducing $\sum$ consequently), we may well suppose that K contains all n-th roots of unity for all $n \leqslant A$. Set

$$Y = \{(1-\alpha T)(1-\beta T) : \ \alpha \text{ and } \beta \text{ are roots of unity of order } \leqslant A\}.$$

The eigenvalues of $\rho_\ell(\mathrm{Frob}_p)$ are root of unity of order $\leqslant A$. Therefore there exist $R(T) \in Y$ such that

$$1 - a_p T + \chi(p) T^2 \equiv R(T) \bmod \lambda_\ell.$$

Since Y is finite and L is infinite, there must exist some $R(T) \in Y$ such that the above congruence is satisfied for an infinite number of ℓ's. This implies that such a congruence is in fact an equality. Thus the polynomials $1 - a_p T + \chi(p) T^2$ all lie in Y. Now let

$$\sum{}' = \{\ell \in L : \ell > A, \ (R, S \in Y, \ R \neq S) \Rightarrow R \not\equiv S \bmod \lambda_\ell\}.$$

Since $\sum \setminus \sum'$ is infinite, $\sum'$ is finite. Choose $\ell \in L'$. It follows that $\gcd(|G_\ell|, \ell) = 1$ and therefore the identical representation $G_\ell \to \mathrm{GL}_2(\mathbb{F}_\ell)$ is the reduction modulo λ_ℓ of a representation $G_\ell \to \mathrm{GL}_2(\mathcal{O}_{\lambda_\ell})$ where $\mathcal{O}_{\lambda_\ell}$ is the valuation ring of λ_ℓ in K, namely we have a commutative diagram

$$\begin{tikzcd} G_\ell \arrow[r] \arrow[dr] & GL_2(\mathcal{O}_{\lambda_\ell}) \arrow[d] \\ & GL_2(\mathbf{F}_\ell) \end{tikzcd}$$

We then compose the representation $G_\ell \to \mathrm{GL}_2(\mathcal{O}_{\lambda_\ell})$ with the projection $G_\mathbb{Q} \to G_\ell$, we get a representation $\rho : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_{\lambda_\ell})$ which by construction is unramified outside Nl.

If $p \nmid Nl$, the eigenvalues of $\rho(\mathrm{Frob}_p)$ are roots of unity of order $\leqslant A$, because $\rho(G_\mathbb{Q}) \cong G_\ell$ and $|G_\ell| \leqslant A$. Therefore $\det(Id_2 - \rho(\mathrm{Frob}_p)T) \in Y$. On the other hand, by construction:

$$\det(Id_2 - \rho(\mathrm{Frob}_p)T) \equiv 1 - a_p T + \chi(p) T^2 \bmod \lambda_\ell.$$

Since $1 - a_p T + \chi(p) T^2 \in Y$ and $\ell \in L'$, the last congruence is an equality. Now repeat the same construction by choosing another $\ell' \in L'$. We obtain a second representation $\rho' : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_{\lambda_\ell})$ which has the same properties as ρ but for $p \nmid Nl'$. This implies that

$$\det(Id_2 - \rho(\mathrm{Frob}_p)T) = \det(Id_2 - \rho'(\mathrm{Frob}_p)T) \quad \text{for all } p \nmid N\ell\ell'.$$

By theorem 36, it easily follows that ρ and ρ' are isomorphic as representation over $\mathrm{GL}_2(K)$ and so they are isomorphic also as complex representations. Moreover, since ρ is unramified at ℓ' and symmetrically ρ' is unramified at ℓ', then both ρ and ρ' are unramified outside N and

$$\det(Id_2 - \rho(\mathrm{Frob}_p)T) = 1 - a_p T + \chi(p) T^2 \quad , \forall p \nmid N.$$

The last thing to prove is that ρ is irreducible. Suppose that it is not, then there exists two 1-dimensional representations $\chi_1, \chi_2 : G_\mathbb{Q} \to \mathbb{C}^*$ such that $\rho \cong \chi_1 + \chi_2$. It follows that $\chi = \chi_1\chi_2$ and $a_p = \chi_1(p) + \chi_2(p)$ for $p \nmid N$ and both χ_1 and χ_2 are unramified outside N. Then we have:

$$\sum_{p\in\mathcal{P}} |a_p|^2 p^{-s} = 2\sum_{p\in\mathcal{P}} p^{-s} + \sum \chi_1(p)\overline{\chi_2(p)}p^{-s} + \sum_{p\in\mathcal{P}} \overline{\chi_1(p)}\chi_2(p)p^{-s}.$$

We should have $\chi_1\overline{\chi_2} \neq \mathbf{1}$, because otherwise we would have $\chi = \chi_1\chi_2 = \chi_1^2$ and so $\chi(-1) = 1$ but the character χ is supposed to be odd. Therefore, since the character $\chi_1\overline{\chi_2}$ is not trivial, we have:

$$\sum_{p\in\mathcal{P}} \overline{\chi_1}(p)\chi_2(p)p^{-s} = O(1),$$
$$\sum_{p\in\mathcal{P}} \chi_1(p)\overline{\chi_2}(p)p^{-s} = O(1).$$

On the other hand, it is well-known that:

$$\sum_{p\in\mathcal{P}} p^{-s} = \log\left(\frac{1}{s-1}\right) + O(1).$$

We then obtain that

$$\sum_{p\in\mathcal{P}} |a_p|^2 p^{-s} = 2\log\left(\frac{1}{s-1}\right) + O(1)$$

which is in contradiction with the theorem 24 so we get the conclusion. ■

Corollary 3. *Let f be a modular form of weight 1 and character χ. Then for all primes p, the coefficient $a_p(f)$ is a sum of two roots of unity. In particular:*

$$|a_p(f)| \leqslant 2.$$

Proof: By the Deligne-Serre's theorem, $a_p(f)$ is equal to $tr(\mathrm{Frob}_p)$, hence the sum of its two eigenvalues. We saw that the eigenvalues of $\sigma(\mathrm{Frob}_p)$ are roots of unity. The result follows immediately. ■

4 The Birch and Swinnerton-Dyer conjecture and the Rankin-Selberg method

Let K be a number field and let E be an elliptic curve over K. We state the theorem of Mordell-Weil and discuss about the structure of the group of points of E and give the definition of $r_K(E)$, i.e. the rank of E over K. Then we state the Birch and Swinnerton-Dyer conjecture (henceforth abbreviated BSD conjecture). If we assume the BSD conjecture, we can prove a more general form of it, i.e. the twisted BSD conjecture. In the last part of this chapter, we gather some numerical evidence for the twisted BSD.

4.1 The Mordell-Weil theorem and the Birch and Swinnerton-Dyer conjecture

Let K be a number field and let E be an elliptic curve over K. The points of E over K has an abelian group structure denoted $E(K)$. Mordell and Weil proved the important theorem below:

Theorem 55. *The group $E(K)$ is finitely generated.*

Proof: See for example [17]. ■

The Mordell-Weil theorem tells us that the *Mordell-Weil* group $E(K)$ has the form

$$E(K) \cong E(K)_{\text{tors}} \oplus \mathbb{Z}^r$$

where the torsion subgroup $E(K)_{\text{tors}}$ is finite and the *rank* r of $E(K)$ (denoted also by $r_K(E)$) is a nonnegative integer. It is relatively easy to compute the torsion subgroup but there is no known procedure that is guaranteed to yield the rank $r_K(E)$.

The L-series of an elliptic curve is a generating function that records information about the reduction of the curve modulo every prime. Consider an elliptic curve E over $\mathbb{Q}$ with a general Weierstrass equation E defined over $\mathbb{Q}$:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_1, ...a_6 \in \mathbb{Q}$$

View two integral Weierstrass equations as equivalent if they are related by a *general admissible change of variable* over $\mathbb{Q}$:

$$x = u^2x^{'} + r, \quad y = u^3y' + su^2x^{'} + t \quad u, r, s, r \in \mathbb{Q},\ u \neq 0.$$

After an admissible change of variable of the form $(x, y) = (u^2x^{'}, u^3y^{'})$ we can assume that the coefficients a_i's are integer. For each prime p, let $v_p(E)$ denote the smallest power of p appearing in the discriminant of any integral Weierstrass equation equivalent to E, i.e. the minimum of a set of nonnegative integers

$$v_p(E) = \min\{v_p(\Delta(E')) : \quad E^{'} \text{ integral, equivalent to } E\}.$$

Define the *global minimal discriminant* of E to be

$$\Delta_{\min}(E) = \prod_p p^{v_p(E)}.$$

This is a finite product since $v_p = 0$ for all $p \nmid \Delta(E)$. One can show that the p-adic valuation of the discriminant can be minimized to $v_p(E)$ simultaneously for all p under an admissible change of variable. That is, E is isomorphic over $\mathbb{Q}$ to an integral model $E^{'}$ with discriminant $\Delta(E^{'}) = \Delta_{\min}(E)$. This is the *global minimal Weierstrass equation* $E^{'}$, the model of E to reduce modulo primes.

Consider the reduction map modulo $p\mathbb{Z}$:

$$\widetilde{\ } : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} = \mathbf{F}_p.$$

This map reduces a global minimal Weierstrass equation E to a Weierstrass equation $\widetilde{E}$ over $\mathbf{F}_p$ and this defines an elliptic curve over $\mathbf{F}_p$ if and only if $p \nmid \Delta_{\min}(E)$. The reduction modulo p is called
1) *good [nonsingular, stable]* if $\widetilde{E}$ is again an elliptic curve,

a) *ordinary* if $\widetilde{E}[p] = \mathbb{Z}/p\mathbb{Z}$,

b) *supersingular* if $\widetilde{E}[p] = \{0\}$,

2) *bad [singular]* if $\widetilde{E}$ is not an elliptic curve, in which case it has only one singular point,

a) *multiplicative [semistable]* if $\widetilde{E}$ has a node,

b) *additive [unstable]* if $\widetilde{E}$ has a cusp.

Define the *algebraic conductor* of E:

$$N_E = \prod_p p^{f_p},$$

where

$$f_p = \begin{cases} 0 & \text{if} \quad \text{E has good reduction at p,} \\ 1 & \text{if} \quad \text{E has multiplicative reduction at p} \quad , \\ 2 & \text{if} \quad \text{E has additive reduction at p and} \quad p \nmid \{2,3\}, \\ 2+\delta_p & \text{if} \quad \text{E has additive reduction at p and} \quad p \in \{2,3\}. \end{cases}$$

Here $\delta_2 \leqslant 6$ and $\delta_3 \leqslant 3$. There is also a closed-form formula for f_p. (see [17])

Denote $\widetilde{E}(\mathbf{F}_p)$ the elliptic curve $\widetilde{E}$ over $\mathbf{F}_p$.

Definition 56. *Let E be an elliptic curve over $\mathbb{Q}$. Assume E is in reduced form. Let p be a prime and let $\widetilde{E}$ be the reduction of E modulo p. Then define*

$$\begin{aligned} a_1(E) &= 1 \\ a_p(E) &= p+1-|\widetilde{E}(\boldsymbol{F}_p)|. \end{aligned} \tag{44}$$

The coefficients $a_{p^e}(E)$ satisfy the same recurrence as the coefficients $a_{p^e}(f)$ of a normalised eigenform in $\mathcal{S}_2(\Gamma_0(N))$ (see [6] section 8.3):

$$a_{p^e}(E) = a_p(E)a_{p^{e-1}}(E) - \mathbf{1}_E(p)pa_{p^{e-2}}(E) \quad \textit{for all } e \geqslant 2$$

Here $\mathbf{1}_E$ is the trivial character modulo the algebraic conductor N_E of E. We extend the definition for all positive integers m by setting

$$a_{mn}(E) = a_m(E)a_n(E) \quad \textit{if } (m,n) = 1. \tag{45}$$

Theorem 57. ***(Modularity Theorem, Version a_p)*** *Let E be an elliptic curve over $\mathbb{Q}$ with conductor N_E. Then for some newform $f \in \mathcal{S}_2(\Gamma_0(N_E))$,*

$$a_p(f) = a_p(E) \quad \textit{for all primes } p.$$

This version of the Modularity theorem rephrases in terms of L-function. Recall that if $f \in \mathcal{S}(\Gamma_0(N))$ is a newform then its L-function is

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s} = \prod_p \frac{1}{1 - a_p(f)p^{-s} + \mathbf{1}_N(p)p^{1-2s}},$$

with convergence in a right half plane. Define the *Hasse-Weil L-function of E* as:

$$\begin{aligned} L(E,s) &= \sum_{n=1}^{\infty} \frac{a_n(E)}{n^s} \\ &= \prod_p \frac{1}{1 - a_p(E)p^{-s} + \mathbf{1}_E(p)p^{1-2s}} \end{aligned} \tag{46}$$

where $\mathbf{1}_E$ is the trivial character modulo the conductor N_E. This L-function encodes all the solution-counts $a_p(E)$.

