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Doctoral Thesis

A Shimura-Shintani correspondence for rigid analytic cocycles of higher weight

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Abstract

Let p be a prime number and let $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$. In their paper *Singular moduli for real quadratic fields: a rigid analytic approach*, Darmon and Vonk introduced rigid meromorphic and analytic cocycles, which can be seen as Γ-invariant modular symbols with values in the Γ-module of rigid meromorphic and analytic functions on Drinfeld's p-adic upper half-plane. Let $k \geq 3$ be an odd integer. We construct a Shimura-Shintani style correspondence $\mathcal{C}$ from a certain $\bar{\mathbb{Q}}$-space of weight $k + 1/2$ cusp forms to the space of rigid analytic cocycles of weight $2k$.

The classical Shimura-Shintani correspondence can be given by pairing modular forms of weight $2k$ (resp. $k + 1/2$) with a holomorphic kernel function $\Omega_k(z, \tau)$ via the Petersson inner product, getting modular forms of weight $k + 1/2$ (resp. $2k$). The function $\Omega_k(z, \tau)$ is a modular form of weight $k + 1/2$ as a function of τ and its Fourier coefficient for any $D > 0$ is a certain weight $2k$ cusp form $f_{k,D}(z)$ for $\mathrm{SL}_2(\mathbb{Z})$.

We at first classify certain rigid meromorphic cocycles of weight $2k$, then for any positive discriminant D we define a weight $2k$ rigid analytic cocycle $J_{k,D}$. These cocycles are p-adic analogues of the forms $f_{k,D}$ and the correspondence $\mathcal{C}$ is given by constructing a weight $k + 1/2$ cusp form $\hat{\Omega}_k(q)$ with coefficients in the space of weight $2k$ rigid analytic cocycles. The D-th coefficient of $\hat{\Omega}_k(q)$ is essentially $J_{k,D}$ if D is not a square.

Our strategy is the following. We define at first certain level p counterparts for the cusp forms $f_{k,D}$, i.e. we define forms $f_{k,D}^{(p)} \in S_{2k}(\Gamma_0(p))$. The forms $f_{k,D}^{(p)}$ are essentially the D-th coefficient of a weight $k + 1/2$ cusp form $\bar{\Omega}_k(q)$ with coefficients in $S_{2k}(\Gamma_0(p))$. We construct $\hat{\Omega}_k(q)$ by retrieving $J_{k,D}$ from $f_{k,D}^{(p)}$ via two linear maps. The output of the first map is the modular symbol given by the periods of $f_{k,D}^{(p)}$. When computing these periods, we get a result which is an analogue of a result of Kohnen and Zagier on the periods of $f_{k,D}$. The second map is a Schneider-Teitelbaum lift for rigid analytic cocycles. This map first appeared in the work of Schneider and Teitelbaum and was exploited in the setting of rigid analytic cocycles of weight two by Darmon and Vonk. We extend this construction to higher weight.

Abrégé

Soit p un nombre premier et soit $\Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p])$. Dans l'article *Singular moduli for real quadratic fields: a rigid analytic approach*, Darmon et Vonk ont introduit les cocycles rigides méromorphes et analytiques, qui peuvent s'interpréter comme des symboles modulaires invariant pour l'action du groupe Γ et à valeurs dans le Γ-module des fonctions rigides méromorphes et analytiques sur le demi-plan supérieur p-adique de Drinfeld. Soit $k \geq 3$ un entier impair. Nous construisons une correspondance $\mathcal{C}$ à la Shimura-Shintani entre un certain $\bar{\mathbb{Q}}$-espace vectoriel de formes paraboliques de poids $k + 1/2$ et l'espace des cocycles rigides analytiques de poids $2k$.

La correspondance classique de Shimura-Shintani peut se définir comme le produit scalaire de Petersson entre une forme modulaire de poids $2k$ (resp. $k + 1/2$) avec une certaine fonction $\Omega_k(z, \tau)$, donnant lieu à une forme modulaire de poids $k + 1/2$ (resp. $2k$). La fonction $\Omega_k(z, \tau)$ est une forme modulaire de poids $k + 1/2$ pour la variable τ et son coefficient de Fourier pour $D > 0$ est une certaine forme parabolique $f_{k,D}(z)$ pour $\mathrm{SL}_2(\mathbb{Z})$ de poids $2k$.

Dans un premier temps, nous classifions certains cocycles rigides méromorphes de poids $2k$, ensuite nous définissons des cocycles rigides analytiques $J_{k,D}$ de poids $2k$ pour tout discriminant D positif. Ces cocycles sont des analogues p-adiques des formes $f_{k,D}$, et la correspondance $\mathcal{C}$ est définie à l'aide d'une forme parabolique $\hat{\Omega}_k(q)$ de poids $k + 1/2$ à coefficients dans l'espace de cocycles rigides analytiques de poids $2k$. Le D-ième coefficient de $\hat{\Omega}_k(q)$ est essentiellement $J_{k,D}$, si D n'est pas un carré parfait.

La stratégie est la suivante. Nous définissons d'abord des formes $f_{k,D}^{(p)} \in S_{2k}(\Gamma_0(p))$ qui sont des analogues de niveau p pour les formes paraboliques $f_{k,D}$. Les formes $f_{k,D}^{(p)}$ sont essentiellement le D-ième coefficient d'une certaine forme parabolique $\bar{\Omega}_k(q)$ de poids $k + 1/2$ à coefficients dans $S_{2k}(\Gamma_0(p))$. Nous construisons $\hat{\Omega}_k(q)$ en obtenant $J_{k,D}$ à partir de $f_{k,D}^{(p)}$ via deux applications linéaires. L'image de la première application est le symbole modulaire donné par les périodes de $f_{k,D}^{(p)}$. Le calcul de ces périodes, nous permet d'obtenir un résultat analogue à un résultat de Kohnen et Zagier sur les périodes de $f_{k,D}$. La seconde application est un relèvement de Schneider-Teitelbaum pour les cocycles rigides analytiques. Cette application a fait sa première apparition dans des articles de Schneider et Teitelbaum et a été exploitée dans le cadre des cocycles rigides analytiques de poids deux par Darmon et Vonk. Nous étendons cette construction au cas général de poids k supérieur.

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Contents

1 Introduction

Let $D > 0$ be a real quadratic discriminant, and let

$$f_{k,D}(z) := \sum_{\mathrm{disc}(Q)=D} Q(z,1)^{-k}, \qquad k > 2 \text{ even}, \quad z \in \mathbb{C}, \quad \mathrm{Im}(z) > 0, \tag{1}$$

where the sum runs over the integral binary quadratic forms $Q(z,1) = az^2 + bz + c$ of discriminant D. This function, which was first considered in [Za, Appendix 2], is a weight $2k$ cusp form on $\mathrm{SL}_2(\mathbb{Z})$, i.e. an element of $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. In [KZ1], it is shown to be the D-th Fourier coefficient of the holomorphic kernel function realising the Shimura-Shintani correspondence $\mathcal{S}$ from the "Kohnen plus space" $S^+_{k+1/2}$ of cusp forms of weight $k + \frac{1}{2}$ on $\Gamma_0(4)$ having a Fourier development of the form

$$g(z) = \sum_{n\geq 1} c(n)q^n, \text{ with } \qquad c(n) = 0 \text{ unless } n \equiv 0 \text{ or } 1 \pmod 4.$$

More precisely, Theorem 2 of loc.cit. asserts that for each fixed z in the usual upper half-plane $\mathcal{H}$, the generating series

$$\Omega_k(z,\tau) := \sum_{D>0} D^{k-1/2} f_{k,D}(z) e^{2\pi i D\tau} \tag{2}$$

belongs to $S^+_{k+1/2}$ as a function of $\tau \in \mathcal{H}$. To any $g \in S^+_{k+1/2}$, the correspondence $\mathcal{S}$ associates an element of $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ which, up to a multiplicative constant, is given by

$$\mathcal{S}(g)(z) = \frac{1}{6}\int_{\Gamma_0(4)\backslash\mathcal{H}} g(\tau)\overline{\Omega_k(-\overline{z},\tau)} v^{k-3/2} du dv.$$

Let p be a prime number such that $p \equiv 3$ $(mod\ 4)$ and let $k \geq 3$ be an *odd* integer. The goal of this thesis is to exhibit an analogous kernel function $\hat{\Omega}_k$ of weight $k+1/2$ and level $4p^2$, in which the space $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is replaced by the space of *rigid analytic cocycles* of weight

$2k$ on Ihara's group $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ introduced in [DV1]. In Section 7 we will explain why such a function $\hat{\Omega}_k$ gives rise to a correspondence $\mathcal{C}$ from the space $S_{k+1/2}^{(\bar{\mathbb{Q}})}(\Gamma(4p^2))$ of weight $k+1/2$ cusp forms of level $4p^2$ with Fourier coefficients in $\bar{\mathbb{Q}}$ to the space of weight $2k$ rigid analytic cocycles.

One of the themes of [DV1] is that rigid analytic cocycles enjoy a strong parallel with classical modular forms, while lending themselves to complementary applications, notably to the analytic construction of class fields of real quadratic fields via their "values" at real multiplication points of Drinfeld's p-adic upper half-plane. The counterparts for rigid analytic cocycles of (1) and (2) fit into program of developing the analogy between modular forms and rigid analytic cocycles initiated in [DV1].

This thesis also fits in the nascent "p-adic Kudla program", an emerging p-adic version of the classical Kudla program. While the latter investigates relations between automorphic forms and generating series of cycles on Shimura varieties, its p-adic counterpart explores connections between automorphic forms and generating series of cycles constructed by p-adic analytic means. As an example, in [DV2] Darmon and Vonk relate Fourier coefficients of certain weakly holomorphic modular forms to divisors of rigid meromorphic cocycles, which can then be viewed as real quadratic counterparts of Borcherds' singular theta lifts. More precisely, Darmon and Vonk start with a weakly holomorphic modular form ψ of weight $1/2$ on $\Gamma_0(4p)$ and associate to it a rigid meromorphic cocycle whose singularities are concentrated at real multiplication (RM) points on Drinfeld's p-adic upper half-plane $\mathcal{H}_p = \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$. The singularities of this rigid meromorphic cocycle are determined by the principal part of ψ and the result of [DV2] adds evidence in favor of the analogy between rigid meromorphic cocycles and meromorphic functions whose divisors are concentrated at CM points, such as those arising in the image of Borcherds' lift. Indeed, in [B1] Borcherds associated to a weight $1/2$ weakly holomorphic modular form ϕ a certain $\mathrm{SL}_2(\mathbb{Z})$-invariant real analytic function with logarithmic singualrities concentrated at CM points in $\mathcal{H}$ and determined by the principal part of ϕ.

Similar correspondences have been studied by Bruinier and Ono ([BrO]), Schwagenscheidt ([Schw]), Oda ([Oda]), and many others. These correspondences are usually defined via some theta kernel, and we will do the same by defining $\hat{\Omega}_k$. In particular, for non square discriminants we will define a rigid analytic cocycle $J_{k,D}$ which is an analogue of the Zagier form $f_{k,D}$. If D is not a square, this cocycle plays for the correspondence $\mathcal{C}$ the same role played by $f_{k,D}$ for the classical Shimura-Shintani correspondence, and in particular $J_{k,D}$ gives the D-th coefficient of $\hat{\Omega}_k$ in the same way as $f_{k,D}$ gives the D-th coefficient of Ω_k.

1.1 Statement of results and thesis outline

Recall that $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ is the Ihara group. Let $\mathcal{A}_k$ (resp. $\mathcal{M}_k$) be the additive group of rigid analytic (resp. meromorphic) functions on $\mathcal{H}_p$, endowed with the "weight k action" of Γ given by

$$h|\gamma(z) = (cz+d)^{-k} h\left(\frac{az+b}{cz+d}\right), \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Precise definitions of rigid analytic and meromorphic functions can be found in [DT] (Sections 1 and 2) and [GVdP] (Chapter 2). For the purpose of this thesis, a rigid analytic cocycle of weight k is a function

$$J : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \to \mathcal{A}_k$$

satisfying the "modular symbol properties"

$$J\{r,s\} = -J\{s,r\} \quad \text{and} \quad J\{r,s\} + J\{s,t\} = J\{r,t\}, \qquad \text{for all } r,s,t \in \mathbb{P}_1(\mathbb{Q}),$$

together with the Γ-invariance condition

$$J\{\gamma r, \gamma s\}|\gamma = J\{r,s\}, \text{ for all } \gamma \in \Gamma = \mathrm{SL}_2(\mathbb{Z}[1/p]).$$

In other words, a rigid analytic cocycle is an element of $\mathrm{MS}^\Gamma(\mathcal{A}_k)$, the space of Γ-invariant modular symbols with values in $\mathcal{A}_k$. Similarly, rigid meromorphic cocycles are elements of $\mathrm{MS}^\Gamma(\mathcal{M}_k)$. This definition is equivalent to the one given in [DV1] using parabolic cohomology (see [DV1], Corollary 1.10, for a proof of this fact).