Then one can state another version for Modularity theorem:

Theorem 58. ***(Modularity Theorem, Version L)*** *Let E be an elliptic curve over $\mathbb{Q}$ with conductor N_E. Then for some newform $f \in \mathcal{S}_2(\Gamma_0(N_E))$,*

$$L(f,s) = L(E,s).$$

By Mordell-Weil theorem, we have

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r.$$

For a modular form f, one has the half plane convergence, analytic continuation, and functional equation of $L(f,s)$. Using the Modularity Theorem version L, we can also get the half plane convergence of $L(E,s)$ (which is $\mathcal{R}(s) > 2$) and the functional equation that determines $L(E,s)$ for $\mathcal{R}(s) < 0$, but the behaviour of $L(E,s)$ at the center of the remaining strip $\{0 \leqslant \mathcal{R}(s) \leqslant 2\}$ is what conjecturally determines the rank of $E(\mathbb{Q})$. The Weak Birch and Swinnerton-Dyer conjecture says that the rank r is equal to the order of vanishing of $L(E,s)$ at $s = 1$:

Conjecture 59. ***(Weak Birch and Swinnerton-Dyer):*** *Let E be an elliptic curve defined over $\mathbb{Q}$. Then the order of vanishing of $L(E,s)$ at $s=1$ is the rank of $E(\mathbb{Q})$. That is, if $E(\mathbb{Q})$ has rank r then*

$$L(E,s) = (s-1)^r g(s) \quad ; \; g(1) \neq 0, \infty.$$

Now let E/K be an elliptic curve and let $v \in M_K$ be a finite place at which E has good reduction. We denote the residue field of K at v by k_v, the reduction of E at v by $\widetilde{E}_v$ and we let $q_v = ||k_v||$ be the norm of the prime ideal corresponding to v. Put

$$a_v = q_v + 1 - ||\widetilde{E}_v(k_v)|| \tag{47}$$

and

$$L_v(T) = 1 - a_v T + q_v T^2 \;\in \mathbb{Z}[T]. \tag{48}$$

Definition 60. *The L-series of E/K is defined by the Euler product*

$$L(E/K,s) = \prod_{v \in M_K^0} L_v(q_v^{-s})^{-1} \tag{49}$$

where M_K^0 is the nonarchimedean absolute values in K.

The product defining $L(E/K,s)$ converges and gives an analytic function for all $\mathcal{R}(s) > \frac{3}{2}$. Its analytic continuation is conjectured as follows:

Conjecture 61. *The L-series $L(E/K,s)$ has an analytic continuation to the entire complex plane and satisfies a functional equation relating its values at s and $2-s$.*

Deuring and Weil proved this conjecture for elliptic curves having complex multiplication. Eichler and Shimura showed that this conjecture is true for all elliptic curves $E/\mathbb{Q}$ which are modular. Later, Wiles proved that all elliptic curves $E/\mathbb{Q}$ are modular. As a consequence, this conjecture is true for all elliptic curves $E/\mathbb{Q}$.

The *conductor* of E/K is the integral ideal of K defined by

$$N_{E/K} = \prod_{v \in M_K^0} \mathfrak{p}_v^{f_v}$$

where the exponent of the conductor f_v is defined by

$$f_v = \begin{cases} 0 & \text{if} \quad \text{E has good reduction at v} \\ 1 & \text{if} \quad \text{E has multiplicative reduction at v} \\ 2+\delta_v & \text{if} \quad \text{E has additive reduction at v} \end{cases},$$

where δ_v is an integer; see [12].

Now we state the BSD conjecture generalised for elliptic curves over any number field K.

Conjecture 62. ***(Birch and Swinnerton-Dyer)*** *Let E be an elliptic curve defined over a number field K. Then the order of vanishing of $L(E/K,s)$ at $s=1$ is the rank of $E(K)$. That is, if $E(K)$ has rank r then*

$$L(E/K,s) = (s-1)^r g(s) \quad ; \; g(1) \neq 0, \infty.$$

4.2 L-functions attached to representations

Assume that K is a number field such that the extension K/Q is Galois and denote $\mathcal{O}_K$ the ring of integers of K. Let p be a a rational prime. The ideal of $\mathcal{O}_K$ generated by p can be factorised as a product of maximal ideals of $\mathcal{O}_K$. In fact, we have:

$$\begin{array}{rcl} p\mathcal{O}_K & = & (\mathfrak{p}_1...\mathfrak{p}_g)^e \\ \mathcal{O}_K/\mathfrak{p}_i & \cong & \mathbf{K}_{p^f} \quad \text{for } i = 1, ..., g \\ efg & = & [\mathbf{K} : \mathbb{Q}]. \end{array}$$

e is called the *ramification degree* which says how many times each maximal ideal $\mathfrak{p}$ of $\mathcal{O}_K$ that lies over p repeats as a factor of $p\mathcal{O}_K$. We say that a prime p ramify in K if its ramification degree e is > 1.
The *residue degree* f is the dimension of the *residue field* $k_p = \mathcal{O}_K/\mathfrak{p}$ as a vector space over $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$ for any $\mathfrak{p}$ over p.
The *decomposition index* g is the number of distinct $\mathfrak{p}$ over p.

Example 63. *Let N be a positive integer and let $K = \mathbb{Q}(\mu_N)$ where $\mu_N = e^{2\pi i/N}$. Then $[K : \mathbb{Q}] = \phi(N)$ and the extension $K/\mathbb{Q}$ is Galois with Galois group isomorphic to $(\mathbb{Z}/N\mathbb{Z})^*$. The isomorphism is given by*

$$\begin{gathered} Gal(K/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^* \\ (\mu_N \mapsto \mu_N^a) \mapsto a(\mod N). \end{gathered}$$

On can show that cyclotomic integers are $\mathcal{O}_K = \mathbb{Z}[\mu_N]$. A prime p ramifies in K if and only if $p|N$. For a prime $p \nmid N$ one can write $p\mathcal{O}_K = \mathfrak{p}_1...\mathfrak{p}_g$ and its residue degree f is equal to the order of p (mod N) in $(\mathbb{Z}/N\mathbb{Z})^*$.

For each maximal ideal $\mathfrak{p}$ of $\mathcal{O}_F$ lying over p, the **decomposition group** of $\mathfrak{p}$ is the subgroup of the Galois group that fixes $\mathfrak{p}$ as a set

$$\mathcal{D}_\mathfrak{p} = \{\sigma \in \mathrm{Gal}(K/\mathbb{Q}) : \sigma(\mathfrak{p}) = \mathfrak{p}\}.$$

The decomposition group $\mathcal{D}_\mathfrak{p}$ has order ef so $[\mathrm{Gal}(K/\mathbb{Q}) : \mathcal{D}_\mathfrak{p}] = g$. One can define a well-defined action of $\mathcal{D}_\mathfrak{p}$ on $k_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$:

$$\begin{array}{rcl} \mathcal{D}_\mathfrak{p} \times k_\mathfrak{p} & \to & k_\mathfrak{p} \\ (\sigma, x + \mathfrak{p}) & \mapsto & \sigma(x) + \mathfrak{p}, \end{array}$$

where $x \in \mathcal{O}_K$. The **inertia group** of $\mathfrak{p}$ is the kernel of the above action:

$$I_\mathfrak{p} = \{\sigma \in \mathrm{Gal}(K/\mathbb{Q}) : \ \sigma(x) \equiv x \text{ for all } x \in \mathcal{O}_K\}.$$

Obviously, $I_\mathfrak{p} \subseteq \mathcal{D}_\mathfrak{p}$. The inertia group $I_\mathfrak{p}$ has order e, so it is trivial for all $\mathfrak{p}$ lying over any unramified p. The kernel of composition map $\mathbb{Z} \to \mathcal{O}_K \to \mathcal{O}_K/\mathfrak{p}$ is $p\mathbb{Z}$. So we can view $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$ as a subfield of $k_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$ then there is an injection

$$\mathcal{D}_\mathfrak{p}/I_\mathfrak{p} \to \mathrm{Gal}(k_\mathfrak{p}/\mathbf{F}_p).$$

Since both groups have order f, this map is in fact an isomorphism. Any Galois group of an finite field is cyclic, so the is an element σ_p that generates $\mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p)$:

$$\mathrm{Gal}(k_{\mathfrak{p}}/\mathbf{F}_p) = \langle \sigma_p \rangle .$$

By isomorphism, the quotient $\mathcal{D}_{\mathfrak{p}}/I_{\mathfrak{p}}$ has a generator that maps to σ_p. Any representative of this generator in $\mathcal{D}_p$ is called a **Frobenius element** of $\mathrm{Gal}(K/\mathbb{Q})$ and denoted $\mathrm{Frob}_{\mathfrak{p}}$. It satisfies:

$$x^{\mathrm{Frob}_{\mathfrak{p}}} \equiv x^{p} \mod \mathfrak{p} \text{ for all } x \in \mathcal{O}_K.$$

When the number field K is Galois over $\mathbb{Q}$, the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ acts transitively on the maximal ideals lying over p, i.e. for any two maximal ideal $\mathfrak{p}$ and $\mathfrak{p}^{'}$, there is an automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $\sigma(\mathfrak{p}) = \mathfrak{p}^{'}$. Therefore

$$\mathcal{D}_{\sigma(\mathfrak{p})} = \sigma^{-1}\mathcal{D}_{\mathfrak{p}}\sigma, \quad \mathcal{I}_{\sigma(\mathfrak{p})} = \sigma^{-1}\mathcal{I}_{\mathfrak{p}}\sigma.$$

It follows that

$$\mathrm{Frob}_{\sigma(\mathfrak{p})} = \sigma^{-1}\mathrm{Frob}_{\mathfrak{p}}\sigma.$$

In particular, if the Galois group is abelian, then $\mathrm{Frob}_{\mathfrak{p}} = \mathrm{Frob}_{\mathfrak{p}'}$ for any $\mathfrak{p}$ and $\mathfrak{p}^{'}$ primes above p. Hence $\mathrm{Frob}_{\mathfrak{p}}$ ($\mathcal{D}_{\mathfrak{p}}$ and $\mathcal{I}_{\mathfrak{p}}$) for any $\mathfrak{p}$ lying over p can be denoted Frob_p (respectively $\mathcal{D}_p$ and $\mathcal{I}_p$).

Now, consider an artin representation:

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}(V)$$

where V is a complex vector space of dimension n. Define V^{I_p} to be the subspace of V on which $\rho(I_p)$ acts as the identity. One can see that the characteristic polynomial of $\rho(\mathrm{Frob}_p)|_{V^{I_p}}$ only depends on its conjugacy class. Therefore we can define the *Artin L-function of* (V, ρ) as follows:

$$L(\rho, s) := \prod_p \frac{1}{\det(\mathrm{Id}_n - \rho(\mathrm{Frob}_p)|_{V^{I_p}}.p^{-s})}.$$

Whenever $\rho(I_p) = Id_n$, we say that ρ is *unramified* at p. In this case, $\rho(\mathrm{Frob}_p)|_{V^{I_p}}$ acts on all of V.

For example, if we take $\rho_{\mathrm{triv}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\mathbb{C})$ the trivial representation, then $L(\rho_{\mathrm{triv}}) = \zeta(s)$.
Analogously, we can define an L-function attached to a Galois representation with ℓ-adic coefficients. Assume

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}(V)$$

where V is a $\mathbb{Q}_\ell$-vector space of dimension n. We need to restrict our attention to ℓ-adic representations where the characteristic polynomial of $\rho(\mathrm{Frob}_p)|_{V^{I_p}}$ has rational coefficients. Then we can define similarly:

$$L(\rho, s) := \prod_p \frac{1}{\det(\mathrm{Id}_n - \rho(\mathrm{Frob}_p)|_{V^{I_p}}.p^{-s})}.$$

4.3 The twisted Birch and Swinnerton-Dyer conjecture

Let E be an elliptic curve over $\mathbb{Q}$ and let τ be a continuous and irreducible complex representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Assume $\ker(\tau) = (\overline{\mathbb{Q}}/K)$ where K is a number field. Let ρ_E denote the 2-dimensional Galois representation of the elliptic curve E, namely, the p-adic Tate module of E.