In Section 2 we will classify rigid meromorphic cocycles of even weight satisfying a certain condition.

Given $D > 0$ as above, let $\mathcal{F}_D(\mathbb{Z}[1/p])$ denote the set of binary quadratic forms of discriminant D with coefficients in $\mathbb{Z}[1/p]$, equipped with its natural action of Γ. Given a quadratic form $Q(x,y) = ax^2 + bxy + cy^2 \in \mathcal{F}_D(\mathbb{Z}[1/p])$, let r_1 and r_2 denote the so-called *first* and *second roots* of $Q(z,1)$, defined by

$$r_1 = \frac{-b+\sqrt{D}}{2a}, \qquad r_2 = \frac{-b-\sqrt{D}}{2a}, \tag{3}$$

where $\sqrt{D}$ denotes the positive square root of D. Let $\gamma_Q := (r_1, r_2)$ denote the hyperbolic geodesic going from r_1 to r_2, and for any pair $(r,s) \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$, let (r,s) likewise denote the hyperbolic geodesic joining r to s on $\mathcal{H}$. The choice of an orientation on $\mathcal{H}$ (following the usual "right hand rule" for instance) determines an intersection pairing between hyperbolic geodesics, which is denoted $\gamma_1 \cdot \gamma_2$, and belongs to $\{-1, 0, 1\}$.

In Section 3 we will define the counterpart of Zagier's form $f_{k,D}(z)$ of (1) in the setting of rigid analytic cocycles via the following theorem:

Theorem. *Let $k \geq 1$ be odd. For all $(r,s) \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$, the infinite sum*

$$J_{k,D}\{r,s\}(z) := \sum_{Q \in \mathcal{F}_D(\mathbb{Z}[1/p])} (\gamma_Q \cdot (r,s)) \cdot Q(z,1)^{-k}$$

converges to a rigid meromorphic function of $z \in \mathcal{H}_p$, which is rigid analytic when $(\frac{D}{p}) = 1$. The function

$$J_{k,D} : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \to \mathcal{M}_{2k}$$

is a rigid meromorphic cocycle of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z}[1/p])$.

Assume now that the prime p is congruent to 3 modulo 4. In that case, each $D > 0$ for which $(\frac{D}{p}) = 1$ admits a canonical "positive" square root in $\mathbb{Q}_p$, which is a perfect square in $\mathbb{Q}_p$. Our main theorem is:

Theorem. *Let $k \geq 3$ be odd. If D is not a square and $\left(\frac{D}{p}\right) = 1$, then $D^{k-1/2}J_{k,D}$ is the D-th coefficient of a weight $k + 1/2$ cusp form $\hat{\Omega}_k(q)$ of level $4p^2$ with coefficients in* $\mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$. *The D-th coefficient of $\hat{\Omega}_k(q)$ vanishes if $\left(\frac{D}{p}\right) \neq 1$.*

The proof of this will be completed in Section 7 and the tools for it will be defined in the preceding sections. In Section 6 we will define a level p analogue $f_{k,D}^{(p)} \in S_{2k}(\Gamma_0(p))$ of the Zagier form $f_{k,D}$. The forms $f_{k,D}^{(p)}$ belong to a certain $\bar{\mathbb{Q}}$-subspace $\mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}})$ of $S_{2k}(\Gamma_0(p))$ and in Section 7 we will package them into a generating series by:

Theorem. *Let $k \geq 3$ be odd. Consider the series $\bar{\Omega}_k(q) = \sum_{D>0} D^{k-1/2} f_{k,D}^{(p)} \cdot q^D$, where D ranges over discriminants with $(\frac{D}{p}) = 1$. Then $\bar{\Omega}_k$ is a weight $k + 1/2$ cusp form of level $4p^2$ with coefficients in $\mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}})$.*

In Section 6 we will compute the period polynomials of $f_{k,D}^{(p)}$, getting a result analogous to Theorem 4 of [KZ2]. We will also define a Schneider-Teitelbaum lift in the setting of rigid analytic cocycles of higher weight in Section 4. The classical Schneider-Teitelbaum lift already appeared in [Sch] and [Te], and was extended to rigid analytic cocycles of weight 2 in [DV2]. We extended to higher weight the construction of this map and of its left inverse.

To summarize, the thesis is organized as follows:

- Section 2 classifies rigid meromorphic cocycles of even weight satisfying a certain condition.

- Section 3 introduces the analogue $J_{k,D}$ in rigid analytic cocycles of the Zagier form $f_{k,D}$.

- Section 4 defines a Schneider-Teitelbaum lift ST for rigid analytic cocycles of higher weight.
- Section 5 defines a left inverse for ST and computes certain p-adic residues of $J_{k,D}$.
- Section 6 defines a level p analogue of the Zagier form $f_{k,D}$ and computes its period polynomials.
- Section 7 constructs the theta kernel $\hat{\Omega}_k$ and shows how this gives the correspondence $\mathcal{C}$ that we seek.

1.2 Some notation

The notion of modular symbol is used heavily in this thesis, so we give its precise definition below.

Definition 1.1. *Let H be a subgroup of $\mathrm{SL}_2(\mathbb{Q})$ and let Ω be a module over H, where we denote the group action by $\omega|h$ for $\omega \in \Omega$ and $h \in H$. A modular symbol with values in Ω is a function $m : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \to \Omega$ such that*

$$m\{r,s\} = -m\{s,r\} \quad \textit{and} \quad m\{r,s\} + m\{s,t\} = m\{r,t\}, \qquad \textit{for all } r,s,t \in \mathbb{P}_1(\mathbb{Q}).$$

A modular symbol m is said to be H-invariant if

$$m\{hr, hs\}|\gamma = m\{r,s\}, \qquad \textit{for all } h \in H.$$

Given a binary quadratic form $Q(x,y) = ax^2 + bxy + cy^2$ of discriminant $D > 0$, we will often adopt the notation $Q = [a,b,c]$. Throughout the thesis we will denote by r_1 and r_2 the so-called *first* and *second roots* of $Q(z,1)$, defined by

$$r_1 = \frac{-b + \sqrt{D}}{2a}, \qquad r_2 = \frac{-b - \sqrt{D}}{2a},$$

where $\sqrt{D}$ denotes the positive square root of D. The notions of the geodesic γ_Q and the intersection number $\gamma_Q \cdot (r,s)$ given above will also be consistent throughout the thesis. A binary quadratic form $[a,b,c]$ with positive discriminant will be called *simple* if $ac < 0$, which implies that the two roots have opposite sign. A form $[a,b,c]$ such that $gcd(a,b,c) = 1$ will be called *primitive*.

We will denote by p a prime number that will be assumed to be congruent to 3 modulo 4 in Section 7 only.

We will use certain concepts from rigid analytic geometry such as the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$ and the reduction map $\mathcal{H}_p \to \mathcal{T}$. Some references that cover this material are [DT] (Section 1 and 2) and [GVdP] (Chapter 1 and 2). We will now fix some notation about these concepts. We will denote by v_0 the standard vertex of $\mathcal{T}$, which is the vertex associated to the lattice $\mathbb{Z}_p^2$. Let $\mathcal{T}^{\leq n}$ be the subgraph of $\mathcal{T}$ containing all the vertices at distance at most n from v_0, as well as all the edges containing two such vertices. The affinoid subdomain of $\mathcal{H}_p$ given by points reducing to $\mathcal{T}^{\leq n}$ will be denoted by $\mathcal{H}_p^{\leq n}$. This affinoid subdomain is obtained by removing $(p+1)p^n$ open disks of radius p^{-n} from $\mathbb{P}^1(\mathbb{C}_p)$. Similarly, we will denote by $\mathcal{T}^{<n}$ the subgraph of $\mathcal{T}$ containing all vertices at distance at most $n-1$ from v_0, as well as all the edges containing at least one of these vertices. The wide open subspace of $\mathcal{H}_p$ made of all the points reducing to $\mathcal{T}^{<n}$ will be denoted by $\mathcal{H}_p^{<n}$. Let $\mathcal{T}_0$ be the set of vertices of $\mathcal{T}$, let $\mathcal{T}_1$ be the set of edges and let $\mathcal{T}_1^*$ be the set of ordered edges of $\mathcal{T}$. A vertex is said to be even (resp. odd) if it has an even (resp. odd) distance from v_0. An ordered edge is said to have an even (resp. odd) orientation if its source is an even (resp. odd) vertex. We will denote by $\mathcal{T}_1^+$ the set of edges which have an even orienation and by $\mathcal{T}_1^-$ the set of edges which have an odd orientation. The standard edge of $\mathcal{T}$ with positive orientation will be denoted by e_0. For any $e \in \mathcal{T}_1^*$, we will denote by $\bar{e}$ the same edge taken with the opposite orientation and by $s(e)$ the *source* of e, i.e. the vertex where e starts.

2 Classification of certain rigid meromorphic cocycles of weight $2k$

In this section we classify rigid *meromorphic* cocycles of weight $2k$ satisfying a certain condition on their poles and residues. This is heavily inspired by the classification of rigid meromorphic cocycles of weight two carried out in [DV1] and most of the notation in this section is taken from there. The integer k will be assumed to be odd in order to prove Theorem 2.2, but the other results of this section hold for any $k \geq 0$.

Definition 2.1. *A rigid meromorphic period function of weight k is the value at $\{0, \infty\}$ of a rigid meromorphic cocyle of weight k. The space of such functions will be denoted by $\mathcal{R}_k$.*

A pair $(a/b, c/d) \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$ is called unimodular if $ad - bc = \pm 1$. Any pair in $\mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$ can be decomposed as a sequence of unimodular pairs and Γ acts transitively on such pairs. This implies that, in order to give a classification for $\mathrm{MS}^{\Gamma}(\mathcal{M}_k)$, it is enough to classify $\mathcal{R}_k$. A function $\varphi \in \mathcal{R}_k$ satisfies the following identities:

$$\varphi|(1+S) = 0, \quad \varphi|(1+U+U^2) = 0, \quad \varphi|D = \varphi, \tag{4}$$

where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}.$$

This follows from the modular symbol properties. Moreover, the matrix

$$P := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$$

induces an involution ϖ_p on $\mathcal{R}_k$, defined by

$$\varpi_p(\varphi)(z) := -\varphi|P(z) = -p^{k/2}\varphi(pz).$$

A rigid meromorphic period function is said to be p-even (resp. p-odd) if it satisfies

$$\varpi_p(\varphi) = \varphi, \quad (\text{resp } \varpi_p(\varphi) = -\varphi).$$

Note that $P^2 = pD$.

Definition 2.2. *A point $\tau \in \mathcal{H}_p$ is said to be a real multiplication (RM) point if $\mathbb{Q}(\tau)$ is a real quadratic field. The set of such points is be denoted by $\mathcal{H}_p^{RM}$. The Galois conjugate of $\tau \in \mathcal{H}_p^{RM}$ is denoted by τ'.*

Fix an embedding of the real quadratic field $\mathbb{Q}(\tau)$ into $\mathbb{R}$. We will denote by $\sqrt{D}$ the square root of D in $\mathbb{Q}_p$ corresponding via the fixed embedding to the positive real root of D.

Definition 2.3. *Let $\omega \in \mathcal{H}_p^{RM}$ and let $r, s \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$. Let (r, s) be the hyperbolic geodesic joining r and s in the upper-half plane $\mathcal{H}$. Let γ_ω be the hyperbolic geodesic joining ω and ω'. We say that ω is linked to (r, s) if γ_ω intersects (r, s).*

Let $\tau \in \mathcal{H}_p^{RM}$ and fix an embedding of $\mathbb{Q}(\tau)$ into $\mathbb{R}$. Let

$$\Sigma_\tau(0, \infty) := \{\omega \in \Gamma \cdot \tau \ \text{ such that } \ \omega\omega' < 0\}.$$

In more generality, one can define

$$\Sigma_\tau(r, s) := \{\omega \in \Gamma \cdot \tau \ \text{ such that } \ \omega \ \text{ is linked to } (r, s)\}.$$

The set $\Sigma_\tau(r, s)$ is endowed with a sign function $\delta_{r,s} : \Sigma_\tau(r, s) \to \pm 1$. The value $\delta_{r,s}$ depends on whether ω is "inside" or "outside" the semicircle (r, s) (for more details, see Equation (19) of [DV1]). Note that $\delta_{0,\infty}(\omega)$ coincides with the sign of ω in $\mathbb{R}$. The intersection of $\Sigma_\tau(r, s)$ with any affinoid in $\mathcal{H}_p$ is finite.