We shall be interested in the twisted L-function

$$L(E, \tau, s) := L(\rho_E \otimes \tau, s).$$

We give a version of the Birch Swinnerton-Dyer conjecture saying that the order of vanishing of $L(E, \tau, s)$ at $s = 1$ is equal to the multiplicity of τ in the representation of $G_{\mathbb{Q}}$ on $\mathbb{C} \otimes_{\mathbb{Z}} E(K)$. By Mordell-Weil theorem, we have $E(K) \cong E(K)_{\mathrm{tors}} \oplus \mathbb{Z}^r$, therefore:

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{Z}} E(K) &\cong \mathbb{C} \otimes_{\mathbb{Z}} (E(K)_{\mathrm{tors}} \oplus \mathbb{Z}^r) \\ &\cong (\mathbb{C} \otimes_{\mathbb{Z}} E(K)_{\mathrm{tors}}) \oplus (\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}^r) \\ &\cong \mathbb{C}^r. \end{aligned}$$

For, $E(K)_{\mathrm{tors}}$ is a finite abelian group of the form $\mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus ... \oplus \mathbb{Z}/m_v\mathbb{Z}$ where $m_v | ... \; | m_2 \; | m_1$ and for any integer m, we have

$$\mathbb{C} \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{C}/m\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{C}/m\mathbb{C} \cong 0.$$

There is a natural strengthening of the Birch and Swinnerton-Dyer conjecture as follows:

Conjecture 64. *Assume the Birch Swinnerton-Dyer conjecture. Then*

$$ord_{s=1} L(E, \tau, s) = \langle \tau, \mathbb{C} \otimes_{\mathbb{Z}} E(K) \rangle = \text{multiplicity of } \tau \text{ in } \mathbb{C} \otimes_{\mathbb{Z}} E(K)$$

where K is the finite extension of $\mathbb{Q}$ which is fixed by the kernel of τ.

Proof: See [13] page 127. ■

Remark 65. *If we replace τ by trivial representation, we recover the BSD conjecture:*

$$ord_{s=1} L(E, \mathbf{1}, s) = ord_{s=1} L(E, s) = r_{\mathbb{Q}}(E) \; .$$

In fact, any rational point P is fixed by all elements of $G_{\mathbb{Q}}$. But if $P \notin \mathbb{Q}^2$, then there is an element $\sigma \in G_{\mathbb{Q}}$ such that $\sigma(P) \neq P$. Therefore the multiplicity of trivial representation in $\mathbb{C} \otimes_{\mathbb{Z}} E(K)$ is equal to $r_{\mathbb{Q}}(E)$.

In the next section, using the Deligne-Serre theorem 32, we shall compute and present some numerical evidence for this theorem .

4.4 Some numerical evidence for the generalized BSD conjecture

We saw that the vanishing order of $L(E,\tau,s)$ at $s=1$ is equal to the multiplicity of τ in $\mathbb{C}\otimes_{\mathbb{Z}} E(K)$. In this section, we take for τ the representation arising from a modular form of weight 1. In fact, by the Deligne-Serre theorem 32, for any cusp form $g=\sum_{n=1}^{\infty} b_n q^n \in \mathcal{S}_1(\Gamma_0(N),\chi)$ of weight 1 and character χ, one can associate an odd, continuous and irreducible Galois representation $\rho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ such that

$$\mathrm{char}(\rho_g(\mathrm{Frob}_p)) = X^2 - b_p X + \chi(p) \quad \text{for any } p \nmid N.$$

Assume $\ker(\rho_g) = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ where K is a number field. We abuse the notation and denote ρ_g also for the induced representation $\mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$. The conjecture 64 implies:

$$\mathrm{ord}_{s=1} L(E,\rho_g,s) \stackrel{?}{=} \langle \rho_g, \mathbb{C}\otimes E(K)\rangle = \text{ multiplicity of } \rho_g \text{ in } \mathbb{C}\otimes E(K)$$

We aim to compute the constant term of $L(E,\rho_g,s)$ at $s=1$. Thus, if we assume the BSD, we deduce:

$$L(E,\rho_g,1)=0 \quad \stackrel{?}{\Longrightarrow} \quad \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho_g, \mathbb{C}\otimes E(K)) \neq 0.$$

Let f be a modular form of weight 2 (of trivial character) attached to an elliptic curve $E/\mathbb{Q}$:

$$f=\sum_{n=1}^{\infty} a_n q^n \in \mathcal{S}_2(\Gamma_0(N))$$

where a_n's are integers, hence $f=f^*$. Let g be a modular form of weight 1 attached to an Artin representation $\rho : Gal(K/\mathbb{Q}) \to GL_2(\mathbb{C})$ by the Deligne-Serre theorem:

$$g=\sum_{n=1}^{\infty} b_n q^n \in \mathcal{S}_1(\Gamma_0(N'),\chi)$$

where $N'|N$. We wish to compute the special value of $L(E,\rho_g,s)=L(f\otimes g,s)$ at $s=1$. Writing the formula (32) for these choices of f and g, we have for $\mathcal{R}(s)>\frac{3}{2}$:

$$\left\langle \widetilde{E}_1'(z,s-1;\chi^{-1};N)g(z), f(z)\right\rangle_{2,N} = \frac{2\Gamma(s)}{(4\pi)^s}\mathcal{D}(f,g,s). \tag{50}$$

The series of $\mathcal{D}(f,g,s)=\displaystyle\sum_{n=0}^{\infty}\frac{a_n b_n}{n^s}$ is convergent for $\mathcal{R}(s)>\frac{3}{2}$, but we have seen that

$$L(f\otimes g,s) = L(\chi,2s-1)\mathcal{D}(f,g,s).$$

Dirichlet showed that $L(\chi,s)$ can be extended to a meromorphic function on the whole complex plane and $L(\chi,1)\neq 0$ if χ is not trivial. We also saw that $L(f\otimes g,s)$ can be extended to an entire function. So $\mathcal{D}(f,g,s)$ is a meromorphic function witch has the same vanishing property as $L(f\otimes g,s)$ at $s=1$. Therefore, we need only to to compute the value of $\mathcal{D}(f,g,s)$ at $s=1$ by the formula (50):

$$\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z), f(z)\right\rangle_{2,N} = \frac{1}{2\pi}\mathcal{D}(f,g,1)\ . \tag{51}$$

Hence

$$\begin{aligned} L(f\otimes g,1) &= L(\chi,1)\mathcal{D}(f,g,1) \\ &= 2\pi L(\chi,1)\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z),f(z)\right\rangle_{2,N} . \end{aligned} \tag{52}$$

Notice that $\widetilde{E}_1'(z;\chi^{-1};N)g(z)$ is a cusp form of weight 2 and character trivial, i.e. $\widetilde{E}_1'(z;\chi^{-1};N)g(z)$ belongs to $\mathcal{S}_2(\Gamma_0(N))$. We wish to find a suitable basis for the vector space $\mathcal{S}_2(\Gamma_0(N))$ in order to compute $\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z),f(z)\right\rangle_{2,N}$.

By definition, the space of newforms at level N is the orthogonal complement of the space of oldforms with respect to the Petersson inner product:

$$\mathcal{S}_2(\Gamma_1(N))^{\text{new}} = (\mathcal{S}_2(\Gamma_1(N))^{old})^{\perp}.$$

On the other hand:

$$\mathcal{S}_2(\Gamma_1(N)) = \bigoplus \mathcal{S}_2(\Gamma_0(N),\chi)$$

where the sum is over all Dirichlet characters modulo N. Recall that $\mathcal{S}_2(\Gamma_1(N),\mathbf{1}) = \mathcal{S}_2(\Gamma_0(N))$. Then

$$\mathcal{S}_2(\Gamma_1(N))^{\text{new}} \cap \mathcal{S}_2(\Gamma_0(N)) = (\mathcal{S}_2(\Gamma_1(N))^{old})^{\perp} \cap \mathcal{S}_2(\Gamma_0(N))$$

or:

$$\mathcal{S}_2(\Gamma_0(N))^{\text{new}} = (\mathcal{S}_2(\Gamma_0(N))^{old})^{\perp}.$$

Assume

$$\begin{aligned} &\dim\,(\mathcal{S}_2(\Gamma_0(N))^{\text{old}}) = w\ , \\ &\dim\,(\mathcal{S}_2(\Gamma_0(N))^{\text{new}}) = v\ . \end{aligned}$$

Then

$$\dim\,(\mathcal{S}_2(\Gamma_0(N))) = d = w+v.$$

From the Spectral Theorem of linear algebra, given a commuting family of normal operators on a finite-dimensional inner product space, the space has an orthogonal basis of simultaneous eigenvectors for the operators. In our case, the vector space $\mathcal{S}_2(\Gamma_0(N))$ is finite-dimensional and Hecke operators T_n and $\langle n\rangle$ commutate and are normal relative to the Petersson inner product on $\mathcal{S}_2(\Gamma_0(N))$ for all n coprime to N:

Theorem 66. *The space $\mathcal{S}_2(\Gamma_0(N))$ has an orthogonal basis of simultaneous eigenforms for all the Hecke operators $\{\langle n\rangle, T_n :\ (n,N)=1\}$.*

Now consider $B_1 = \{f = f_1,\ f_2,\ ...,\ f_v\}$ the basis of eigenforms for the space of newforms $\mathcal{S}_2(\Gamma_0(N))^{\text{new}}$ where we can assume $f = f_1$ (since f is a newform by modularity theorem.) Take any basis $B_2 = \{f = f_{v+1},\ f_{v+2},\ ...,\ f_d\}$ for the space of oldforms $\mathcal{S}_2(\Gamma_0(N))^{\text{old}}$. Then $B = B_1 \cup B_2$ is a basis for $\mathcal{S}_2(\Gamma_0(N))$. By definition of the space of newforms:

$$\langle f_i(z), f_j(z)\rangle_{2,N} = 0$$

for any $f_i \in B_1$ and $f_j \in B_2$. Now write the modular form $\widetilde{E}_1'(z;\chi^{-1};N)g(z)$ as a linear combination of the elements of the basis B:

$$\widetilde{E}_1'(z;\chi^{-1};N)g(z) = \alpha_1 f + \alpha_2 f_2 + ... + \alpha_d f_d.$$

(Recall that we set $f = f_1$) It follows:

$$\begin{aligned}\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z), f(z)\right\rangle_{2,N} &= \sum \alpha_i \left\langle f_i(z), f(z)\right\rangle_{2,N} \\ &= \alpha_1 \left\langle f(z), f(z)\right\rangle_{2,N}. \end{aligned} \tag{53}$$

Combining this with (52), we get

$$L(f\otimes g,1) = 2\pi\alpha_1 L(\chi,1)\left\langle f(z), f(z)\right\rangle_{2,N} \ . \tag{54}$$

Notice that $\langle f(z), f(z)\rangle_{2,N}$ is nonzero. In summary,

$$\begin{aligned}\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho_g, \mathbb{C}\otimes E(K)) \neq 0 &\overset{?}{\Leftrightarrow} \mathrm{ord}_{s=1} L(E,\rho_g,s) > 0 \\ &\Leftrightarrow L(f\otimes g,1) = 0 \\ &\Leftrightarrow \mathcal{D}(f,g,1) = 0 \\ &\Leftrightarrow \left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z), f(z)\right\rangle_{2,N} = 0 \\ &\Leftrightarrow \alpha_1 = 0.\end{aligned}$$

Take any $g \in \mathcal{S}_1(\Gamma_0(N_g),\chi)$. Then consider all elliptic curves of conductor N with $N_g|N$. Using the Sage database, we can compute the value α_1 relating to each $L(f\otimes g,1)$ and as a consequence, we can observe if $L(f\otimes g,1) = 0$.

For any $d|\frac{N}{N_g}$, one can also compute $L(f(z)\otimes g(dz),1)$, since $g(dz) \in \mathcal{S}_1(\Gamma_0(dN_g),\chi) \subset \mathcal{S}_1(\Gamma_0(N),\chi)$. However the representation associated to $g(dz)$ using the Deligne-Serre theorem is the same as one associated to $g(z)$. Hence the vanishing order of $L(f(z)\otimes g(dz),s)$ at $s=1$ doesn't give any new information about $\langle \rho_g, \mathbb{C}\otimes_{\mathbb{Z}} E(K)\rangle$.

Proposition 67. *For $\mathcal{R}(s) > \frac{1}{2}$, we have:*

$$\left\langle \widetilde{E}_1'(z,s-1;\chi^{-1};N)g(dz), f(z)\right\rangle_{2,N} = 2\frac{a_d}{d^s}\frac{\Gamma(s)}{(4\pi)^s}\mathcal{D}(f,g,s) \ . \tag{55}$$

In particular at $s=1$, we obtain:

$$\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(dz), f(z)\right\rangle_{2,N} = \frac{a_d}{d}\frac{1}{2\pi}\mathcal{D}(f,g,1) \ .$$

Proof: Using again Rankin's unfolding trick, one can show:

$$\begin{aligned}
\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(dz), f^*(z)\right\rangle_{2,N} &= \int_{y=0}^{\infty}\int_{x=0}^{1} y^{1+s}g(dz)f(-\bar{z})\frac{dxdy}{y^2}\\
&= \int_{y=0}^{\infty}\int_{x=0}^{1} y^{1+s}\left(\sum_{n\geqslant 1} b_n e^{2\pi i dnz}\right)\left(\sum_{m\geqslant 1} a_m e^{-2\pi i m\bar{z}}\right)\frac{dxdy}{y^2}\\
&= \int_{y=0}^{\infty}\int_{x=0}^{1} y^{1+s}\sum_{n,m\geqslant 1} b_n a_m e^{2\pi i n(dx+iy)} e^{-2\pi i m(dx-iy)}\frac{dxdy}{y^2}\\
&= \int_{y=0}^{\infty} y^{1+s}\sum_{n,m\geqslant 1} b_n a_m e^{-2\pi(dn+m)y}\left(\int_{x=0}^{1} e^{2\pi i(dn-m)x}dx\right)\frac{dy}{y^2}
\end{aligned}$$

The integral in the parenthesis is equal to the Kronecker delta $\delta_{(dn,m)}$. So the last line is equal to:

$$\begin{aligned}
&= \int_{y=0}^{\infty} y^{1+s}\sum_{n\geqslant 1} a_{dn}b_n e^{-2\pi(dn+dn)y}\frac{dy}{y^2}\\
&= \sum_{n\geqslant 1} a_{dn}b_n\int_{y=0}^{\infty} y^{s}e^{-4\pi dny}\frac{dy}{y}\\
&= \left(\sum_{n\geqslant 1}\frac{a_{dn}b_n}{(4\pi dn)^s}\right)\int_{0}^{\infty} u^s e^{-u}\frac{du}{u}\ . \qquad (56)
\end{aligned}$$

If $d = p_1^{\alpha_1}p_2^{\alpha_2}...p_v^{\alpha_v}$, since $d|N$ where N is the conductor of the elliptic curve corresponding to f, we have $a_{p_j n} = a_{p_j}a_n$ for any $j = 1, ..., v$, thus $a_{dn} = a_d a_n$ and

$$(56) = \frac{a_d}{d^s}\left(\sum_{n\geqslant 1}\frac{a_n b_n}{(4\pi n)^s}\right)\int_{0}^{\infty} u^s e^{-u}\frac{du}{u}.$$

Putting $s = 1$ gives the required result. ■

We conclude that

$$\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z), f(z)\right\rangle_{2,N} = \frac{a_d}{d}\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z), f(z)\right\rangle_{2,N}. \qquad (57)$$

Thus $\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z), f(z)\right\rangle_{2,N}$ and $\left\langle \widetilde{E}_1'(z;\chi^{-1};N)g(z), f(z)\right\rangle_{2,N}$ vanish together unless $a_d \neq 0$ which can happen for some elliptic curves. The computations done in Sage confirm this formula.

Let $g \in \mathcal{S}_1(\Gamma_0(N_g), \chi)$ where χ is an odd character. Using the Deligne-Serre theorem, we get a 2-dimensional representation $\rho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ associated to g. Let $\tilde{\rho}_g : G_{\mathbb{Q}} \to \mathrm{PGL}_2(\mathbb{C})$ be the projective representation obtained from ρ_g. We say ρ_g is a *dihedral* representation if its image $\mathrm{im}(\tilde{\rho}_g) \subset \mathrm{PGL}_2(\mathbb{C})$ is isomorphic to the dihedral group D_n of order $2n$ for some $n \geqslant 2$. A dihedral representation is irreducible. (see [16])

Let C_n be a cyclic subgroup of D_n of order n. If $n \geqslant 3$, C_n is uniquely determined. The composition:

$$w : G_{\mathbb{Q}} \xrightarrow{\tilde{\rho}_g} D_n \longrightarrow D_n/C_n = \{\pm 1\}$$

can be viewed as a 1-dimensional complex representation of $G_{\mathbb{Q}}$ of order 2. Let $\ker(w) = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ where K is a number field. Since the order of w is 2, the index $[G_{\mathbb{Q}} : \mathrm{Gal}(\overline{\mathbb{Q}}/K)]$ is equal to 2. Hence K is a quadratic extension of $\mathbb{Q}$. Put $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K) \subset G_{\mathbb{Q}}$, then $\tilde{\rho}_g(G_K) \subset C_n$. Since $[D_n : C_n] = [\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \mathrm{Gal}(\overline{\mathbb{Q}}/K)] = 2$, we have $\tilde{\rho}_g(G_K) = C_n$ a cyclic group. Therefore $\rho_g(G_K)$ is an abelian group. Consider

$$G_K \longrightarrow G_K/\ker\rho_g \xrightarrow{\rho_g} \mathrm{GL}_2(\mathbb{C})$$

where ρ_g is denoted also for the induced representation $\rho_g : G_K/\ker\rho_g \to \mathrm{GL}_2(\mathbb{C})$. Since $G_K/\ker\rho_g \cong \rho_g(G_K)$ is abelian, the representation $\rho_g|_{G_K} : G_K/\ker\rho_g \to \mathrm{GL}_2(\mathbb{C})$ is reducible. From this, one can easily see that the representation $\rho_g|_{G_K} : G_K \to \mathrm{GL}_2(\mathbb{C})$ is also reducible. One can write

$$\begin{aligned} \rho_g|_{G_K} :\quad G_K &\to \mathrm{GL}_2(\mathbb{C}) \\ \gamma &\mapsto \begin{pmatrix} \chi(\gamma) & 0 \\ 0 & \chi'(\gamma) \end{pmatrix} \end{aligned}$$

for some 1-dimensional representations χ and χ' of G_K. If σ lies in the non-identity coset of $G_{\mathbb{Q}}/G_K$, then $\chi' = \chi_\sigma$ where:

$$\chi_\sigma(\gamma) = \chi(\sigma\gamma\sigma^{-1}), \quad \gamma \in G_K.$$

Moreover, $\rho_g = \mathrm{Ind}_{K/\mathbb{Q}}(\chi)$.

Suppose, conversely, that we start with a quadratic number field $K/\mathbb{Q}$ corresponding to a character w of $G_{\mathbb{Q}}$ and a 1-dimensional linear representation χ of G_K. Let $\rho = \mathrm{Ind}_{K/\mathbb{Q}}(\chi)$, and let $\tilde{\rho}$ be the associated projective representation of $G_{\mathbb{Q}}$. If σ generates $\mathrm{Gal}(K/\mathbb{Q})$, let χ_σ be as above. Finally, let $\mathfrak{m}$ be the conductor of χ and d_K be the discriminant of K.

Proposition 68. *With the above notation:*
a) The following are equivalent:
i) ρ is irreducible;
ii) ρ is dihedral;
iii) $\chi \neq \chi_\sigma$.
b) The conductor of ρ is $|d_K|.N_{K/\mathbb{Q}}(\mathfrak{m})$.
c) ρ is odd if and only if one of the following holds:
i) K is imaginary.
ii) K is real and χ has signature +,- at infinity, that is, if c and c' are Frobenius elements at the two real places of K then $\chi(c) \neq \chi(c')$.
d) If $\tilde{G_{\mathbb{Q}}} = D_n$, then n is the order of $\chi^{-1}\chi_\sigma$.

Proof: see [16]. ■

Let $\rho = \mathrm{Ind}_{K/\mathbb{Q}}(\chi)$ be a dihedral representation of $G_{\mathbb{Q}}$ where K is imaginary and χ is unramified. Hence, we can view χ as a character of the ideal class group of $\mathcal{O}_K$. For any ideal $\mathfrak{a}$ of $\mathcal{O}_K$, the ideal $\mathfrak{a}.\sigma(\mathfrak{a})$ is principal, so $\chi \neq \chi_\sigma$ if and only if $\chi^2 \neq 1$. Therefore an imaginary quadratic field K gives rise to a dihedral representation of $G_{\mathbb{Q}}$ if its ideal class group is not an elementary abelian 2-group, i.e. $(\mathbb{Z}/2\mathbb{Z})^r$. The smallest value of $|d_K|$ for which this happens is 23. Let CL_K be the ideal class group of K and H be the Hilbert class field of K. There is an isomorphism $\mathrm{Gal}(H/K) \cong \mathrm{CL}_K$. For any character of $\mathrm{Gal}(H/K)$, the induced representation of $\mathrm{Gal}(H/\mathbb{Q})$ is dihedral and irreducible.

We present the results shown in tables 1 to 14 via some examples.

Example 69. *Consider the modular form g of level $p = 23$ discussed in the example 5 of section 1.2. For any elliptic curve of level N with $23|N$, we can compute $L(f \otimes g, 1)$ where f is the modular form arising from an elliptic curve of conductor N.*

There is no elliptic curve of conductor $N = 23$. However for level $N = 2 * 23$, there is one elliptic curve $E = [1, -1, 0, -10, -12]$ (up to isogeny) for which $L(f \otimes g, 1) \neq 0$. One can also consider $L(f(z) \otimes g(2z), 1) \neq 0$ since $g(2z) \in \mathcal{S}_1(\Gamma_0(2 * 23))$. In the table (1), we see that $L(f(z) \otimes g(2z), 1) \neq 0$ (in the column $g(d_1 z)$). This supports the formula (55), since the second fourier coefficient of the modular form attached to E is nonzero ($a_2 = -1$). Then the twisted BSD conjecture implies:

$$\textit{multiplicity of } \rho_g \textit{ in } \mathbb{C} \otimes E(H) \stackrel{?}{=} 0$$

where H is the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$; The class number of $\mathbb{Q}(\sqrt{-23})$ is 3 so $[H : \mathbb{Q}(\sqrt{-23})] = 3$ hence $im(\tilde{\rho}_g) = D_3$.