Lemma 2.1. *Let $Q = [A, B, C]$ be a primitive binary quadratic form with positive discriminant $D = D_0 p^m$, where $p \nmid D_0$ and $m > 0$. Let z be a point in the affinoid $\mathcal{H}_p^{\leq n}$ for some fixed n. If $n < m/2$, then*

$$\left|\frac{1}{Q(z,1)^k}\right| < p^{2nk}.$$

Proof. Note that, if p does not split in $\mathbb{Q}(\sqrt{D})$, then the roots r_1, r_2 of Q reduce to a point of $\mathcal{T}$ at distance $m/2$ from the standard vertex (see for example Proposition 1.1 of [DV1]), therefore $|z - r_i| > 1/p^n$ if $m/2 > n$. If p splits in $\mathbb{Q}(\sqrt{D})$, the inequality $|z - r_i| > 1/p^n$ still holds, because of equations (1.2.4) of [DT]. If $p \nmid A$, the lemma follows immediately as

$$\frac{1}{Q(z,1)^k} = \frac{1}{(Az^2 + Bz + C)^k} = \frac{1}{A^k (z - r_1)^k (z - r_2)^k}.$$

Now assume that $p|A$. Note that p must divide B, as $m > 0$. This implies that $p \nmid C$, otherwise Q would not be primitive. Assume $A > 0$ (if $A < 0$, we proceed similarly), so that $A = \frac{D_0 p^m - B^2}{4|C|}$ and the two roots r_1 and r_2 of $[A, B, C]$ can be written as

$$\frac{-B \pm p^{m/2}\sqrt{D_0}}{2A} = \frac{2|C|(-B \pm p^{m/2}\sqrt{D_0})}{D_0 p^m - B^2}.$$

Hence we get

$$\frac{1}{A^k (z - r_1)^k (z - r_2)^k} = \frac{4|C|^k}{(z(p^{m/2}\sqrt{D_0} + B) - 2|C|)^k (z(p^{m/2}\sqrt{D_0} - B) + 2|C|)^k}. \tag{5}$$

We can write $val_p(z - 2|C|/B) \leq n - 2val_p(B)$ (this is one of the equations defining $\mathcal{H}_p^{\leq n}$, see for example equation (1.2.4) of [DT]). Hence we have $val_p(zB - 2|C|) \leq n - val_p(B) \leq n$, and so the norm of (5) is smaller than p^{2nk} for $n < m/2$. $\square$

To any $\omega \in \mathcal{H}_p^{RM}$ we can associate a binary quadratic form Q_ω with integer coprime coefficients such that $Q_\omega(\omega, 1) = 0$ and ω is the first root of Q_ω. The correspondence $\omega \leftrightarrow Q_\omega$ is bijective. The discriminant of Q_ω is the discriminant of ω, which is a positive integer $D_\omega = p^{n_\omega} D_{\omega,0}$ where p does not divide $D_{\omega,0}$. Given a binary quadratic form Q_ω of positive discriminant, recall the definition of the geodesic γ_{Q_ω} given in Section 1.1, after 3.

Theorem 2.1. *For any $\tau \in \Gamma \setminus \mathcal{H}_p^{RM}$, the infinite sum*

$$\varphi_\tau(z) := \sum_{\omega \in \Sigma_\tau(0,\infty)} (\gamma_{Q_\omega} \cdot (0, \infty)) \frac{D_\omega^{k/2}}{Q_\omega(z,1)^k}.$$

converges uniformly on affinoid subsets to a rigid meromorphic period function of weight $2k$.

Proof. We will assume that τ reduces to a vertex of the Bruhat-Tits tree $\mathcal{T}$, as the case in which τ reduces to an edge can be treated in a similar way. We will show that the restriction of φ_τ to any affinoid $\mathcal{H}_p^{\leq n}$ can be written as the limit for $h \to \infty$ of a Cauchy sequence $\{\varphi_\tau^{(h)}\}$ of rational functions and that it has finitely many poles in $\mathcal{H}_p^{\leq n}$. Indeed, Proposition 1.1 of [DV1] implies that φ_τ is the limit for $h \to \infty$ of the rational functions

$$\varphi_\tau^{(h)}(z) := \sum_{\omega \in \Sigma_\tau^{\leq h}} (\gamma_{Q_\omega} \cdot (0, \infty)) \frac{D_\omega^{k/2}}{Q_\omega(z,1)^k}, \qquad \text{where} \quad \Sigma_\tau^{\leq h} := \Sigma_\tau(0,\infty) \cap \mathcal{H}_p^{\leq h},$$

as p^{n_ω} grows for $\omega \in \Sigma_\tau(0,\infty) - \Sigma_\tau^{\leq h}$, when h grows. If $\omega \in \Sigma_\tau^{\leq h}$, then ω reduces to a vertex of $\mathcal{T}$ at distance $N \leq h$ from the standard vertex v_0. If $N > n$, then ω does not belong to $\mathcal{H}_p^{\leq n}$ and $Q_\omega(z,1)^{-k}$ is regular on $\mathcal{H}_p^{\leq n}$.

For $z \in \mathcal{H}_p^{\leq n}$ with n fixed and h big enough, Lemma 2.1 implies that

$$\Big| \frac{D_\omega^{k/2}}{Q_\omega(z,1)^k} \Big|_p < p^{k(2n - n_\omega/2)}.$$

Hence $\{\varphi_\tau^{(h)}\}_h$ is a Cauchy sequence. We have proven that $\varphi_\tau(z)$ converges to a rigid meromorphic function on $\mathcal{H}_p$. Similarly the sums

$$\Phi_\tau\{r,s\} := \sum_{\omega\in\Sigma_\tau(r,s)} (\gamma_{Q_\omega}\cdot(r,s))\frac{D_\omega^{k/2}}{Q_\omega(z,1)^k}$$

converge to rigid meromorphic functions. We will now prove that

$$\Phi_\tau : \mathbb{P}_1(\mathbb{Q})\times\mathbb{P}_1(\mathbb{Q}) \to \mathcal{M}_{2k}$$

is a Γ-invariant modular symbol. The modular symbol property holds because any pair (ω,ω') is linked to the pair (r,t) if and only if it is linked either to (r,s) or (s,t) but not to both. (If it is linked to both, then the two intersection numbers have opposite signs).

Now we show the Γ-invariance condition, i.e. $\Phi_\tau\{\delta^{-1}r,\delta^{-1}s\} = \Phi_\tau\{r,s\}|\delta$ for any $\delta\in\Gamma$, where the action of Γ on $\mathcal{M}_{2k}$ is the weight $2k$ one. Given a form $Q=[A,B,C]$ with roots ω,ω' and a matrix $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we compute

$$Q(z,1)^{-k}|\delta = \Big(\frac{1}{A(z-\omega)^k(z-\omega')^k}\Big)|\delta = \frac{(a-\omega c)^{-k}(a-\omega' c)^{-k}}{A(z-\delta^{-1}\omega)^k(z-\delta^{-1}\omega')^k}.$$

Letting $Q|\delta = [A',B',C']$ where the action of Γ on quadratic forms is the usual one, the expression above can be rewritten as

$$\frac{1}{A'(z-\delta^{-1}\omega)^k(z-\delta^{-1}\omega')^k} = (Q|\delta)(z,1)^{-k},$$

hence

$$\Phi_\tau\{r,s\}(z)|\delta = \sum_{\omega\in\Sigma_\tau(r,s)} (\gamma_{Q_\omega}\cdot(r,s))\frac{D_\omega^{k/2}}{(Q_\omega|\delta)(z,1)^k}.$$

Now note that $(\gamma_{Q_\omega} \cdot (\delta^{-1}r, \delta^{-1}s)) = (\gamma_{Q_\omega|\delta^{-1}} \cdot (r, s))$, so

$$\Phi_\tau\{\delta^{-1}r, \delta^{-1}s\}(z) = \sum_{\omega \in \Sigma_\tau(r,s)} (\gamma_{Q_\omega|\delta^{-1}} \cdot (r, s)) \frac{D_\omega^{k/2}}{Q_\omega(z, 1)^k},$$

and the Γ-invariance follows. The theorem follows from the fact $\varphi_\tau = \Phi_\tau\{0, \infty\}$. $\square$

Similarly to [DV1], we provided an explicit collection of rigid meromorphic period functions φ_τ of weight $2k$ indexed by $\tau \in \Gamma \setminus \mathcal{H}_p^{RM}$. However in our case the sets of poles of these functions are given by $\Sigma_\tau(0, \infty) \cup \Sigma_{\tau'}(0, \infty)$, and not only by $\Sigma_\tau(0, \infty)$. For this reason, the rigid meromorphic period functions ϕ that we classify have to satisfy the condition that if $\tau \in \mathcal{H}_p^{RM}$ is a pole then also τ' is a pole, and $res_\tau(\phi) = -res_{\tau'}(\phi)$.

Note that $\varphi_\tau = \varphi_{\tau'}$, since $Q_\omega = -Q_{\omega'}$ and $(\gamma_{Q_\omega} \cdot (0, \infty)) = -(\gamma_{Q_{\omega'}} \cdot (0, \infty))$. However, if we consider the index $\tau \in \Gamma \setminus \mathcal{H}_p^{RM}$ modulo the Galois action, then the functions φ_τ are linearly independent, as they have disjoint set of poles.

Theorem 2.2. *Let $k \geq 1$ be odd. Let ϕ be a rational period function of weight $2k$, and assume that if a point $\tau \in \mathcal{H}_p^{RM}$ is a pole of ϕ, then also τ' is a pole of ϕ. Assume moreover that $res_\tau(\phi) = -res_{\tau'}(\phi)$. Then ϕ is a finite linear combination of the functions φ_τ of Theorem 2.1 and of a rigid analytic period function of weight $2k$.*

Proof. This proof parallels closely the proof of Theorem 1.24 of [DV1].

Let $\phi^\pm := \phi \pm \varpi_p(\phi)$. As $\phi = (\phi^+ + \phi^-)/2$, we can assume without loss of generality that ϕ is p-even or p-odd. Let Σ_ϕ denote the set of poles of ϕ. The identities in (4) imply

$$\omega \in \Sigma_\phi \Rightarrow S(\omega) \in \Sigma_\phi \quad \text{and} \quad U(\omega) \in \Sigma_\phi \text{ or } U^2(\omega) \in \Sigma_\phi. \tag{6}$$

Now let

$$\Sigma_\phi^{<1} := \Sigma_\phi \cap \mathcal{H}_p^{<1}.$$

The set $\Sigma_\phi^{<1}$ is finite because a rigid meromorphic function has finitely many poles on any given affinoid. Moreover, the fact that $\mathrm{SL}_2(\mathbb{Z})$ preserves $\mathcal{H}_p^{<1}$ implies that $\Sigma_\phi^{<1}$ satisfies the conditions in (6) just as Σ_ϕ does. Any finite set satisfying (6) can be written as a finite union

$$\Sigma_\phi^{<1} = \cup_{\tau\in I_\phi} \Sigma_\tau^0(0,\infty),$$

where

$$I_\phi \subset \mathrm{SL}_2(\mathbb{Z}) \setminus (\mathcal{H}_p^{RM} \cap \mathcal{H}_p^{<1}) \quad \text{and} \quad \Sigma_\tau^0(0,\infty) = \Sigma_\tau(0,\infty) \cap \mathcal{H}_p^{<1}.$$

See [DV1] for a proof of this fact. Lemma 4 and Lemma 5 of [CZ] imply that ϕ has only poles of order k on $\mathcal{H}_p^{<1}$ and that if $\tau \in \mathcal{H}_p^{<1}$ is a pole of ϕ, then

$$\mathrm{PP}_\tau(\phi) = res_\tau((z-\tau)^{k-1}\phi) \cdot \mathrm{PP}_\tau(D^{k/2}/Q_\tau(z,1)^k), \tag{7}$$

where PP_τ denotes the principal part of a function at τ. Lemma 4 of [CZ] also implies that, for any $\mathrm{SL}_2(\mathbb{Z})$-orbit $\mathcal{A}$ of a pole $\tau \in \mathcal{H}_p^{<1}$ of ϕ, there exist a constant $C_\mathcal{A}$ such that

$$res_\tau((z-\tau)^{k-1}\phi) = \mathrm{sgn}(\tau) \cdot C_\mathcal{A}. \tag{8}$$

Let $\mathcal{A}_\tau$ be the $\mathrm{SL}_2(\mathbb{Z})$-orbit of a pole τ. As k is odd, equations (7) and (8), together with the conditions on the poles and residues of ϕ in the statement of the Theorem, imply that

$$\phi - \sum_{\tau\in I_\phi} C_{\mathcal{A}_\tau} \varphi_\tau^+$$

is a p-even rigid meromorphic period function having no singularities outside $\mathcal{H}_p^{<1}$. Any p-even or p-odd rigid meromorphic period function which is regular on $\mathcal{H}_p^{<1}$ must be regular on $\mathcal{H}_p$. This is proven in [DV1] for functions of weight 2, but the proof is identical for functions of higher weight. $\square$

3 Rigid analytic cocycles attached to real quadratic fields

In this section, we associate to any real quadratic discriminant D satisfying $(\frac{D}{p}) = 1$ (resp. $(\frac{D}{p}) \neq 1$) a rigid analytic (resp. meromorphic) cocycle $J_{k,D}$ of weight $2k$ for the Ihara group. The cocycle $J_{k,D}$ is one of the main objects studied in this thesis, as it is a p-adic analogue of the Zagier form $f_{k,D}$ mentioned in the introduction and it will be used in Section 7 to construct the correspondence $\mathcal{C}$ that we seek.

Theorem 3.1. *Let $k \geq 1$ be odd. For all $(r, s) \in \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$, the infinite sum*

$$J_{k,D}\{r, s\}(z) := \sum_{Q \in \mathcal{F}_D(\mathbb{Z}[1/p])} (\gamma_Q \cdot (r, s)) \cdot Q(z, 1)^{-k}$$

converges to a rigid meromorphic function of $z \in \mathcal{H}_p$, which is rigid analytic when $(\frac{D}{p}) = 1$. The function

$$J_{k,D}\{r, s\} : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to \mathcal{M}_{2k}$$

is a rigid meromorphic cocycle of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z}[1/p])$.

Proof. Note that $(\gamma_Q \cdot (r, s)) = -(\gamma_{-Q} \cdot (r, s))$, so we do not get cancellation between a form Q and $-Q$ in the sum defining $J_{k,D}$. For any form in $\mathcal{F}_D(\mathbb{Z}[1/p])$, we can clear the denominators, obtaining a form in $\mathcal{F}_{Dp^{2n}}(\mathbb{Z})$ for some $n \in \mathbb{Z}$. Hence the sum in the statement can be rewritten as

$$\sum_{n=0}^{\infty} \Big(\sum_{Q \in \mathcal{F}_{Dp^{2n}}(\mathbb{Z})} (\gamma_Q \cdot (r, s)) \cdot Q(z, 1)^{-k} \Big) p^{nk}. \tag{9}$$

Let z belong to the affinoid $\mathcal{H}_p^{\leq h}$ for some fixed h. We assume at first that $(r, s) = (0, \infty)$, so the inner sum is over simple forms $[A, B, C]$ with $A, B, C \in \mathbb{Z}$, $AC < 0$, thus $B^2 + 4|AC| = Dp^{2n}$. Note that, if p does not split in $\mathbb{Q}(\sqrt{D})$, the roots r_1, r_2 of such a form reduce to a point of $\mathcal{T}$ at distance n from the standard vertex (see for example Proposition 1.1 of [DV1]), therefore $|z - r_i| > 1/p^h$ if $n > h$, so the inner sum is regular on $\mathcal{H}_p^{\leq h}$.

We are going to prove that, for any $h \geq 0$, the outer sum in (9) is the limit of a Cauchy sequence relative to the sup norm on $\mathcal{H}_p^{\leq h}$ and hence converges to a rigid meromorphic function on $\mathcal{H}_p$. To do this, we will show that the norm of the general term of the outer sum is (eventually) going to zero uniformly in $z \in \mathcal{H}_p^{\leq h}$. Indeed, Lemma 2.1 implies that eventually

$$\left|\frac{1}{Q(z,1)^k}\right| < p^{2hk}. \tag{10}$$

As there are only finitely many simple forms of a given discriminant, we get that

$$\sum_{Q \in \mathcal{F}_{Dp^{2n}}(\mathbb{Z})} (\gamma_Q \cdot (0, \infty)) \cdot Q(z,1)^{-k}$$

is a finite sum and the general term of the outer sum in (9) eventually has norm smaller than $p^{k(2h-n)}$, so the series converges to a rigid meromorphic function $J_{k,D}(0,\infty)(z)$, which is analytic when $(\frac{D}{p}) = 1$ (as in this case $\sqrt{D} \notin \mathcal{H}_p$).

The case of a general pair (r, s) follows from the modular symbol property and the Γ-invariance condition, together with the fact that any pair (r, s) can be written as a sum of unimodular pairs and $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on such pairs. The modular symbol property and the Γ-invariance condition are proved with the same computations used at the end of the proof of Theorem 2.1. $\square$

4 A Schneider-Teitelbaum lift for rigid analytic cocycles

The aim of this section is to define a Schneider-Teitelbaum lift for rigid analytic cocycles, i.e. a map

$$\mathrm{ST} : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2}) \to \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k}),$$

where $\mathcal{P}_{2k-2}$ denotes polynomials with coefficients in $\mathbb{C}_p$ and degree at most $2k-2$, endowed with the following "weight $2k-2$ action" of Γ

$$q|\gamma(z) = (cz+d)^{2k-2} q\Big(\frac{az+b}{cz+d}\Big), \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{11}$$

For cocycles of weight 2, this was already done in [DV1]. It will be more convenient to consider the dual space $\mathcal{P}_{2k-2}^\vee := \mathrm{Hom}_{\mathbb{C}_p}(\mathcal{P}_{2k-2}, \mathbb{C}_p)$, which is a Γ-module with the action of Γ given by

$$(\hat{q}|\gamma)(\cdot) = \hat{q}(\,\cdot\,|\gamma^{-1}).$$

The spaces $\mathcal{P}_{2k-2}$ and $\mathcal{P}_{2k-2}^\vee$ are isomorphic as Γ-modules, so we will actually define a map

$$\mathrm{ST}^\vee : \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2}^\vee) \to \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$$

and at the beginning of Section 5 we will write down the corresponding map ST defined on $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$.

Definition 4.1. *A harmonic cocycle with value in a Γ-module Ω is a function $c : \mathcal{T}_1^* \to \Omega$ satisfying*

$$c(\bar{e}) = -c(e), \ \ \textit{and} \quad \sum_{s(e)=v} c(e) = 0, \quad \textit{for all} \quad v \in \mathcal{T}_0 \quad \textit{and} \quad e \in \mathcal{T}_1^*.$$

The space of such harmonic cocycles is denoted by $C_{har}(\Omega)$. The actions of Γ on $\mathcal{T}$ and Ω induce an action on $C_{har}(\Omega)$ given by

$$(c|\gamma)(e) = c(\gamma e)|\gamma.$$

So we want to define a map $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2}^{\vee}) \to \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$, but because of the lemma below this is like defining a map $\mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^{\vee})) \to \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$.

Lemma 4.1. *There is an isomorphism $ev_{e_0} : \mathrm{MS}^{\Gamma}(C_{har}(P_{2k-2}^{\vee})) \to \mathrm{MS}^{\Gamma_0(p)}(P_{2k-2}^{\vee})$.*

Proof. Let ev_{e_0} be the $\Gamma_0(p)$-equivariant map induced by the evaluation of harmonic cocycles on the standard edge e_0 of the Bruhat-Tits $\mathcal{T}$. To clarify, $\mathrm{ev}_{e_0}(c\{r,s\}) = (c\{r,s\})(e_0)$. We first prove the injectivity. If c is a modular symbol in the kernel of ev_{e_0}, then $c\{r,s\}(e_0) = 0$ for all $r, s \in \mathbb{P}_1(\mathbb{Q})$. But Γ acts transitively on $\mathcal{T}_1^+$ so for any edge e we have $e = \gamma^{-1}e_0$ for some $\gamma \in \Gamma$. The definition of the Γ-action on harmonic cocycles, together with the Γ-invariance of the modular symbol c, give:

$$c\{r,s\}(e) = c\{r,s\}(\gamma^{-1}e_0) = (c\{r,s\}|\gamma^{-1})(e_0)|\gamma = (c\{\gamma r, \gamma s\}(e_0))|\gamma = 0.$$

We now prove the surjectivity. Given $c_0 \in \mathrm{MS}^{\Gamma_0(p)}(P_{2k-2}^{\vee})$, define $c \in \mathrm{MS}^{\Gamma}(C_{\mathrm{har}}(P_{2k-2}^{\vee}))$ by setting, for each $e = \gamma^{-1}e_0 \in \mathcal{T}_1^+$,

$$c\{r,s\}(e) := c_0\{\gamma r, \gamma s\}|\gamma.$$

Note that γ is only well-defined up to left multiplication by elements of $\mathrm{Stab}_{\Gamma}(e_0) = \Gamma_0(p)$, but if we substitute γ by $\delta\gamma$ with $\delta \in \Gamma_0(p)$, we get

$$c_0\{\delta\gamma r, \delta\gamma s\}|\delta\gamma = (c_0\{\gamma r, \gamma s\}|\delta^{-1})|\delta\gamma = c_0\{\gamma r, \gamma s\}|\gamma.$$

It is easy to see that $\mathrm{ev}_{e_0}(c) = c_0$. Finally, for any $c \in \mathrm{MS}^\Gamma(C_{\mathrm{har}}(P^\vee_{2k-2}))$ the modular symbol $\mathrm{ev}_{e_0}(c)$ is $\Gamma_0(p)$-invariant, because $\Gamma_0(p)$ fixes e_0 and c is Γ-invariant. $\square$

To any oriented edge e of the Bruhat-Tits tree $\mathcal{T}$, one can associate a p-adic ball $U(e) \subset \mathbb{P}_1(\mathbb{Q}_p)$. This is done in [DT], but we will briefly cover the construction below. We will need the notion of *ends* on the tree $\mathcal{T}$.

Definition 4.2. *Let $P = (l_0, l_1, \dots)$ and $P' = (l'_0, l'_1, \dots)$ be infinite paths of vertices of $\mathcal{T}$ without backtracking. If P and P' differ only by a finite number of vertices, we say that they are equivalent and we write $P \sim P'$. An equivalence class $[P]$ for the relation $\sim$ is called an end of $\mathcal{T}$. The set of all ends is denoted by* $\mathrm{Ends}(\mathcal{T})$.

Let e be the oriented edge running from a vertex l_0 to a vertex l_1 and let

$$U_e := \{ \, [P] \in \mathrm{Ends}(\mathcal{T}) \mid P = (l_0, l_1 \dots) \, \} \, ,$$

which is the subtree of $\mathcal{T}$ given by all ends leaving the oriented edge e. Let $red : \mathcal{H}_p \to \mathcal{T}$ be the reduction map from $\mathcal{H}_p$ to $\mathcal{T}$ (for a precise definition see, for example, [DT]). Let $\Sigma_e = red^{-1}(U_e)$ and let $\bar{\Sigma}_e$ be the closure of Σ_e in $\mathbb{P}^1(\mathbb{C}_p)$, then

$$U(e) := \bar{\Sigma}_e \cap \mathbb{P}^1(\mathbb{Q}_p)$$

is a ball in $\mathbb{P}^1(\mathbb{Q}_p)$. One can show that $U(\gamma e) = \gamma U(e)$ for any $\gamma \in \Gamma$ (see [DT]).

Note that a modular symbol c in $\mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathcal{P}^\vee_{2k-2}))$ is a collection of harmonic cocycles $c\{r, s\}$ indexed by $\mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q})$ and satisfying the modular symbol conditions. This gives a collection of distributions $\mu_{c\{r,s\}}$ on $\mathbb{P}_1(\mathbb{Q}_p)$ defined by

$$\int_{U(e)} P(t) d\mu_{c\{r,s\}}(t) = (c\{r, s\}(e))(P), \tag{12}$$

where $P \in \mathcal{P}_{2k-2}$. Note that $c\{r,s\}(e)$ is an element of $\mathcal{P}_{2k-2}^\vee$, hence it can be evaluated at $P \in \mathcal{P}_{2k-2}$. The distributions given in 12 are basically the same as the ones defined in [Sch] and [Te]. The only difference is that we are generalizing them to the setting of modular symbols.

The map $\mathrm{ST}^\vee : \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^\vee)) \to \mathrm{MS}^\Gamma(\mathcal{A}_{2k})$ will be given by $c \mapsto f$, where

$$f\{r,s\}(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_{c\{r,s\}}(t). \tag{13}$$

We need to show that this expression makes sense, in particular we need to show that our integral extends to a set of functions containing $\frac{1}{z-t}$. This is done in the two following sections. After that, we will show that $f\{r,s\}(z)$ is an element of $\mathcal{A}_k$ and that $f\{r,s\}$ is a Γ-invariant modular symbol.