For $N = 16 * 23$, there are 7 elliptic curves of level N up to isogeny. For two of them, $L(f \otimes g, 1) \neq 0$. For any $d|\frac{N}{23} = 16$, we can also consider $g(dz) \in \mathcal{S}_1(\Gamma_0(16 * 23))$ (in the table 1, the ordered divisors 2,4,8 and 16 are denoted by $d_1 = 2$, $d_2 = 4$, $d_3 = 8$ and $d_4 = 16$ respectively.) Since $a_2 = 0$ for all these 7 elliptic curves, we then have $L(f \otimes g(dz), 1) = 0$. It follows that for these 2 elliptic curves E of conductor $N = 16 * 23 = 368$:

$$\textit{multiplicity of } \rho_g \textit{ in } \mathbb{C} \otimes E(H) \stackrel{?}{=} 0.$$

For the other 5 elliptic curves we have $L(f \otimes g, 1) = 0$, therefore

$$\textit{multiplicity of } \rho_g \textit{ in } \mathbb{C} \otimes E(H) \stackrel{?}{\geqslant} 1.$$

Assuming the BSD conjecture, one has for these 5 elliptic curves:

$$\textit{rank of } E \textit{ over } H \quad = \quad r_H(E) \stackrel{?}{\geqslant} 2.$$

Example 70. *There is only one elliptic curve of conductor $N = 23 * 31 = 693$ up to isogeny. We shall consider two cusp forms arising from theta series:*

$$\begin{aligned} g_1 &\in \mathcal{S}_1(\Gamma_0(23), (\tfrac{-23}{\cdot})) \\ g_2 &\in \mathcal{S}_1(\Gamma_0(31), (\tfrac{-31}{\cdot})). \end{aligned}$$

For both g_1 and g_2 we have:

$$L(f \otimes g_1, 1) = 0$$
$$L(f \otimes g_2, 1) = 0 \ .$$

Therefore assuming the twisted BSD conjecture, one can say:

$$ord_{s=1}L(E, \rho_{g_1}, s) \stackrel{?}{=} \text{ multiplicity of } \rho_{g_1} \text{ in } \mathbb{C} \otimes E(H_1) \geqslant 1$$
$$ord_{s=1}L(E, \rho_{g_2}, s) \stackrel{?}{=} \text{ multiplicity of } \rho_{g_2} \text{ in } \mathbb{C} \otimes E(H_2) \geqslant 1$$

where H_1 (H_2) is the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$ ($\mathbb{Q}(\sqrt{-31})$ respectively). The class number of $\mathbb{Q}(\sqrt{-31})$ is 3 so $[H_2 : \mathbb{Q}(\sqrt{-31})] = 3$ hence $im(\tilde{\rho}_{g_2}) = D_3$.
As an immediate consequence, assuming the BSD conjecture, one has:

$$\begin{aligned} \text{rank of } E \text{ over } H_1 &= r_{H_1}(E) \stackrel{?}{\geqslant} 2 + r_{\mathbb{Q}}(E) = 3 \\ \text{rank of } E \text{ over } H_2 &= r_{H_2}(E) \stackrel{?}{\geqslant} 2 + r_{\mathbb{Q}}(E) = 3. \end{aligned}$$

Example 71. ***(Octahedral type)*** *The space $\mathcal{S}_1(\Gamma_0(283), (\frac{-283}{\cdot}))$ has two cuspforms g_1, g_2 of type S_4 (octahedral) and one cuspform g_3 of type D_3 (dihedral):*

$$\begin{aligned} g_1 &= q + \sqrt{-2}q^2 - \sqrt{-2}q^3 - q^4 - \sqrt{-2}q^5 + 2q^6 - q^7 - q^9 + ... \\ g_2 &= q - \sqrt{-2}q^2 + \sqrt{-2}q^3 - q^4 + \sqrt{-2}q^5 + 2q^6 - q^7 - q^9 + ... \\ g_3 &= q - q^4 - q^7 - q^9 - q^{11} + ... \quad . \end{aligned}$$

Let $\rho_i : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$ be the Galois representation attached to g_i for $i = 1, 2, 3$. Let K be the field corresponding to the kernel of ρ_1 (or equivalently ρ_2) and K' be the field corresponding to the kernel of $\widetilde{\rho}_1$. As before, the field corresponding to the kernel of ρ_3 is $H(\mathbb{Q}(\sqrt{-283}))$, i.e. the Hilbert class field of $\mathbb{Q}(\sqrt{-283})$. Then

$$\mathbb{Q}(\sqrt{-283}) \subset H(\mathbb{Q}(\sqrt{-283})) \subset K' \subset K.$$

*We have $Gal(H(\mathbb{Q}(\sqrt{-283})) : \mathbb{Q}) = S_3 = D_3$ and $Gal(E : \mathbb{Q}) = S_4$. These fields are constructed explicitly as follows. Let $x^3 + 4x - 1 = (x - \alpha)(x - \beta_1)(x - \beta_2)$ where $\alpha \in \mathbb{R}$ and $\beta_2 = \overline{\beta_1}$. Then we have $H = \mathbb{Q}(\alpha, \beta_1, \beta_2)$ and $L = \mathbb{Q}(\sqrt{\alpha}, \sqrt{\beta_1}, \sqrt{\beta_2})$. (see [18]) There is only one elliptic curve E, up to isogeny, of conductor 2*283. Concerning the tables 12 to 14, one can say (again, assuming the BSD conjecture):*

$$ord_{s=1}L(E, \rho_{g_1}, s) \stackrel{?}{=} \text{ multiplicity of } \rho_{g_1} \text{ in } \mathbb{C} \otimes E(K) \geqslant 1$$
$$ord_{s=1}L(E, \rho_{g_2}, s) \stackrel{?}{=} \text{ multiplicity of } \rho_{g_2} \text{ in } \mathbb{C} \otimes E(K) \geqslant 1.$$

Therefore:

$$\text{rank of } E \text{ over } K = r_K(E) \stackrel{?}{\geqslant} 4 + r_{\mathbb{Q}}(E) = 5. \quad \blacksquare$$

Example 72. ***(Octahedral type)*** *There are four newforms on $\Gamma_0(229)$ of weight 1. If g_1, g_2, g_3 and g_4 are these newforms, their first coefficients are:*

$$\begin{aligned} g_1 &= q + q^3 - iq^4 + iq^5 + (i-1)q^7 - iq^{11} - iq^{12} - (1+i)q^{13} + iq^{15} - q^{16} + q^{17} - q^{19} + ... \\ g_2 &= q + (1+i)q^2 - q^3 + iq^4 + iq^5 - (1+i)q^6 + (-1+i)q^{10} - iq^{11} - iq^{12} - iq^{15} + q^{16} \\ &\quad -q^{17} + q^{19} + ... \\ g_3 &= \overline{g_1} \\ g_4 &= \overline{g_2} \quad . \end{aligned}$$

Let χ be the character of order 4 of $(\mathbb{Z}/229\mathbb{Z})^\times$ such that $\chi(2) = i$. Then $g_1, g_2 \in \mathcal{S}_1(\Gamma_0(229), \chi)$ and $g_3, g_4 \in \mathcal{S}_1(\Gamma_0(229), \overline{\chi})$. Let $\rho_i : G_\mathbb{Q} \to GL_2(\mathbb{C})$ be the Galois representation attached to g_i for $i = 1, 2, 3, 4$. They are representations of type S_4 (Octahedral). Let K_i be the field corresponding to the kernel of ρ_i for $i = 1, 2, 3, 4$. If x_1, x_2 and x_3 are the roots of $x^3 - 4x + 1 = 0$, then K_1 is the field generated by the $\sqrt{-3 + 8x_i}$ and K_2 is the field generated by the $\sqrt{4 - 3x_i^2}$ (see [16]). Clearly, $K_3 = K_1$ and $K_4 = K_2$. There are two elliptic curves of conductor $2 * 229$ up to isogeny. Concerning the tables 8 to 11, one can say (again, assuming the BSD conjecture):

$$\begin{aligned} \textit{multiplicity of } \rho_{g_i} \textit{ in } \mathbb{C} \otimes E(K_i) &\stackrel{?}{=} 0 \\ \textit{multiplicity of } \rho_{\overline{g_i}} \textit{ in } \mathbb{C} \otimes E(K_i) &\stackrel{?}{=} 0 \end{aligned}$$

for $i = 1, 2$. ■

If the BSD conjecture is true, one can deduce a more general form of it, namely the twisted BSD conjecture. Using the Deligne-Serre theorem and the Rankin method, we could compute the constant term of $L(E, \rho_g, s)$ at $s = 1$. It is also interesting to find a method in order to compute the coefficients of higher degree and compute the order of $L(E, \rho_g, s)$ at $s = 1$. Then one can compute the rank of E over certain number field K, i.e. $r_K(E)$. The numerical examples in this thesis are done for dihedral representations arising from theta series. It is interesting to find more cusp forms of weight 1 such that the image of its associated projective representation is one of the exceptional groups A_4, S_4 or A_5 and provide more numerical examples. Unfortunately, there seems to be relatively little published regarding explicit computations of weight 1 cusp forms. However, there are some examples in [2], [3], [5], [9], [16] and [18].

Tables

Notation:

$E = [a_1, a_2, a_3, a_4, a_6]$: Elliptic curve E with Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

N: The conductor of E.

d_1, d_2,...are divisors of $\frac{N}{N_g}$ in increasing order where N_g =the level of g.

The column under $g(d_iz)$, $i = 1, 2, ...$, shows the coefficient α_1 in the equation $\widetilde{E}_1'(z; \chi^{-1}; N)g(d_iz) = \alpha_1 f + \alpha_2 f_2 + ... + \alpha_d f_d$ where the modular forms $\widetilde{E}_1'(z; \chi^{-1}; N)$, $f, f_2, ..., f_d$ are as in the text.

For each level N, all elliptic curves, up to isogeny, are listed.

$g(z)=\eta(z)\eta(23z)\in\mathcal{S}_1(\Gamma_0(23),\left(\frac{-23}{\cdot}\right))$, $\mathrm{Im}(\widetilde{\rho}_g)=D_3$, $K=H(\mathbb{Q}(\sqrt{-23}))$								
N	$E=[a_1,a_2,a_3,a_4,a_6]$	$r_{\mathbb{Q}}(E)$	$g(z)$	$g(d_1z)$	$g(d_2z)$	$g(d_3z)$	$g(d_4z)$	$g(d_5z)$
2*23	[1, -1, 0, -10, -12]	0	4/5	$-2/5$				
3*23	[1, 0, 1, -16, -25]	0	0	0				
4*23	[0, 0, 0, -1, 1]	1	1	0	0			
	[0, 1, 0, -18, -43]	0	0	0	0			
5*23	[0, 0, 1, 7, -11]	0	0	0				
6*23	[1, 1, 0, -31, 55]	1	0	0	0	0		
	[1, 0, 1, -771, 1342]	0	0	0	0	0		
	[1, 0, 1, -36, 82]	0	0	0	0	0		
7*23	[1, -1, 1, -124, 560]	0	14/5	2/5				
8*23	[0, -1, 0, 0, 1]	1	3/2	0	0	0		
	[0, -1, 0, -4, 5]	1	0	0	0	0		
	[0, 0, 0, -35, 62]	0	4/3	0	0	0		
	[0, 0, 0, -55, -157]	0	1/6	0	0	0		
9*23	[1, -1, 1, -140, 668]	1	27/8	0	0			
14*23	[1, -1, 0, -238, 1470]	1	0	0	0	0		
	[1, 1, 0, -605, 5117]	0	1/4	-1/8	1/28	-1/56		
	[1, 0, 0, -174, 868]	1	0	0	0	0		
	[1, 1, 1, -14, -23]	0	0	0	0	0		
15*23	[0, 1, 1, -100, 406]	1	45/32	15/32	9/32	3/32		
	[1, 0, 0, -411, -3234]	0	3/2	1/2	-3/10	-1/10		
	[0, -1, 1, -731, -7369]	0	0	0	0	0		
	[0, 1, 1, -1, 1]	1	0	0	0	0		
	[1, 0, 1, -30134, 2010071]	0	3/10	1/10	-3/50	-1/50		
	[0, -1, 1, 30, -97]	0	5/16	-5/48	1/16	-1/48		
16*23	[0, 0, 0, -55, 157]	1	0	0	0	0	0	
	[0, -1, 0, -18, 43]	1	3	0	0	0	0	
	[0, 0, 0, -35, -62]	1	0	0	0	0	0	
	[0, 0, 0, -2723, 54690]	0	0	0	0	0	0	
	[0, 1, 0, 0, -1]	1	0	0	0	0	0	
	[0, 1, 0, -4, -5]	0	1/2	0	0	0	0	
	[0, 0, 0, -1, -1]	0	0	0	0	0	0	
18*23	[1, -1, 0, -2223, -39785]	1	0	0	0	0	0	
	[1, -1, 1, -1532, 23455]	1	0	0	0	0	0	
	[1, -1, 1, -6935, -36241]	0	0	0	0	0	0	
	[1, -1, 1, -284, -1767]	0	0	0	0	0	0	
19*23	[0, -1, 1, 19, 100]	1	0	0				
	[0, -1, 1, 0, -5]	0	0	0				
20*23	[0, -1, 0, -10, 17]	1	5/2	0	0	1/2	0	0
	[0, 0, 0, -8, -12]	0	10/9	0	0	-2/9	0	0
	[0, 1, 0, -46, 529]	1	0	0	0	0	0	0
	[0, 0, 0, -73, 2453]	0	1/20	0	0	-1/100	0	0