Note that the assignment $c\{r,s\} \mapsto c\{0,\infty\}$ is injective and identifies $\mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^\vee))$ with the subset of $C_{\mathrm{har}}(\mathcal{P}_{2k-2}^\vee)$ given by the harmonic cocycles c satisfying the relations

$$c|(1+S) = 0, \quad c|(1+U+U^2) = 0, \quad c|D = c, \tag{14}$$

where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}.$$

This means that, given $c \in C_{\mathrm{har}}(\mathcal{P}_{2k-2}^\vee)$ satisfying the relations above, it is enough to show that the integral

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_{c\{0,\infty\}}(t)$$

makes sense. We will do this using the following fact, which can be found in [Te] (Proposition 9) and [Ort] (Section 3.2). Let r be a fixed integer with $0 \leq r \leq k-2$ and let μ be a distribution on $\mathcal{P}_{2k-2}$ satisfying

$$\Big| \int_{U(e)} (x-a)^n d\mu(x) \Big|_p \leq C(1/p)^{\alpha(e)(n-r)} \quad \text{for } a \in U(e), \infty \notin U(e), 0 \leq n \leq 2k-2, \tag{15}$$

where $\alpha(e) = \inf_{u,v \in U(e)}\{val_p(u-v)\}$ for $\infty \notin U(e)$, and

$$\Big| \int_{U(e)} x^n d\mu(x) \Big|_p \leq C(1/p)^{\alpha(e)(r-n)} \quad \text{for } \infty \in U(e), 0 \notin U(e), 0 \leq n \leq 2k-2, \tag{16}$$

where $\alpha(e) = -\inf_{u,v \notin U(e)}\{val_p(u-v)\}$ for $\infty \in U(e)$. Then the distribution can be extended uniquely to the space A_{2k} of $\mathbb{C}_p$-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic except for a pole at ∞ of order at most $2k-2$. The space A_{2k} is endowed with a weight $2k-2$ action of Γ defined by the same formula (11) giving the weight $2k-2$ action on $\mathcal{P}_{2k-2}$.

We want to show that if c is an harmonic cocycle satisfying the conditions (14), then the distribution given by c satisfies the two conditions above. In the following section, we will prove this for p-adic balls centered at 0 or ∞, while the case of balls centered at a general $a \in \mathbb{Q}_p$ will be covered in Section 4.2.

4.1 The case of balls centered at 0 or ∞

In this section we will first prove condition (15) for $a = 0$ and the prove condition (16). Recall that Γ acts on $C_{\text{har}}(\mathcal{P}_{2k-2}^\vee)$ by $(c|\gamma)(e) = c(\gamma e)|\gamma$. Hence we have

$$\int_{U(e)} P(t) d\mu_{c|\gamma}(t) = (c|\gamma)(e)(P) = c(\gamma e)(P|\gamma^{-1}) = \int_{\gamma U(e)} (P|\gamma^{-1})(t) d\mu_c(t). \tag{17}$$

We will use this property to prove the following lemmas.

Lemma 4.2. *Let c be an element of $C_{har}(\mathcal{P}_{2k-2}^\vee)$ satisfying condition $c|D = c$ from (14). Then inequality (15) holds for μ_c and $a = 0$.*

Proof. All balls of $\mathbb{P}^1(\mathbb{Q}_p)$ centered at zero can be written as translates of $\mathbb{Z}_p$ or $p\mathbb{Z}_p$ via some power of the matrix D. Let e such that $U(e) = \mathbb{Z}_p$ and consider for example $U(D^m e) = \{\tau | \; val_p(\tau) \geq 2m\}$ for some $m \in \mathbb{Z}$. Then property (17) of the integral combined with the D^m-invariance of c gives

$$\int_{U(D^m e)} x^n | D^{-m} d\mu_c = \int_{U(e)} x^n d\mu_c,$$

which is

$$\int_{U(D^m e)} x^n d\mu_c = (1/p)^{-2m(n-(2k-2)/2)} \int_{U(e)} x^n d\mu_c.$$

If instead than $U(e) = \mathbb{Z}_p$ we take $U(e) = p\mathbb{Z}_p$, we get similar inequalities for the balls that have radius given by an odd valuation. Condition (15) then follows for all balls centered at zero and $0 \leq n \leq 2k-2$. $\square$

Lemma 4.3. *Let c be an element of $C_{har}(\mathcal{P}^\vee_{2k-2})$ satisfying condition $c|(1+S) = 0$ from (14). Assume also that inequality (15) holds for μ_c with $a = 0$. Then inequality (16) is also satisfied.*

Proof. Any ball of $\mathbb{P}^1(\mathbb{Q}_p)$ centered at ∞ can be written as $U(Se)$ for some ball $U(e) \subseteq \mathbb{Q}_p$ centered at zero. Then property (17) of the integral together with the fact $c = -c|S$ gives

$$\int_{U(e)} x^n d\mu_c = -\int_{U(Se)} (x^n | S) d\mu_c.$$

So we get

$$\int_{U(Se)} x^n d\mu_c = (-1)^{n+1} \int_{U(e)} x^{2k-2-n} d\mu_c,$$

hence

$$\Big| \int_{U(Se)} x^n d\mu_c \Big|_p \leq C \Big(\frac{1}{p}\Big)^{\alpha(e)(2k-2-n-(2k-2)/2)} = C \Big(\frac{1}{p}\Big)^{\alpha(e)((2k-2)/2-n)} = C \Big(\frac{1}{p}\Big)^{\alpha(Se)((2k-2)/2-n)},$$

and the thesis follows (recall that $\alpha(e)$ and $\alpha(Se)$ have been defined at the end of the previous section). $\square$

4.2 The case of balls centered at $a \in \mathbb{Q}_p$

In this section we prove condition (15) for balls centered at any $a \in \mathbb{Q}_p$.

Lemma 4.4. *Let c be an element of $C_{har}(\mathcal{P}_{2k-2}^\vee)$ satisfying conditions $c|(1+S) = 0$ and $c|D = c$ from (14). Then inequality (15) holds for any $a \in \mathbb{Q}_p$.*

Proof. Let $\alpha = \begin{pmatrix} p^m & a/p^m \\ 0 & p^{-m} \end{pmatrix}$ for an integer m. A general ball in $\mathbb{Q}_p$ can be written as $\alpha(\mathbb{Z}_p)$ or $\alpha(p\mathbb{Z}_p)$.

Let $U(e) = \mathbb{Z}_p$ so that $U(\alpha e) = a + p^{2m}\mathbb{Z}_p$. From property (17) of the integral we get

$$\int_{U(\alpha e)} (x-a)^n d\mu_c = \int_{U(e)} ((x-a)^n|\alpha) d\mu_{c|\alpha} = p^{2m(n-(2k-2)/2)} \int_{U(e)} x^n d\mu_{c|\alpha}.$$

Recall now that we are implicitly identifying $\mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^\vee))$ with the subset of $C_{\mathrm{har}}(\mathcal{P}_{2k-2}^\vee)$ given by the harmonic cocycles c satisfying the relations (14). So, with a slight abuse of notation, $d\mu_c = d\mu_{c\{0,\infty\}}$ and $d\mu_{c|\alpha} = d\mu_{c\{\alpha^{-1}0,\alpha^{-1}\infty\}}$. But the pair $(\alpha^{-1}0, \alpha^{-1}\infty)$ can be written as a finite sum of unimodular pairs, so we get a finite sum

$$\int_{U(\alpha e)} (x-a)^n d\mu_c = p^{2m(n-(2k-2)/2)} \sum_t \int_{U(e)} x^n d\mu_{c\{r_t,s_t\}},$$

where the pairs (r_t, s_t) are unimodular. Now recall that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on unimodular pairs, therefore the sum above can be rewritten as

$$p^{2m(n-(2k-2)/2)} \sum_t \int_{U(e)} x^n d\mu_{c\{0,\infty\}|\beta_t} = p^{2m(n-(2k-2)/2)} \sum_t \int_{U(e)} x^n d\mu_{c|\beta_t},$$

for some $\beta_t \in \mathrm{SL}_2(\mathbb{Z})$. Using again property (17) of the integral we get

$$p^{2m(n-(2k-2)/2)} \sum_t \int_{U(e)} x^n d\mu_{c|\beta_t} = p^{2m(n-(2k-2)/2)} \sum_t \int_{U(\beta_t e)} P_t(x) d\mu_c, \tag{18}$$

where $P_t(x) = x^n|\beta_t^{-1}$ is a polynomial in $\mathcal{P}_{2k-2}$. Now note that $U(\beta_t e) = U(e)$ as $\mathrm{SL}_2(\mathbb{Z})$ stabilizes $\mathbb{Z}_p$. This means that inequality (15) holds for the integrals in the sum above. The norm of these integrals will be bounded by some constant $Cp^0 = C$ because the integrals are on $\mathbb{Z}_p$, and so for $U(e) = \mathbb{Z}_p$ we get

$$\Big| \int_{U(\alpha e)} (x-a)^n d\mu_c \Big|_p \leq C(1/p)^{2m(n-(2k-2)/2)}.$$

If instead we have $U(e) = p\mathbb{Z}_p$, the proof is the same as above, we just need to be more careful as β_t does not necessarily fix $p\mathbb{Z}_p$. However, as $\beta_t \in \mathrm{SL}_2(\mathbb{Z})$, we have that $\beta_t(p\mathbb{Z}_p)$ will also be a ball of radius $1/p$, so the integral in (18) is bounded by $(1/p)^{-(2k-2)/2}$ and the proof proceedes as in the previous case. $\square$

The lemma above, together with what we did in Section 4.1, implies that the integral defined by $c\{r, s\}$ with $c \in \mathrm{MS}^\Gamma(C_{\mathrm{har}}(\mathcal{P}_{2k-2}^\vee))$ can be extended uniquely to the space A_{2k} of $\mathbb{C}_p$-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic except for a pole at ∞ of order at most $2k-2$. So the map

$$c\{r, s\} \mapsto f\{r, s\}(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_{c\{r,s\}}(t)$$

makes sense.

4.3 The end of the proof

In this section we prove that the function $f\{r,s\}(z)$ defined in (13) is rigid analytic and that $f\{r,s\}$ is a Γ-invariant modular symbol.

Proposition 4.1. *Let c be an element of $\mathrm{MS}^{\Gamma}(C_{har}(\mathcal{P}_{2k-2}^{\vee}))$. The function*

$$f\{r,s\}(z)=\int_{\mathbb{P}^1(\mathbb{Q}_p)}\frac{1}{z-t}d\mu_{c\{r,s\}}(t)$$

is an element of $\mathcal{A}_{2k}$.

Proof. We will drop the index $\{r,s\}$ as we do not need it in this proof. Let $\mathcal{A}$ be a connected affinoid of $\mathcal{H}_p$. We can assume that for some n we have

$$\mathcal{A}=\mathbb{P}^1(\mathbb{C}_p)\setminus\bigsqcup_{i=1}^{n}B_i,$$

where $B_i=\{\tau\in\mathbb{P}^1(\mathbb{C}_p)\colon |\tau-a_i|_p<r_i\}$ with $a_i\in\mathbb{P}^1(\mathbb{Q}_p)$ and the r_i's are powers of p. We assume $a_i=\infty$ if $\infty\in B_i$. Let U_i be the intersection between $\mathbb{P}^1(\mathbb{Q}_p)$ and B_i. These intersections give a covering of $\mathbb{P}^1(\mathbb{Q}_p)$ by disjoint compact open balls, hence if we let

$$f_i(z)=\int_{U_i}\frac{1}{z-x}d\mu(x),$$

then $f(z)=\sum_{i=1}^{n}f_i$ and it is enough to show that each f_i is rigid analytic on the complement of B_i. Let us first consider the case $\infty\notin U_i$. Let $z\in\mathbb{P}^1(\mathbb{C}_p)\setminus B_i$, then

$$\frac{1}{z-x}=\sum_{l=0}^{\infty}\frac{1}{(z-a_i)^{l+1}}(x-a_i)^l$$

converges for $x \in B_i$ and we have

$$f_i(z) = \sum_{l=0}^{\infty} \frac{1}{(z-a_i)^{l+1}} \int_{U_i} (x-a_i)^l d\mu(x).$$

This converges uniformly on the complement of B_i by Proposition 9 in [Te]. Consider now the case $\infty \in B_i$. If z is in the complement of B_i then

$$\frac{1}{z-x} = \sum_{l=0}^{\infty} \frac{z^l}{x^{l+1}}$$

converges for $x \in U_i$ and as before

$$f_i(z) = \sum_{l=0}^{\infty} z^l \int_{U_i} \frac{1}{x^{l+1}} d\mu(x)$$

converges uniformly on $\mathbb{P}^1(\mathbb{C}_p) \setminus B_i$. So f is rigid analytic on $\mathcal{A}$ and hence on $\mathcal{H}_p$.

$\square$

Lemma 4.5. *The quantity*

$$\frac{(ct+d)^{2k-2}}{\gamma z - \gamma t} - \frac{(cz+d)^{2k}}{z-t}$$

is a polynomial in t of degree at most $2k-2$.

Proof. We have

$$\begin{aligned}
\frac{(ct+d)^{2k-2}}{\gamma z - \gamma t} - \frac{(cz+d)^{2k}}{z-t} &= \frac{(ct+d)^{2k-1}(cz+d)}{z-t} - \frac{(cz+d)^{2k}}{z-t} \\
&= \frac{cz+d}{z-t}((ct+d)-(cz+d)) \sum_{i=0}^{2k-2} (ct+d)^i (cz+d)^{2k-2-i} \\
&= -c(cz+d) \sum_{i=0}^{2k-2} (ct+d)^i (cz+d)^{2k-2-i},
\end{aligned}$$

so the thesis follows. $\square$

Proposition 4.2. *The expression $f\{r,s\}$ defined in (13) gives an element of* $\mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$.

Proof. We have that $f\{r,s\}+f\{s,t\} = f\{r,t\}$ because to define $f\{r,s\}$ we used a cocycle $c\{r,s\}$ which satisfies the modular symbol condition. Now we need to show that $f\{\gamma r,\gamma s\}(z)=(f\{r,s\}(z))|\gamma^{-1}$. By property (17) of the integral we get

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_{c\{\gamma r,\gamma s\}}(t) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{(ct+d)^{2k-2}}{z-\gamma t} d\mu_{c\{r,s\}}(t).$$

Using now Lemma 4.5 we get that this expression is

$$(c\gamma^{-1}z+d)^{2k}\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{\gamma^{-1}z-t} d\mu_{c\{r,s\}}(t) = (a-cz)^{-2k}\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{\gamma^{-1}z-t} d\mu_{c\{r,s\}}(t),$$

which is $(f\{r,s\}(z))|\gamma^{-1}$, so the thesis follows. $\square$

5 The residue of $J_{k,D}$

Our goal for this section is to define a left inverse for ST and evaluate it at $J_{k,D}$, assuming $\left(\frac{D}{p}\right) = 1$. Consider the pairing on $\mathcal{P}_{2k-2}$ given by

$$\langle T^i, T^j \rangle = \binom{2k-2}{i}^{-1} (-1)^i \delta_{i,2k-2-j}. \tag{19}$$

Definition 5.1. *Let $c \in C_{har}(\mathcal{P}_{2k-2})$. For any edge e of $\mathcal{T}$ we define the components $c_i(e)$ of c by the following expression*

$$c(e)(T) = \sum_{i=0}^{2k-2} \binom{2k-2}{i} c_i(e) T^i.$$

If we identify $\mathcal{P}_{2k-2}$ and $\mathcal{P}_{2k-2}^\vee$ via the pairing above, then the distribution defined by

$$\int_{U(e)} x^i d\mu_c(t) = (-1)^i c_{2k-2-i}(e)$$

for a cocycle $c \in C_{har}(\mathcal{P}_{2k-2})$ agrees with the distribution that was given in (12) for a cocycle in $C_{har}(\mathcal{P}_{2k-2}^\vee)$. Because of what we did in Section 4, this distribution can be extended to A_{2k} and we can define a map

$$\mathrm{ST} : \mathrm{MS}^\Gamma(C_{har}(\mathcal{P}_{2k-2})) \to \mathrm{MS}^\Gamma(\mathcal{A}_{2k})$$

in the same way as we defined $\mathrm{ST}^\vee$ in Section 4. So we have the commutative diagram

$$\begin{tikzcd}
\mathrm{MS}^\Gamma(C_{har}(\mathcal{P}_{2k-2})) \arrow[r, "\sim"] \arrow[d, "\mathrm{ST}"'] & \mathrm{MS}^\Gamma(C_{har}(\mathcal{P}_{2k-2}^\vee)) \arrow[d, "\mathrm{ST}^\vee"] \\
\mathrm{MS}^\Gamma(\mathcal{A}_{2k}) \arrow[r, equal] & \mathrm{MS}^\Gamma(\mathcal{A}_{2k})
\end{tikzcd}$$

We now define a map Res on $\mathrm{MS}^\Gamma(\mathcal{A}_{2k})$ with values in $\mathrm{MS}(C_{har}(\mathcal{P}_{2k-2}))$ by $f\{r,s\} \mapsto c\{r,s\}$ with

$$c\{r,s\}(e)(T) = \sum_{i=0}^{2k-2} \binom{2k-2}{i} (-1)^i \operatorname{Res}_e(z^{2k-2-i} f\{r,s\}(z)dz)T^i, \tag{20}$$

where Res_e denotes the residue with respect to the annulus in $\mathcal{H}_p$ corresponding to the edge e. The lemma below shows that $\operatorname{Res}(f\{r,s\})$ is a Γ-invariant modular symbol, hence we constructed a map

$$\operatorname{Res} : \mathrm{MS}^\Gamma(\mathcal{A}_{2k}) \to \mathrm{MS}^\Gamma(C_{har}(\mathcal{P}_{2k-2})).$$

Lemma 5.1. *The modular symbol defined in (20) is Γ-invariant.*

Proof. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in Γ we need to show that the expression

$$c\{\gamma r, \gamma s\} = \sum_{i=0}^{2k-2} \binom{2k-2}{i} (-1)^i \operatorname{Res}_e(z^{2k-2-i} f\{\gamma r, \gamma s\}(z)dz)T^i \tag{21}$$

equals the expression

$$(c\{r,s\})|\gamma^{-1} = \sum_{i=0}^{2k-2} \binom{2k-2}{i} (-1)^i \operatorname{Res}_{\gamma^{-1}e}(z^{2k-2-i} f\{r,s\}(z)dz)(T^i|\gamma^{-1}). \tag{22}$$

We can rewrite (21) as

$$\sum_{i=0}^{2k-2} \binom{2k-2}{i} (-1)^i \operatorname{Res}_{\gamma^{-1}e}\Big((\gamma z)^{2k-2-i}(-c\gamma z + a)^{-k} f\{r,s\}(z)\frac{dz}{(cz+d)^2}\Big)T^i \tag{23}$$

because of the property of the annular residue $\operatorname{Res}_{\gamma^{-1}e}(f(\gamma z)d(\gamma z)) = \operatorname{Res}_e(f(z)dz)$. The lemma follows after completely expanding (22), (23) and comparing their coefficients. □

Proposition 5.1. *The map* Res *is a left inverse for* ST, *i.e.* $\operatorname{Res} \circ \mathrm{ST} = \mathrm{Id}$.

Proof. To ease the notation, we will drop the index $\{r, s\}$ as it is not needed here. Recall that for a cocycle $c \in C_{har}(\mathcal{P}_{k-2})$ we have $F := \mathrm{ST}(c)(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-t} d\mu_c(t)$ and if $U(e)$ is a ball centered at a and not containing infinity, then one can write a power series expansion for $\mathrm{ST}(c)(z)$ and get

$$\mathrm{Res}_e((z-a)^i F(z)) = \int_{U(e)} (t-a)^i d\mu_c(t),$$

where suitable adjustments can be made if $U(e)$ contains infinity. Given any edge e of $\mathcal{T}$ the components of the cocycle $c_F := (\mathrm{Res} \circ ST)(c)$ are

$$(c_F)_i(e) = (-1)^i \sum_{j=0}^{2k-2-i} \binom{2k-2-i}{j} \mathrm{Res}_e(a^{2k-2-i-j}(z-a)^j F dz).$$

Using the equality above these can be rewritten as

$$(-1)^i \sum_{j=0}^{2k-2-i} \binom{2k-2-i}{j} a^{2k-2-i-j} \int_{U(e)} (t-a)^i d\mu_c(t) = (-1)^i \int_{U(e)} t^i d\mu_c(t),$$

which is just

$$(-1)^i(-1)^i c_i(e) = c_i(e)$$

because of the definition of μ_c. This shows that $c_F = c$ and the proposition follows. $\square$

Recall that there is an isomorphism between $\mathrm{MS}^{\Gamma}(C_{har}(\mathcal{P}_{2k-2}))$ and $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$ which is induced by evaluating a harmonic cocycle at the standard edge e_0. So there is a map

$$\mathrm{Res}_0 : \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k}) \to \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$$

which is the composite of Res and this isomorphism. In the next section we will compute $\mathrm{Res}_0(J_{k,D})$.

5.1 The computation of the residue

Let $\mathcal{F}_D(\mathbb{Z})$ denote, as before, the set of integral binary quadratic forms of discriminant D. A form $Q(x,y) = ax^2 + bxy + cy^2 \in \mathcal{F}_D(\mathbb{Z})$ is called a *Heegner form* if p divides a, and the set of all Heegner forms of discriminant D is denoted $\mathcal{F}_D^{(p)}(\mathbb{Z})$. The *standard annulus* of $\mathcal{H}_p$ is the annulus of $\mathcal{H}_p$ which reduces to the standard vertex e_0 of $\mathcal{T}$. Recall that in this section we are assuming $\left(\frac{D}{p}\right) = 1$. We will need the proposition below.

Proposition 5.2. *Let $Q \in \mathcal{F}_{Dp^{2n}}(\mathbb{Z})$ be a binary quadratic form with integer coefficients and discriminant Dp^{2n}. Then $\mathrm{Res}_{e_0} Q(z,1)^{-k}$ is not zero if and only if Q is an Heegner form and p does not divide the discriminant of Q.*

Proof. This follows from the lemmas below and the p-adic Residue Theorem (see for example [FVdP]). $\square$

Lemma 5.2. *Let $Q = [A,B,C]$ and let r_1 and r_2 be the two roots of Q. Assume that $val_p(r_1) \geq 0$ (resp. $val_p(r_2) \geq 0$) and $val_p(r_2) \leq -1$ (resp. $val_p(r_1) \leq -1$), i.e. one root is "inside" the standard annulus in $\mathcal{H}_p$ and the other one is "outside". Then $p|A$.*

Proof. The sum of the two roots is

$$r_1 + r_2 = \frac{-B + \sqrt{\Delta}}{2A} + \frac{-B - \sqrt{\Delta}}{2A} = \frac{-B}{A},$$

where Δ is the discriminant of Q. Therefore $val_p(-B/A) = val_p(r_1 + r_2) < 0$ and $p|A$. $\square$

Lemma 5.3. *Let $Q \in \mathcal{F}_{Dp^{2n}}(\mathbb{Z})$ with $Q = [A,B,C]$ and let Δ be the discriminant of Q. If Q is an Heegner form and if $p|\Delta$, then $\mathrm{Res}_{e_0}(Q(z,1)^{-k}) = 0$.*

Proof. We know that $p|A$, so we can write $A = ap^{\alpha}$ where $p \nmid a$ and $\alpha > 0$. As p divides the discriminant then we also have $p|B$ and $B = bp^{\beta}$ with $p \nmid b$ and $\beta > 0$. There are now three cases.

If $2\beta > \alpha$ then we can write $\Delta = p^{\alpha}(b^2p^{2\beta-\alpha} - 4aC)$. Note that α must be even. The roots of Q can be written as

$$\frac{-bp^{\beta} \pm p^{\alpha/2}\sqrt{b^2p^{2\beta-\alpha} - 4aC}}{2ap^{\alpha}},$$

so the p-adic valuation of both roots is $-\alpha/2 \leq -1$. Hence both r_1 and r_2 are "outside" the standard annulus and $\mathrm{Res}_{e_0}(Q(z,1)^{-k}) = 0$ by the Theorem of Residues in $\mathcal{H}_p$.

If $2\beta < \alpha$ then we have $\Delta = p^{2\beta}(b^2 - 4ap^{\alpha-2\beta}C)$. We can aslo write $\alpha = 2\beta + x$ fore some $x > 0$ and the two roots of $Q(z,1)$ are

$$\frac{-bp^{\beta} \pm p^{\beta}\sqrt{b^2 - 4ap^xC}}{2ap^{2\beta+x}} = \frac{-b \pm \sqrt{b^2 - 4ap^xC}}{2ap^{\beta+x}}.$$

Note that $p \nmid C$, so the valuation of one of the roots must be $-\beta < 0$. This implies that the valuation of the other root must be $-\beta - x$, because $val_p(r_1r_2) = -x - 2\beta$. Therefore also in these case the valuation of both roots is at most -1, hence they are "outside" the standard annulus and the residue of $Q(z,1)^{-k}$ is zero also in this case.

Finally let us consider the case $2\beta = \alpha$. We have $\Delta = p^{2\beta}(b^2 - 4aC)$ and the two roots can be written as

$$\frac{-b \pm \sqrt{b^2 - 4aC}}{2ap^{\beta}}.$$

Then $val_p(-b + \sqrt{b^2 - 4aC}) = 0$, which implies that $val_p(-b - \sqrt{b^2 - 4aC}) = 0$. Therefore the valuation of both roots is $-\beta$ and the residue of $Q(z,1)^{-k}$ is again zero. $\square$

Lemma 5.4. *Let Q be a Heegner form and assume $p \nmid \Delta$. Then one of the roots of Q is "inside" the standard annulus of $\mathcal{H}_p$ and the other one is "outside".*

Proof. We let again $Q = [A, B, C]$, with $A = p^\alpha a$ and $p \nmid a$. Then either $val_p(-B+\sqrt{\Delta}) = \alpha$ or $val_p(-B - \sqrt{\Delta}) = \alpha$. If we consider the product $r_1 r_2$ we see that in the first case $val_p(r_1) = 0$ and $val_p(r_2) = -\alpha$, while in the second case $val_p(r_1) = -\alpha$ and $val_p(r_2) = 0$. □

In order to make a distinction between the cases in which a root is inside or outside the standard annulus, it will be important to fix a choice of a square root of D in $\mathbb{Q}_p$ and to introduce the notation of first (resp. second) root of Q *in a p-adic sense*. Note that we are still assuming that D is a square modulo p.