Table 1: $g\in\mathcal{S}_1(\Gamma_0(23),\left(\frac{-23}{\cdot}\right))$

$g(z)=\eta(z)\eta(23z)\in\mathcal{S}_1(\Gamma_0(23),\left(\frac{-23}{\cdot}\right))$, $\mathrm{Im}(\widetilde{\rho}_g)=D_3$, $K=H(\mathbb{Q}(\sqrt{-23}))$									
N	$E=[a_1,a_2,a_3,a_4,a_6]$	$r_{\mathbb{Q}}(E)$	$g(z)$	$g(d_1z)$	$g(d_2z)$	$g(d_3z)$	$g(d_4z)$	$g(d_5z)$	$g(d_6z)$	$g(d_7z)$
21*23	[0, 1, 1, 2, 1]	0	0	0	0	0				
	[0, 1, 1, -96, -457]	0	7/50	7/150	-1/50	-1/150				
22*23	[1, 0, 1, -48, -130]	1	33/14	-33/28	-3/14	3/28				
	[1, 0, 1, -397, -3072]	0	3/2	-3/4	3/22	-3/44				
	[1, -1, 0, -290561, 60356981]	0	0	0	0	0				
	[1, -1, 0, -935, 11229]	1	0	0	0	0				
	[1, 0, 0, -86, 292]	1	33/26	33/52	3/26	3/52				
	[1, -1, 1, -4, -1]	1	0	0	0	0				
24*23	[0, -1, 0, -1144, -14516]	1	0	0	0	0	0	0	0	0
	[0, -1, 0, -46648, 3893500]	0	27/28	0	-9/28	0	0	0	0	0
	[0, -1, 0, -752, 6972]	1	3/2	0	-1/3	0	0	0	0	0
	[0, -1, 0, -56, -132]	0	3/4	0	-1/4	0	0	0	0	0
	[0, 1, 0, -2944, 60512]	0	0	0	0	0	0	0	0	0
25*23	[0, 1, 1, -18, 24]	1	75/14	0	0					
	[0, 0, 1, 175, -1344]	1	0	0	0					
	[1, -1, 1, -55, 72]	1	15/2	0	0					
	[1, -1, 0, -2, 1]	1	0	0	0					
	[0, -1, 1, -458, 3943]	1	0	0	0					
26*23	[1, -1, 0, 44, 496]	1	13/10	-13/20	1/10	-1/20				
	[1, -1, 0, -1802, 29898]	1	13/8	-13/16	1/8	-1/16				
	[1, 1, 1, 4, -1443]	1	39/34	39/68	-3/34	-3/68				
	[1, 1, 1, -14, -27]	0	0	0	0	0				
27*23	[1, -1, 1, -14, -16]	1	27/8	0	0	0				
	[1, -1, 0, -123, 548]	0	0	0	0	0				
28*23	[0, -1, 0, 2, -7]	1	7/2	0	0	1/2	0	0		
	[0, 1, 0, 6, -43]	1	0	0	0	0	0	0		
31*23	[1, 0, 1, -1, 1]	1	0	0						
33*23	[1, 1, 1, -1238, -17152]	1	0	0	0	0				
	[1, 0, 0, -93104, -10942305]	1	99/32	33/32	-9/16	-3/16				

Table 2: $g\in\mathcal{S}_1(\Gamma_0(23),\left(\frac{-23}{\cdot}\right))$

$g(z) \in \mathcal{S}_1(\Gamma_0(31), \left(\frac{-31}{\cdot}\right))$, $\mathrm{Im}(\widetilde{\rho}_g) = D_3$, $K = H(\mathbb{Q}(\sqrt{-31}))$								
N	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g(z)$	$g(d_1 z)$	$g(d_2 z)$	$g(d_3 z)$	$g(d_4 z)$	$g(d_5 z)$
2*31	[1, -1, 1, -331, 2397]	0	0	0				
4*31	[0, 1, 0, -2, 1]	1	1	0	0			
	[0, 0, 0, -17, -27]	0	0	0	0			
5*31	[0, -1, 1, -840, -9114]	1	3/2	3/10				
	[1, 1, 1, -26, -62]	0	5/4	-1/4				
	[0, -1, 1, -1, 1]	1	0	0				
6*31	[1, 1, 0, -83, -369]	0	0	0	0	0		
	[1, 0, 1, -17, -28]	0	3/7	-3/14	1/7	-1/14		
	[1, 0, 0, -1395, -20181]	0	0	0	0	0		
8*31	[0, 1, 0, 0, 1]	1	0	0	0	0		
	[0, 1, 0, -32, -32]	0	0	0	0	0		
	[0, 0, 0, 1, -1]	1	1/5	0	0	0		
10*31	[1, 0, 0, -2046, 15376]	1	5/8	5/16	-1/8	-1/16		
	[1, 1, 1, -1066, -13841]	0	0	0	0	0		
12*31	[0, -1, 0, -6, 9]	1	0	0	0	0	0	0
	[0, 1, 0, -2, 9]	1	3/2	0	1/2	0	0	0
	[0, 1, 0, -164, 756]	0	0	0	0	0	0	0
	[0, 1, 0, -250914, -48460347]	0	0	0	0	0	0	0
14*31	[1, -1, 0, -47, 133]	1	0	0	0	0		
	[1, -1, 1, -2364, -43641]	0	0	0	0	0		
	[1, 0, 0, -139, 465]	1	0	0	0	0		
	[1, 0, 0, -3374, -75754]	0	0	0	0	0		
	[1, 1, 1, -522, 4373]	0	0	0	0	0		
15*31	[1, 0, 0, -170, 837]	1	45/16	15/16	9/16	3/16		
	[1, 1, 0, -162, 729]	1	0	0	0	0		
16*31	[0, 0, 0, 1, 1]	1	0	0	0	0	0	
	[0, 0, 0, -5291, -148134]	1	2	0	0	0	0	
	[0, 0, 0, -17, 27]	1	1	0	0	0	0	
	[0, -1, 0, -2, -1]	0	0	0	0	0	0	
	[0, -1, 0, 0, -1]	0	1/2	0	0	0	0	
	[0, -1, 0, -32, 32]	0	1	0	0	0	0	
18*31	[1, -1, 0, -12555, 544887]	1	27/10	-27/20	0	0	0	0
	[1, -1, 0, 0, 2]	1	0	0	0	0	0	0
	[1, -1, 0, -2976, -61750]	0	0	0	0	0	0	0
	[1, -1, 0, -48, 288]	0	9/10	-9/20	0	0	0	0
	[1, -1, 1, -434, -7343]	1	9/10	9/20	0	0	0	0
	[1, -1, 1, -149, 749]	1	0	0	0	0	0	0
	[1, -1, 1, -2, -53]	0	0	0	0	0	0	0
	[1, -1, 1, -752, 9213]	0	9/22	9/44	0	0	0	0
20*31	[0, 0, 0, 8, 4]	1	0	0	0	0	0	0
	[0, 0, 0, -1207, 9006]	1	0	0	0	0	0	0
	[0, 1, 0, -101, 359]	1	5/3	0	0	-1/3	0	0
23*31	[1, 0, 1, -1, 1]	1	0					

Table 3: $g \in \mathcal{S}_1(\Gamma_0(31), \left(\frac{-31}{\cdot}\right))$

$g(z) = \eta(z)\eta(47z) \in \mathcal{S}_1(\Gamma_0(47), \left(\frac{-47}{\cdot}\right))$, $\mathrm{Im}(\widetilde{\rho}_g) = D_5$, $K = H(\mathbb{Q}(\sqrt{-47}))$								
N	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g(z)$	$g(d_1z)$	$g(d_2z)$	$g(d_3z)$	$g(d_4z)$	$g(d_5z)$
2*47	[1, -1, 1, -10, -9]	0	0	0				
3*47	[0, 1, 1, -12, 2]	1	-6/7	-2/7				
	[1, 1, 1, -143, -718]	0	1/2	-1/6				
	[1, 0, 0, -752, 7875]	0	0	0				
	[0, -1, 1, -1, 0]	1	0	0				
	[0, 1, 1, -26, -61]	0	0	0				
5*47	[1, 1, 1, -3551, -82926]	0	0	0				
	[1, 1, 1, -5, 0]	1	0	0				
	[0, -1, 1, 4, 1]	0	0	0				
6*47	[1, 1, 1, -255, 1461]	1	3/16	3/32	-1/16	-1/32		
	[1, 1, 1, -3502, -81181]	0	0	0	0	0		
7*47	[1, 1, 1, 246, -1376]	0	-49/90	7/90				
9*47	[0, 0, 1, -9, 10]	1	0	0	0			
	[0, 0, 1, -237, 1404]	1	-33/16	0	0			
	[0, 0, 1, -12, 4]	1	9/16	0	0			
	[1, -1, 0, -6768, -212625]	1	3/2	0	0			
	[1, -1, 0, -1287, 18094]	0	0	0	0			
	[0, 0, 1, -111, -171]	0	0	0	0			
	[0, 0, 1, -81, -277]	0	9/16	0	0			
10*47	[1, 1, 0, -97, 281]	1	0	0	0	0		
	[1, 0, 1, -44, 106]	1	-5/4	5/8	1/4	-1/8		
	[1, 0, 1, -6348, 132618]	0	-3/2	3/4	-3/10	3/20		
	[1, -1, 1, -117, 141]	1	5/7	5/14	1/7	1/14		
	[1, 1, 1, -11, 9]	1	0	0	0	0		
	[1, 0, 0, -176, -844]	0	0	0	0	0		
11*47	[0, 0, 1, -16, -26]	0	-11/36	-1/36				
	[0, -1, 1, -52, -3863]	1	0	0				
	[0, -1, 1, 36, -3]	0	0	0				
12*47	[0, -1, 0, -221, -1191]	1	4/5	0	-4/15	0	0	0
	[0, 1, 0, -517, -4681]	1	0	0	0	0	0	0
13*47	[0, 0, 1, -1, 1]	0	0	0				

Table 4: $g \in \mathcal{S}_1(\Gamma_0(47), \left(\frac{-47}{\cdot}\right))$

$g(z) = \eta(z)\eta(71z) \in \mathcal{S}_1(\Gamma_0(71), \left(\frac{-71}{\cdot}\right))$,	$\mathrm{Im}(\widetilde{\rho}_g) = D_7$,	$K = H(\mathbb{Q}(\sqrt{-71}))$				
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g(z)$	$g(d_1 z)$	$g(d_2 z)$	$g(d_3 z)$
2*71	[1, 1, 0, -1, -1]	1	0	0		
	[1, -1, 0, -41, -91]	0	-4/9	2/9		
	[1, -1, 0, -2626, 52244]	0	2/81	-1/81		
	[1, -1, 1, -12, 15]	1	-2/9	-1/9		
	[1, 0, 0, -58, -170]	0	0	0		
3*71	[1, 0, 1, -15, 19]	0	0	0		
5*71	[0, 1, 1, -95, -396]	0	0	0		
6*71	[1, 1, 0, -286, 1780]	1	0	0	0	0
	[1, 0, 1, -23007, 1341682]	0	0	0	0	0
	[1, 0, 0, -230, -5202]	0	0	0	0	0
7*71	[1, 1, 0, 25, -14]	1	0	0		
8*71	[0, -1, 0, -72, -212]	0	-2/5	0	0	0
9*71	[1, -1, 1, -131, -520]	0	15/8	0	0	0
11*71	[0, 0, 1, -1378, 347]	0	0	0		
	[0, 0, 1, -808, 8840]	1	0	0		

Table 5: $g \in \mathcal{S}_1(\Gamma_0(71), \left(\frac{-71}{\cdot}\right))$

$g(z) = \eta(z)\eta(167z) \in \mathcal{S}_1(\Gamma_0(167), \left(\frac{-167}{\cdot}\right))$,	$\mathrm{Im}(\widetilde{\rho}_g) = D_{11}$,	$K = H(\mathbb{Q}(\sqrt{-167}))$		
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g(z)$	$g(d_1 z)$
2*167	[1, -1, 1, -1, -1]	0	0	0
3*167	[1, 1, 0, -12, -15]	0	6/23	-2/23
7*167	[1, -1, 0, -1, 2]	0	2/7	2/49