Definition 5.2. *Let $\sqrt{D}$ be a fixed choice of a square root of D in $\mathbb{Q}_p$ and let $Q = ax^2 + bxy + cy^2$ be a binary quadratic form with coefficients in $\mathbb{Q}_p$. The first root r_1 and the second root r_2 of Q in a p-adic sense are defined as*

$$r_1 = \frac{-b+\sqrt{D}}{2a}, \quad r_2 = \frac{-b-\sqrt{D}}{2a}.$$

Remark 5.1. The notion in the definition above is different from the notion of first and second root used so far and given in the introduction, because it depends on a square root of D in $\mathbb{Q}_p$ rather than on the positive real root. This notion will be only used in Definition 5.3 below.

Definition 5.3. *Let Q be as in the definition above and assume $p|A$. Let r_1, r_2 be the fisrt and second root of Q in a p-adic sense. We will denote by $(\gamma_Q \cdot e_0)$ the "p-adic intersection number" defined as*

$$(\gamma_Q \cdot e_0) = \begin{cases} 1 & \textit{if } val_p(r_1) \geq 0, \ \ val_p(r_2) \leq -1, \\ -1 & \textit{if } val_p(r_1) \leq -1, \ \ val_p(r_2) \geq 0. \end{cases}$$

In other words $(\gamma_Q \cdot e_0) = 1$ *(resp.* -1*) if* r_1 *is inside the standard annulus (resp. outside) and* r_2 *outside (resp. inside). Note that if we change the choice of the square root of* D *in* $\mathbb{Q}_p$ *then* $(\gamma_Q \cdot e_0)$ *changes sign.*

Proposition 5.3. *Let* $k \geq 1$ *be odd. For all* $\{r, s\} \in \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q})$*, the finite sum*

$$\kappa_{k,D}\{r,s\}(z) = \binom{2k-2}{k-1} \frac{1}{D^{k-1}\sqrt{D}} \sum_{Q \in \mathcal{F}_D^{(p)}(\mathbb{Z})} (\gamma_Q \cdot (r,s))(\gamma_Q \cdot e_0) \cdot Q(z,1)^{k-1}$$

is a polynomial of degree $2k-2$ *with coefficients in* $\mathbb{C}_p$*. The function*

$$\kappa_{k,D} : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to \mathcal{P}_{2k-2}$$

is an element of $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$*.*

Proof. This is true because Heegner forms are fixed by $\Gamma_0(p)$ and because $(\gamma_Q \cdot e_0) = (\gamma_{Q_\delta} \cdot e_0)$, where $Q_\delta = Q|\delta \in \mathcal{F}_D^{(p)}$ and the action of $\Gamma_0(p)$ on binary quadratic forms is the usual one. Indeed let r_1 and r_2 be the first and second root of Q. Then one can check that $\delta^{-1}r_1$ and $\delta^{-1}r_2$ are the first and second root of Q_δ, respectively. Because the matrix δ^{-1} fixes the standard affinoid of $\mathcal{H}_p$, the p-adic intersection number does not change. □

We are now ready to compute the residue of $J_{k,D}$.

Theorem 5.1. *Let* $k \geq 1$ *be odd and let* $\mathrm{Res}_0 : \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k}) \to \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$ *be the map defined in Section 5. Then*

$$\mathrm{Res}_0(J_{k,D}) = \kappa_{k,D}.$$

Proof. Let $P := \mathrm{Res}_0(J_{k,D})$, so

$$P\{r,s\}(T) = \sum_{i=0}^{2k-2} \binom{2k-2}{i} (-1)^i \,\mathrm{Res}_{e_0}(z^{2k-2-i} J_{k,D}\{r,s\}(z)dz) T^i.$$

We want to show that the expression above equals $\kappa_{k,D}\{r,s\}(T)$. We will drop the index $\{r,s\}$ as it is not necessary in this proof. Given a Heegner form $Q = [A,B,C]$, let $H(T) = Q(T,1)^{k-1}$ and let $H^{(2k-2-i)}$ denote the coefficient of degree $2k-2-i$ of the polynomial $H(T)$. Because only the Heegner forms give a contribution in the computation of the residue, it is enough to show that

$$\binom{2k-2}{i}(-1)^i \operatorname{Res}_{e_0}\Big(\frac{z^i}{Q(z,1)^k}\Big) = (\gamma_q \cdot e_0)\frac{H^{(2k-2-i)}}{D^{k-1}\sqrt{D}}\binom{2k-2}{k-1}.$$

From now on we will assume that $(\gamma_q \cdot e_0) = 1$, so that $\operatorname{Res}_{e_0}(z^i/Q(z,1)^k) = \operatorname{Res}_{r_1}(z^i/Q(z,1)^k)$. This is enough because if it was $(\gamma_q \cdot e_0) = -1$, then we would just take the residue with respect to r_2, getting the opposite sign. We can write

$$\operatorname{Res}_{e_0}\Big(\frac{z^i}{Q(z,1)^k}\Big) = \sum_{l=0}^{M}\binom{i}{l}\frac{r_1^{i-l}}{A^k}\operatorname{Res}_{e_0}\Big(\frac{(z-r_1)^{l-k}}{(z-r_2)^k}\Big),$$

where $M = \min(i, k-1)$. The upper bound for l is M because $\operatorname{Res}_{e_0}((z-r_1)^{l-k}/(z-r_2)^k) = 0$ if $l \geq k$.

To compute the residues in the sum above we express $(z-r_1)^{l-k}/(z-r_2)^k$ as a power series in $(z-r_1)$, getting

$$\frac{(z-r_1)^{l-k}}{(z-r_2)^k} = \frac{(z-r_1)^{l-k}}{(r_1-r_2)(k-1)!}\sum_{j=k-1}^{\infty}\frac{j(j-1)(j-2)...(j-k+2)(z-r_1)^{j-k+1}}{(r_2-r_1)^j}.$$

This expression can be obtained by formally differentiating the geometric series which gives the expansion of $(z-r_2)^{-1}$. Therefore

$$\operatorname{Res}_{e_0}\Big(\frac{(z-r_1)^{l-k}}{(z-r_2)^k}\Big) = \binom{2k-2-l}{k-1}\frac{(-1)^l}{(r_1-r_2)^{2k-1-l}},$$

and

$$\operatorname{Res}_{e_0}\Big(\frac{z^i}{Q(z,1)^k}\Big) = \sum_{l=0}^{M}\binom{i}{l}\binom{2k-2-l}{k-1}\frac{(-1)^l r_1^{i-l}}{A^k}\Big(\frac{A}{\sqrt{D}}\Big)^{2k-1-l},$$

where we used the equality $r_1 - r_2 = \sqrt{D}/A$.

On the other hand, we have

$$H^{(2k-2-i)} = A^{k-1} \sum_{l=0}^{N} \binom{k-1}{l} \binom{k-1}{2k-2-i-l} (-r_1)^{k-1-l} (-r_2)^{i+l-k+1}, \tag{24}$$

where $N = \min(2k-2-i, k-1)$. Now using the equality $r_2 = r_1 - \sqrt{D}/A$ we can rewrite (24) as

$$A^{k-1} \sum_{l=0}^{N} \Big(\binom{k-1}{l} \binom{k-1}{2k-2-i-l} r_1^{k-1-l} (-1)^i \sum_{h=0}^{i+l-k+1} \binom{i+l-k+1}{h} r_1^h \Big(\frac{\sqrt{D}}{-A}\Big)^{i+l-k+1-h} \Big).$$

From the above computations we see that to complete the proof we need to show that

$$\binom{2k-2}{i} \sum_{l=0}^{M} \binom{i}{l} \binom{2k-2-l}{k-1} (-1)^l r_1^{i-l} \Big(\frac{\sqrt{D}}{A}\Big)^l \tag{25}$$

is equal to

$$\binom{2k-2}{k-1} \sum_{l=0}^{N} \Big(\binom{k-1}{l} \binom{k-1}{2k-2-i-l} r_1^{k-1-l} \sum_{h=0}^{i+l-k+1} \binom{i+l-k+1}{h} r_1^{i+l-k+1-h} (-1)^h \Big(\frac{\sqrt{D}}{A}\Big)^h \Big). \tag{26}$$

We can see the two expressions above as polynomials in $(\sqrt{D}/A)$. These polynomials have the same degree, because if $M = k-1$ then $N = 2k-2-i$, while if $M = i$ then $N = k-1$. Assume $M = k-1$ (the case $M = i$ can be treated similarly).

The coefficient of degree l of the polynomial in (25) is

$$\binom{2k-2}{i} \binom{i}{l} \binom{2k-2-l}{k-1} (-1)^l r_1^{i-l}, \tag{27}$$

while the coefficient of degree l of the polynomial in (26) is

$$\binom{2k-2}{k-1} \sum_{t=0}^{2k-2-i} \binom{k-1}{t} \binom{k-1}{2k-2-i-t} \binom{i+t-k+1}{l} (-1)^l r_1^{i-l}. \tag{28}$$

Therefore to conclude the proof it is enough to show the equality

$$\binom{2k-2}{k-1}\sum_{t=0}^{2k-2-i}\binom{k-1}{t}\binom{k-1}{2k-2-i-t}\binom{i+t-k+1}{l}=\binom{2k-2}{i}\binom{i}{l}\binom{2k-2-l}{k-1}. \tag{29}$$

We will prove this equality by applying some binomial identities to its left hand side. If we let $a := k-1$ and $b := 2k-2-t-i$, then the left hand side of (29) becomes

$$\binom{2k-2}{k-1}\sum_{t=0}^{2k-2-i}\binom{a-b}{l}\binom{a}{b}\binom{a}{t}=\binom{2k-2}{k-1}\binom{k-1}{l}\sum_{t=0}^{2k-2-i}\binom{k-1-l}{2k-2-i-t}\binom{k-1}{t}, \tag{30}$$

where the equality follows from the binomial identity $\binom{a}{b}\binom{a-b}{l}=\binom{a}{l}\binom{a-l}{b}$.

Now we let $n = 2k-2-i$ and (30) becomes

$$\binom{2k-2}{k-1}\binom{k-1}{l}\sum_{t=0}^{n}\binom{k-1-l}{n-t}\binom{k-1}{t}=\binom{2k-2}{k-1}\binom{k-1}{l}\binom{2k-2-l}{2k-2-1}, \tag{31}$$

where the last equality holds because of the Chu-Vandermonde indentity, i.e.

$$\sum_{t=0}^{n}\binom{m}{t}\binom{s-m}{n-t}=\binom{s}{n},$$

with $m = k-1-l$ and $s = 2k-2-l$ in our case.

Now it is easy to see that (31) is equal to the right hand side of (29), so the theorem follows. $\square$

6 A Zagier form of level p

Let k continue to be an *odd* integer, and assume also $k \geq 3$. The Zagier form $f_{k,D}(z)$ of the introduction admits an analogue in level p, given by

$$f_{k,D}^{(p)}(z) = \sum_{Q \in \mathcal{F}_D^{(p)}(\mathbb{Z})} \frac{(\gamma_Q \cdot e_0)}{Q(z,1)^k},$$

where $\mathcal{F}_D^{(p)}(\mathbb{Z})$ was defined at the beginning of Section 5.1. The coefficient $(\gamma_Q \cdot e_0)$ ensures that there is no cancellation between the forms Q and $-Q$. Moreover, this coefficient will play an important role in the proof of Theorem 7.2.