Table 6: $g \in \mathcal{S}_1(\Gamma_0(167), \left(\frac{-167}{\cdot}\right))$

$g(z) = \eta(z)\eta(191z) \in \mathcal{S}_1(\Gamma_0(191), \left(\frac{-191}{\cdot}\right))$,	$\mathrm{Im}(\widetilde{\rho}_g) = D_{13}$,	$K = H(\mathbb{Q}(\sqrt{-191}))$		
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g(z)$	$g(d_1 z)$
3*191	[0, 1, 1, -4, -2]	0	1/5	1/15
5*191	[1, -1, 1, -16663, 832042]	0	23/44	-23/220

Table 7: $g \in \mathcal{S}_1(\Gamma_0(191), \left(\frac{-191}{\cdot}\right))$

$g_1 \in \mathcal{S}_1(\Gamma_0(229), \chi)$ where $\chi(2) = i$, $\mathrm{Im}(\widetilde{\rho}_{g_1}) = S_4$				
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g_1(z)$	$g_1(d_1 z)$
2*229	[1, -1, 0, -19, 37]	1	-(1/12)i	(1/24)i
	[1, 1, 1, -16, -15]	1	-3/10	-3/20
5*229	[1, 0, 0, -596, 5551]	1	-40/87	8/87

Table 8: $g_1 \in \mathcal{S}_1(\Gamma_0(229), \chi)$

$g_2 \in \mathcal{S}_1(\Gamma_0(229), \chi)$ where $\chi(2) = i$, $\mathrm{Im}(\widetilde{\rho}_{g_2}) = S_4$				
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g_2(z)$	$g_2(d_1 z)$
2*229	[1, -1, 0, -19, 37]	1	(1/12)(1-i)	(-1/24)(1-i)
	[1, 1, 1, -16, -15]	1	(-1/2)(1+i)	(-1/4)(1+i)
5*229	[1, 0, 0, -596, 5551]	1	(10/87)(1-i)	(-2/87)(1-i)

Table 9: $g_2 \in \mathcal{S}_1(\Gamma_0(229), \chi)$

$g_3 = \overline{g_1} \in \mathcal{S}_1(\Gamma_0(229), \overline{\chi})$ where $\chi(2) = i$, $\mathrm{Im}(\widetilde{\rho}_{g_3}) = S_4$				
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g_3(z)$	$g_3(d_1 z)$
2*229	[1, -1, 0, -19, 37]	1	(1/12)i	-(1/24)i
	[1, 1, 1, -16, -15]	1	-3/10	-3/20
5*229	[1, 0, 0, -596, 5551]	1	-40/87	8/87

Table 10: $g_3 \in \mathcal{S}_1(\Gamma_0(229), \overline{\chi})$

$g_4 = \overline{g_2} \in \mathcal{S}_1(\Gamma_0(229), \overline{\chi})$ where $\chi(2) = i$, $\mathrm{Im}(\widetilde{\rho}_{g_4}) = S_4$				
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g_4(z)$	$g_4(d_1 z)$
2*229	[1, -1, 0, -19, 37]	1	(1/12)(1+i)	(-1/24)(1+i)
	[1, 1, 1, -16, -15]	1	(-1/2)(1-i)	(-1/4)(1-i)
5*229	[1, 0, 0, -596, 5551]	1	(10/87)(1+i)	(-2/87)(1+i)

Table 11: $g_4 \in \mathcal{S}_1(\Gamma_0(229), \overline{\chi})$

$g_1 \in \mathcal{S}_1(\Gamma_0(283), \left(\frac{-283}{\cdot}\right))$, $\mathrm{Im}(\widetilde{\rho}_{g_1}) = S_4$				
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g_1(z)$	$g_1(d_1 z)$
2*283	[1, -1, 0, -2, 4]	1	0	0
3*283	[1, 0, 0, 1, -1]	0	1/13	1/39

Table 12: $g_1 \in \mathcal{S}_1(\Gamma_0(283), \left(\frac{-283}{\cdot}\right))$

$g_2 \in \mathcal{S}_1(\Gamma_0(283), \left(\frac{-283}{\cdot}\right))$, $\mathrm{Im}(\widetilde{\rho}_{g_2}) = S_4$				
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g_2(z)$	$g_2(d_1 z)$
2*283	[1, -1, 0, -2, 4]	1	0	0
3*283	[1, 0, 0, 1, -1]	0	1/13	1/39

Table 13: $g_2 \in \mathcal{S}_1(\Gamma_0(283), \left(\frac{-283}{\cdot}\right))$

$g_3 \in \mathcal{S}_1(\Gamma_0(283), \left(\frac{-283}{\cdot}\right))$, $\mathrm{Im}(\widetilde{\rho}_{g_3}) = S_3$, $K = H(\mathbb{Q}(\sqrt{-283}))$				
$\boldsymbol{N}$	$E = [a_1, a_2, a_3, a_4, a_6]$	$r_{\mathbb{Q}}(E)$	$g_3(z)$	$g_3(d_1 z)$
2*283	[1, -1, 0, -2, 4]	1	1/2	-1/4
3*283	[1, 0, 0, 1, -1]	0	1/13	1/39

Table 14: $g_3 \in \mathcal{S}_1(\Gamma_0(283), \left(\frac{-283}{\cdot}\right))$

Codes

```
sage: Nf=3*144   # the level of the modular form f arising from the elliptic curve EC
sage: Ng=144     # the level of the modular form g of weight 1
sage: etiquette=0    # index for the elements of the basis S_1(Gamma_1(Ng))

sage: BoP=53  ## bits of precision

sage: S = CuspForms(Gamma0(Nf),2,base_ring =ComplexField());
sage: bound=S.sturm_bound()
sage: M = ModularForms(Gamma0(Nf),2,base_ring =ComplexField());
sage: m= ModularForms(Gamma0(Nf),2).dimension()
sage: d= CuspForms(Gamma0(Nf),2).dimension()
sage: S.set_precision(bound)
sage: BS= S.basis()
sage: M.set_precision(bound)
sage: BM= M.basis()

sage: EC = EllipticCurve([1, 1, 0, -12, -15]);   # elliptic curve of conductor Nf
sage: Erank= EC.rank();
sage: EConductor= EC.conductor();
sage: f = EC.modular_form();

# computing the coefficients of the Eisenstein series E for dihedral representation
# the character is not supposed to be primitive
sage: def EisensteinCoeffP(n,Level,Mod):
...       l=Level/Mod
...       if n==0:
...          sum = (l)*quadratic_L_function__exact(0, -Mod)/2
...          Div = prime_divisors(Level/Mod)
...          for i in range (0, len(Div)):
...                sum= sum * (1-kronecker(Div[i],Ng)/Div[i] )
...          return sum
...       sum=0
...       for c in range(1,n+1):
...           if (n % c == 0):
...              for d in range(1,GCD(l,c)+1):
...                  if GCD(l,c) % d  == 0:
...                     sum = sum+d*moebius(l/d)*kronecker(l/d,Ng)*kronecker(c/d,Ng)
...       return sum

sage: def EisensteinP(Level, Mod, prec):
...       if Level % Mod <> 0:
...          return false
...       R.<q> = PowerSeriesRing(ComplexField(BoP))
...       E=0
...       for h in range(0,prec+1):
...           E = E + EisensteinCoeffP(h,Level,Mod)*q^(h)
...       return E + O(q^prec)

sage: def EisensteinCoeff(n,Level,Mod, char):
```

```
...        const=0
...        if n==0:
...           for a in range (0,Mod):
...               const= const + a * char(a)
...           return -const/(2*Mod)
...        sum=0
...        for d in range(1,n+1):
...            if n % d == 0:
...               sum = sum + char(d)
...        return sum

sage: def Eisenstein(Level,Mod , char, prec):
...        R.<q> = PowerSeriesRing(ComplexField(BoP))
...        E=0
...        for h in range(0,prec):
...            E = E + EisensteinCoeff(h,Level,Mod,char) * q^( (Level/Mod) * h)
...        return E + O(q^prec)

# computing the coefficients of g of level 283
sage: def CoeffIter283(n, etiquette):
...       L.<z> = NumberField(x^2+1)
...       RootI=L.complex_embeddings()[1](z)
...       K.<w> = NumberField(x^2-2)
...       Sq2=K.complex_embeddings()[1](w)
...       C= Matrix([[2 , 3 , 5 , 7 , 11 , 13 , 17 , 19 , 23 , 29 , 31 , 37 , 41 , 43 , 47 ,
53 , 59 , 61 , 67 , 71 , 73 , 79 , 83 , 89 , 97 , 101 , 103 , 107 , 109 , 113 ,
127 , 131 , 137 , 139 , 149 , 151 , 157 , 163 , 167 , 173 , 179 , 181 , 191 ,  193 , 197 ,
283 , 293 , 307 , 311 , 313 , 317, 331, 337, 347, 349 ,353 ,359, 367, 373, 379, 383,
389, 397, 401, 409 ],
[-Sq2*RootI, Sq2*RootI, Sq2*RootI, -1, 1, 1, 0, -Sq2*RootI, -1, -1,
-Sq2*RootI, 0, 1, -Sq2*RootI, Sq2*RootI, 0,1, 1, 0, 0, 0, 0 , -2, -1, 1 ,
0 , -1 , 0  , Sq2*RootI , 0, 0 , 0 , 1 , Sq2*RootI , -Sq2*RootI ,
-1 , -1 , -1 , Sq2*RootI , -Sq2*RootI , 1 , 0 , 0 , 0 , 0, 1 , -1 , Sq2*RootI , 0 , 0 ,
-1 , 0 , 0 , 1 , -1,  -1 , -1 , 1 , -Sq2*RootI , 0, 1 , 1, 1, 0 , -Sq2*RootI, 1, 0 , 1 , 0, -1,
1, 0, 0, -1, 1,0, -1, Sq2*RootI  , 0, 0]] )
...       D = Matrix([[2 , 3 , 5 , 7 , 11 , 13 , 17 , 19 , 23 , 29 , 31 , 37 , 41 , 43 , 47 ,
53 , 59 , 61 , 67 , 71 ,73 , 79 , 83 , 89 , 97 ,101 , 103 , 107 , 109 , 113 ,
127 , 131 , 137 , 139 , 149 , 151 , 157 , 163 , 167 , 173 , 179 , 181 , 191 ,  193 , 197 ,
199 , 211 , 223 , 227 , 229 , 233 , 239 , 241 , 251 , 257, 263 , 269 , 271 , 277 , 281 ,
283 , 293 , 307 , 311 , 313 ],
[0,0,0,-1,-1, -1, 0, 0, -1, -1,0,0,-1 ,0 ,0,0,-1,-1,0,2,2,0,2,-1,-1,2,-1,0,0,2, 2,0,-1,0,0,
-1,-1,-1,0,0, -1,2,0,0,0,-1,-1,0,2,-0 ,-1,0,0,-1,-1 , -1,-1,-1,0,2 ,2,-1,-1,2,0 ]] )
...       if n==1:
...          return 1
...       if n in Primes():
...          for k in range (0,65):
...              if etiquette<2:
...                 if C[0][k]==n:
...                    return C[1][k]
...              else:
...                 if D[0][k]==n:
...                    return D[1][k]
```

```
...      else:
...         F=factor(n)
...         l=len(F)
...         r=1
...         if l==1:
...            if F[0][1]==1:
...               return CoeffIter283(F[0][0], etiquette) *
 CoeffIter283(F[0][0]^(F[0][1]-1), etiquette) - kronecker(-283,F[0][0])
...            if F[0][1]>1:
...               return CoeffIter283(F[0][0], etiquette) *
CoeffIter283(F[0][0]^(F[0][1]-1), etiquette) -
kronecker(-283,F[0][0])*CoeffIter283(F[0][0]^(F[0][1]-2), etiquette)
...         else:
...            for i in range (0,len(F)):
...               r = r * CoeffIter283(F[i][0]^F[i][1], etiquette)
...            return r


# computing the coefficients of g
sage: def g(Level, Mod):
...       R.<q> = PowerSeriesRing(ComplexField(BoP))
...       l = bound
...       a=Level/ Mod
...       Ng=Mod
...       g=0
...       if Ng % 24 == 23:
...           if etiquette==0:
...              for m in range (-l,l):
...                 for n in range (-l,l):
...                    if 0<(Ng+1)/24*m^2+m*n+6*n^2 < l:
...                        g = g + (1/2)* q^(a*((Ng+1)/24*m^2+m*n+6*n^2));
...                    if 0<(Ng+25)/24*m^2+5*m*n+6*n^2 < l:
...                        g = g - (1/2)* q^(a*((Ng+25)/24*m^2+5*m*n+6*n^2));
...       if etiquette==1:
...           if Ng==47:
...              g= q - q^3 - q^6 - q^8 + q^9 + O(q^12)
...              e = DirichletGroup(47)
...              psi=e.list()
...              g= ComputeCoeff( g, Eisenstein(Ng,Ng, psi[23], bound) ,bound)
...              if Level>Mod:
...                 gP=0
...                 for r in range (1,(bound/a)+1):
...                     gP = gP + g[r] * q^(a*r);
...                 g=gP
...       if etiquette==0:
...           if Ng==229:
...              e = DirichletGroup(229)
...              psi=e.list()
...              L.<z> = NumberField(x^2+1)
...              RootI=L.complex_embeddings()[0](z)
...              g= q + q^3 - RootI*q^4 + RootI*q^5 + (RootI - 1)*q^7 - RootI*q^11
 - RootI*q^12 +(-RootI - 1)*q^13 +RootI*q^15 - q^16 + q^17 - q^19 + q^20
```

```
  + (RootI - 1)*q^21 + (RootI + 1)*q^23 - q^27 + (RootI + 1)*q^28 -RootI*q^33
  + (-RootI - 1)*q^35 + (-RootI - 1)*q^39 + q^43 - q^44 + (RootI - 1)*q^47 - q^48
  - RootI*q^49 + q^51 + (RootI - 1)*q^52 + 2*q^53 + q^55 +
 (0)*(RootI + 1)*(q^2 + (RootI - 1)*q^3 + (RootI + 1)*q^4 - q^6 - RootI*q^7
 + RootI*q^10 + q^13 + (-RootI - 1)*q^15 + (-RootI + 1)*q^16 + (RootI - 1)*q^17
 + (-RootI + 1)*q^19 + (RootI - 1)*q^20 - RootI*q^21 - RootI*q^22 - 2*q^23 +
 (-RootI + 1)*q^27 - q^28 - RootI*q^30 - q^31 + q^32 + (RootI + 1)*q^33 - q^34
 + q^35 + q^38 + q^39 + RootI*q^41 + (RootI - 1)*q^43 + (-RootI + 1)*q^44+
(-RootI - 1)*q^46 + (RootI + 1)*q^49 - RootI*q^52 + (RootI - 1)*q^53 + q^54) +O(q^57)
...               g= ComputeCoeff( g, Eisenstein(Ng,Ng, psi[57], bound) ,bound)
...               if Level>Mod:
...                  gP=0
...                  for r in range (1,(bound/a)+1):
...                      gP = gP + g[r] * q^(a*r);
...                  g=gP
...        if Ng==144:
...            for m in range (-20,1+20):
...               for n in range (-20,1+20):
...                   if m%3==1:
...                      if n%3==0:
...                         if (m+n)%2==1:
...                            g = g + ((-1)^n)*q^(a*(m^2+n^2))
...        if Ng==31:
...            for m in range (-l,l):
...               for n in range (-l,l):
...                  if 0<(m^2+m*n+8*n^2) < l:
...                      g = g + (1/2)* q^(a*(m^2+m*n+8*n^2));
...                  if 0<2*m^2+m*n+4*n^2 < l:
...                      g = g - (1/2)* q^(a*(2*m^2+m*n+4*n^2));
...        if etiquette==0:
...           if Ng==124:
...              e = DirichletGroup(124)
...              psi=e.list()
...              L.<z> = NumberField(x^2-x+1)
...              zeta = L.complex_embeddings()[1](z)
...              g= q - q^4 + (zeta - 1)*q^5 - zeta*q^6 + (-zeta + 1)*q^13
+ zeta*q^14 + q^16 -zeta*q^17 + (-zeta + 1)*q^20 + (zeta - 1)*q^21 + (zeta - 1)*q^22
 + zeta*q^24 + q^30 - q^33 - zeta*q^37 - zeta*q^38 + (zeta - 1)*q^41
+ i* (q^2 + zeta^5*q^3 - zeta^5*q^7 - q^8 + (zeta^5 - 1)*q^10 + (-zeta^5 + 1)*q^11 -
zeta^5*q^12 - q^15 + zeta^5*q^19 + (-zeta^5 + 1)*q^26 - q^27 + zeta^5*q^28 -
q^31 + q^32 - zeta^5*q^34 + q^35 + q^39 + (-zeta^5 + 1)*q^40 + (zeta^5 - 1)*q^42
+ zeta^5*q^43 + (zeta^5 - 1)*q^44) + O(q^45)
...              g= ComputeCoeff( g, Eisenstein(p,p, psi[41], bound) ,bound)
...              if Level>Mod:
...                 gP=0
...                 for r in range (1,(bound/a)+1):
...                     gP = gP + g[r] * q^(a*r);
...                 g=gP
...        if Ng==283:
...           for r in range (1,l):
...               g = g + CoeffIter283(r,etiquette) * q^(a*r);
...        return g + O(q^l)
```

```
# computing all coefficients of g smaller than "bound"
sage: def ComputeCoeff(g,E,bound):
...      R.<q> = PowerSeriesRing(ComplexField(BoP),bound)
...      Sp = CuspForms(Gamma0(p),2,base_ring =ComplexField());
...      Basisp= Sp.basis()
...      dp= CuspForms(Gamma0(p),2).dimension()
...      boundp=2*Sp.sturm_bound()
...      gE1=g*E
...      gE=0
...      for i in range(1,boundp+1):
...          for j in range (0,dp):
...              if Order(Basisp[j],boundp)== i:
...                 gE = gE + gE1[i] * Basisp[j]
...      f1=0
...      f2=E[0]
...      for r in range (1,bound):
...         f1 = f1 + gE[r] * q^(r);
...         f2 = f2 + E[r] * q^(r);
...      v=f1/f2
...      return v

sage: def Order (f,s):
...     Ord=1
...     for i in range (1,s):
...        if f[i]==1:
...           return Ord
...        else:
...           Ord=Ord+1

sage: def MulBy(f, a):
...       R.<q> = PowerSeriesRing(ComplexField(BoP))
...       h=0
...       for i in range(1,(bound/a)+1):
...           for j in range (0,d):
...               if Order(BS[j],bound)== a * i:
...                  h = h + f[i] * BS[j]
...       return h

#test: gE is a modular form?
sage: def Test(Series, Modularform):
...       Cr=0
...       for i in range (0, bound):
...           if abs( Modularform[i] - Series[i]) > 1.1e-6:
...              Cr=1
...       if Cr==1:
...          print "gE is not a modular form"

sage: MatSpa = MatrixSpace(ComplexField(BoP),d)
sage: BasisMS = MatSpa.basis()
sage: VecSpa = VectorSpace(QQbar,bound)
```

```
sage: W = matrix(ComplexField(BoP),d,d)
sage: B = matrix(ComplexField(BoP),d,bound)
sage: OrtA=B.new_matrix()
sage: BB = matrix(ComplexField(BoP),3*d,bound)
sage: Alak=BB.new_matrix()

sage: NewF=Newforms(Gamma0(Nf),2, names='alpha')
sage: l=len(NewF)
sage: redundant=0
sage: counter=0
sage: cs=0

#Computing a basis of newforms for S_{2}(Gamma_{Nf})
sage: D= divisors(Nf)
...    for z in range (0,len(D)):
...        Nd= Nf/D[z]
...        NewF=Newforms(Gamma0(Nd),2, names='alpha')
...        l=len(NewF)
...        Mult= divisors (D[z])
...        MultLen=len(Mult)
...        for i in range (0,l):
...           K=NewF[i].base_ring()
...           redundant=0
...           if K.degree() > 1:
...              u=len (K.complex_embeddings());
...              for j in range (0,u):
...                 h=0
...                 for s in range (0,d):
...                     Ord=Order(BS[s],bound )
...                     t=K.complex_embeddings()[j](NewF[i][Ord])
...                     h= h + t*BS[s]
...                 for v in range(cs,counter):
...                     if OrtA.row(v)==h.coefficients(bound):
...                        redundant=1
...                 if redundant==0:
...                    OrtA.set_row(counter, h.coefficients(bound))
...                    counter = counter+1
...                 if MultLen>1:
...                    for r in range (1,MultLen):
...                       if redundant==0:
...                           OrtA.set_row(counter, MulBy(h , Mult[r]).coefficients(bound))
...                           counter = counter+1
...                 cs=counter
...           else:
...              OrtA.set_row(counter, NewF[i].coefficients(bound) )
...              counter = counter+1
...              if MultLen>1:
...                 for r in range (1,MultLen):
...                    OrtA.set_row(counter, MulBy(NewF[i] , Mult[r]).coefficients(bound) )
...                    counter = counter+1
...              cs=counter
```

```
sage: Columns = vector([ Order(BS[i], bound) for i in range (0,d)])
sage: G=W.new_matrix()
sage: for j in range (0,d):
...      G.set_column(j, OrtA.column(Columns[j]-1))

sage: fVector = vector ([ f[Columns[i]] for i in range (0,d)])
sage: Y=vector(ComplexField(BoP), d)
sage: for j in range (0,d):
...      Y.set(j, fVector[j])

# Set the Eisenstein series E in order to compute \alpha_{1} in the formula
#  <Eg,f>=\alhpa_{1}f+\alpha_{2}f_{2}+...+\alpha_{d}f_{d}
sage: if Ng==144:
...      e = DirichletGroup(144)
...      psi=e.list()
# Eisenstein series for Nf=3*144
...      E= 1*Eisenstein(Nf,Ng, psi[1], bound)
            -(psi[1](3))*Eisenstein(Nf,Ng, psi[1], bound)

sage: if Ng==283:
...      e = DirichletGroup(283)
...      psi=e.list()
# Eisenstein series for Nf=2*283
...      E= Eisenstein(Nf,Ng, psi[141], bound)
            - (psi[141](2)/2)*Eisenstein(Ng,Ng, psi[141], bound)

sage: if Ng==229:
...      e = DirichletGroup(229)
...      psi=e.list()
# Eisenstein series for Nf=2*229
...      E= 2*Eisenstein(Nf,Ng, psi[57], bound)
            - (psi[57](2)/1)* Eisenstein(Nf,Ng, psi[57], bound)

sage: if Ng==124:
...      e = DirichletGroup(124)
...      psi=e.list()
...      E= 1*Eisenstein(Nf,Ng, psi[41], bound)

sage: if Ng==47:
...      if etiquette==1:
...         e = DirichletGroup(47)
...         psi=e.list()
...         E= EisensteinP(Nf, Ng, bound)

#sage: E= EisensteinP(Nf, Ng, bound)
sage: print "E=",E

# computing the inverse of G by a numerical method
sage: Id=identity_matrix(d)
sage: GInv=transpose(G)/(G.norm(1)*G.norm(Infinity));
sage: for i in range(0,40):
...      GInv=GInv*(2*Id-G*GInv)
```

```
# Computing the coefficient \alpha_{1}
sage: Div= divisors(Nf/Ng)
sage: Output = Matrix(ComplexField(BoP),4*len(Div),d);
sage: for i in range (0, len(Div)):
...       print "Div[i]*Ng=", Div[i]*Ng
...       Z=vector(ComplexField(BoP), d)
...       print "g=", g(Div[i]*Ng , Ng)
...       gE1= E * g(Div[i]*Ng , Ng)
...       gE = MulBy(gE1,1)
...       Test(gE1, gE)
...       gEVector = vector([ gE[Columns[i]] for i in range (0,d)])
...       for j in range (0,d):
...           Z.set(j, gEVector[j])
...       X=Z*GInv
...       print (X[0].real()).nearby_rational(max_error=0.00001)
# X[0],X[1],... correspond to \alpha_{1} related to elliptic curves of conductor Nf
```

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