Proposition 6.1. *The function $f_{k,D}^{(p)}(z)$ is a weight $2k$ cusp form for $\Gamma_0(p)$.*

Proof. This follows from the fact that $\Gamma_0(p)$ fixes Heegner forms. Indeed for any $\delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\Gamma_0(p)$ we have

$$(f_{k,D}^{(p)}|\delta)(z) = \sum_{Q \in \mathcal{F}_D^{(p)}(\mathbb{Z})} \frac{(\gamma_Q \cdot e_0)}{Q_\delta(z,1)^k},$$

where $Q_\delta = Q|\delta \in \mathcal{F}_D^{(p)}$ and the action of $\Gamma_0(p)$ on binary quadratic forms is the usual one. Moreover when we proved Proposition 5.3 we showed that $(\gamma_Q \cdot e_0) = (\gamma_{Q_\delta} \cdot e_0)$. Note that the series defining $f_{k,D}^{(p)}(z)$ converges by the argument used to prove the convergence of the Eisenstein series. This is covered for example in [Z123], Section 2.1. The series moreover gives a cusp form because it converges absolutely uniformly on compact sets and each term tends to zero when z tends to ∞. This completes the proof. $\square$

An *Eichler cocycle* of weight $2k$ is an element of the space $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathcal{P}_{2k-2})$ of $\mathrm{SL}_2(\mathbb{Z})$-invariant modular symbols with values in $\mathcal{P}_{2k-2}$. More generally, an Eichler cocycle of weight $2k$ and level p is an element of the space $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$. The relevance of Eichler cocycles to modular forms arises from the Eichler-Shimura isomorphism, which to any cusp form f of weight k for a congruence group Γ associates the Eichler cocycle of weight k defined by

$$\kappa_f\{r,s\} := \int_r^s f(z)(x-z)^{k-2}dz, \tag{32}$$

where the integral is over the geodesic in the upper half plane joining r and s. The right hand side is a polynomial in x, and κ_f is an element of $\mathrm{MS}^{\Gamma}(\mathcal{P}_{2k-2})$. Furthermore, the assignment $f \mapsto \kappa_f$ induces a Hecke equivariant vector space isomorphism between the space $S_k(\Gamma)$ of cusp forms of weight k for Γ and the space $\mathrm{MS}^{\Gamma}(\mathcal{P}_{2k-2})$. Some references for this material are [GS] (Section 4), [KZ2], and [Dar2] (Chapter 2).

Let f be a weight $2k$ cusp form for $\Gamma_0(p)$ (or $\mathrm{SL}_2(\mathbb{Z})$). To f we associate the modular symbol

$$\bar{\kappa}_f\{r,s\} := \kappa_f\{r,s\} - \kappa_f\{-r,-s\}|\widetilde{I},$$

where

$$\widetilde{I} := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 6.1. *The modular symbol $\bar{\kappa}_f$ is $\Gamma_0(p)$-invariant, i.e. it belongs to $\mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2})$.*

Proof. Let $\gamma \in \Gamma_0(p)$. The result follows from the following equalities

$$\begin{aligned} \kappa_f\{-\gamma r, -\gamma s\} &= \kappa_f\{\widetilde{I}\gamma r, \widetilde{I}\gamma s\} \\ &= \kappa_f\{\widetilde{I}\gamma\widetilde{I}(-r), \widetilde{I}\gamma\widetilde{I}(-s)\} \\ &= \kappa_f\{-r,-s\}|\widetilde{I}\gamma^{-1}\widetilde{I}, \end{aligned}$$

where we used the fact that $\widetilde{I}\gamma\widetilde{I} \in \Gamma_0(p)$ in the last equality. □

An important theme of [KZ2] is that the forms $f_{k,D}(z)$ have *rational periods.* Indeed

Kohnen and Zagier computed the *even* period of $f_{k,D}(z)$, which is essentially equal to $\bar{\kappa}_f\{0,\infty\}$, up to a constant. They did not use the notation of modular symbols, however their result can be formulated as

$$\bar{\kappa}_{f_{k,D}}\{0,\infty\} = \binom{2k-2}{k-1}\frac{\pi}{D^{k-1}\sqrt{D}}\sum_{Q\in\mathcal{F}_D(\mathbb{Z})}(\gamma_Q\cdot(0,\infty))\cdot Q(x,1)^{k-1} \tag{33}$$

up to a constant, for k *even* and D non square. Note that the sum above is finite because of the coefficient $(\gamma_Q\cdot(0,\infty))$. Hence one can associate to the forms $f_{k,D}$ an Eichler cocycle in $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathcal{P}_{2k-2})$ whose values are polynomials with rational, indeed integral, coefficients.

We now prove an analogous and more general result for $f^{(p)}_{k,D}$, in the case where k is *odd*. Note that the polynomial in (33) is an even polynomial and it is indeed the *even* period of $f_{k,D}$. The polynomial $\bar{\kappa}_{f^{(p)}_{k,D}}\{0,\infty\}$ of Theorem 6.1 below is instead an odd polynomial, and it is basically the *odd* period of $f^{(p)}_{k,D}$.

Theorem 6.1. *If D is not a square, then*

$$\bar{\kappa}_{f^{(p)}_{k,D}}\{r,s\}(x) = 3\pi\sqrt{-1}\binom{2k-2}{k-1}\frac{1}{D^{k-1}\sqrt{D}}\sum_{Q\in\mathcal{F}^{(p)}_D(\mathbb{Z})}(\gamma_Q\cdot(r,s))(\gamma_Q\cdot e_0)\cdot Q(x,1)^{k-1},$$

where $\sqrt{-1}$ denotes the square root of -1 in $\mathbb{C}$.

Proof. Note that the sum in the statement is finite because of the coefficient $(\gamma_Q\cdot(r,s))$. We will start assuming $r,s\in\mathbb{Q}$. Let $w=2k-2$. Then

$$\begin{aligned}
\bar{\kappa}_{f^{(p)}_{k,D}}\{r,s\}(x) &= \int_r^s f^{(p)}_{k,D}(z)(x-z)^{2k-2}dz - \int_{-r}^{-s} f^{(p)}_{k,D}(z)(-x-z)^{2k-2}dz \\
&= \sum_{i=0}^{w}\binom{w}{i}x^{w-i}\Big[\int_r^s(-z)^i\cdot f^{(p)}_{k,D}(z)dz - \int_{-r}^{-s} z^i\cdot f^{(p)}_{k,D}(z)dz\Big] \\
&= \sum_{i=0}^{w}\binom{w}{i}x^{w-i}\Big[\sum_{Q\in\mathcal{F}^{(p)}_D(\mathbb{Z})}\int_r^s\frac{(\gamma_Q\cdot e_0)\cdot(-z)^i}{Q(z,1)^k} - \sum_{Q\in\mathcal{F}^{(p)}_D(\mathbb{Z})}\int_{-r}^{-s}\frac{(\gamma_Q\cdot e_0)\cdot z^i}{Q(z,1)^k}\Big].
\end{aligned}$$

The last equality holds because the series defining $f_{k,D}^{(p)}$ converges absolutely uniformly on suitable compact sets containing the semicircle joining r to s. Now note that this also implies that the series appearing in the last expression converge absolutely, hence we can rewrite the expression as

$$\begin{aligned}
&\sum_{i=0}^{w}\binom{w}{i}x^{w-i}\Big[\sum_{Q\in\mathcal{F}_D^{(p)}(\mathbb{Z})}(\gamma_Q\cdot e_0)\cdot\Big(\int_r^s\frac{(-z)^i}{Q(z,1)^k}-\int_{-r}^{-s}\frac{z^i}{Q(z,1)^k}\Big)\Big]\\
=\;&\sum_{i=0}^{w}\binom{w}{i}x^{w-i}\Big[\sum_{Q\in\mathcal{F}_D^{(p)}(\mathbb{Z})}(\gamma_Q\cdot e_0)\cdot\Big(\int_r^s\frac{(-z)^i}{Q(z,1)^k}-\int_{-r}^{-s}\frac{z^i}{\widetilde{Q}(z,1)^k}\Big)\Big],
\end{aligned}$$

where if $Q=[a,b,c]$ then $\widetilde{Q}:=[-a,b,-c]$. The last equality holds because we can rearrange the terms of a series which converges absolutely. By Proposition 6.2 and Lemma 6.2 below, we can rewrite the difference of the integrals above as

$$\begin{aligned}
&3\pi\sqrt{-1}\bigg((-1)^i\frac{(\gamma_Q\cdot(r,s))\cdot A_{Q,1}^{(i)}+G_{(r,s)}^{Q,i}}{a^k}-\frac{(\gamma_{\widetilde{Q}}\cdot(-r,-s))\cdot A_{\widetilde{Q},1}^{(i)}+G_{(-r,-s)}^{\widetilde{Q},i}}{(-a)^k}\bigg)\\
=\;&2(-1)^i a^{-k}3\pi\sqrt{-1}\cdot(\gamma_Q\cdot(r,s))\cdot A_{Q,1}^{(i)}.
\end{aligned}$$

For $r,s\in\mathbb{Q}$ the Theorem then follows by formally comparing this expression with Theorem 5.1.

The case $\{r,s\}=\{0,\infty\}$ follows from Theorem 5 of [KZ2]. Indeed for any cusp form f, the polynomial $\bar{\kappa}_f\{0,\infty\}(x)$ is odd, and our result for $\{0,\infty\}$ can be read from the odd part of the polynomial in Theorem 5 of [KZ2].

The case $\{r,\infty\}$ follows because $\bar{\kappa}_f\{r,\infty\}(x)=\bar{\kappa}_f\{r,0\}(x)+\bar{\kappa}_f\{0,\infty\}(x)$. □

Remark 6.1. The period polynomials of Theorem 6.1 basically have the same expression as the polynomials $\kappa_{k,D}\{r,s\}$ defined in Proposition 5.3. However, the period polynomials $\bar{\kappa}_{f_{k,D}^{(p)}}\{r,s\}(x)$ have coefficients in $\mathbb{C}$ and the square root of D appearing in their formula is the complex one. The polynomials $\kappa_{k,D}\{r,s\}$ instead have coefficients in $\mathbb{C}_p$ and the square root of D appearing in their definition is an element of $\mathbb{Q}_p$.

We devote the rest of this section to proving the results mentioned in the proof of Theorem 6.1. Given a binary quadratic form $Q = [a, b, c]$, let r_1 and r_2 denote, as usual, its *first* and *second* root. Consider the following partial fraction decomposition

$$\frac{z^i}{(z-r_1)^k(z-r_2)^k} = \frac{A^{(i)}_{Q,1}}{(z-r_1)} + \ldots + \frac{A^{(i)}_{Q,k}}{(z-r_1)^k} + \frac{B^{(i)}_{Q,1}}{(z-r_2)} + \ldots + \frac{B^{(i)}_{Q,k}}{(z-r_2)^k}.$$

For any $r, s \in \mathbb{Q}$, let $G^{Q,i}_{(r,s)}$ be defined as

$$\begin{aligned} G^{Q,i}_{(r,s)} \quad &:= \quad \sum_{l=2}^{k} \Big(\frac{A^{(i)}_{Q,l}}{(1-l)(s-r_1)^{l-1}} + \frac{B^{(i)}_{Q,l}}{(1-l)(s-r_2)^{l-1}} - \frac{A^{(i)}_{Q,l}}{(1-l)(r-r_1)^{l-1}} - \frac{B^{(i)}_{Q,l}}{(1-l)(r-r_2)^{l-1}} \Big) \\ &+ \quad A^{(i)}_{Q,1} \ln \left| \frac{s-r_1}{s-r_2} \right| - A^{(i)}_{Q,1} \ln \left| \frac{r-r_1}{s-r_2} \right|. \end{aligned}$$

Proposition 6.2. *Let $r, s \in \mathbb{Q}$ and assume that the discriminant of Q is not a square. Then*

$$\int_r^s \frac{z^i}{(z-r_1)^k(z-r_2)^k} dz = 3\pi\sqrt{-1} \cdot (\gamma_Q \cdot (r,s)) \cdot A^{(i)}_{Q,1} + G^{Q,i}_{(r,s)},$$

where $\sqrt{-1}$ denotes the square root of -1 in $\mathbb{C}$ and the integral is taken over the geodesic in the upper-half plane joining r and s.

Proof. We will denote by (r, s) the geodesic in the upper-half plane joining r and s, which is a semicircle having r and s as endpoints and oriented from r to s.

Case $(\gamma_Q \cdot (r, s)) = 0$

In this case either both roots lie "inside" (r, s) or "outside" of it. If both r_1 and r_2 are outside the semicircle, then the Residue Theorem implies that the integral on (r, s) is the same as the line integral from r to s, which is exactly $G^{Q,i}_{(r,s)}$.

If both roots are inside (r, s) we proceed similarly but this time we need to add detours around the poles. Assume also that $r < r_1 < r_2 < s$ to fix things. Let ρ be a positive number with $4\rho < s - r$. Consider the path given by the union of the segment joining r and $r_1 - \rho$, the semicircle in the lower half-plane of radius ρ and center r_1, the segment joining $r_1 + \rho$ and $r_2 - \rho$, the semicircle in the lower half-plane of radius ρ centered at r_2, and the segment joining $r_2 + \rho$ with s. Call this path S_ρ. Then the Residue Theorem implies

$$\int_r^s \frac{z^i}{(z-r_1)^k(z-r_2)^k} dz = \int_{S_\rho} \frac{z^i}{(z-r_1)^k(z-r_2)^k} dz = G^{Q,i}_{(r,s)}.$$

Case $(\gamma_Q \cdot (r,s)) \neq 0$

In this case one root is inside the semicircle (r,s) and the other is outside. Call y the positive root. We will proceed similarly to the previous case, but we need to add a detour around y. Let the path S_ρ be defined as the union of the segment joining r with $y-\rho$, the semicircle in the lower half-plane of radius ρ centered at y, and the segment joining $y+\rho$ with s. By the Residue Theorem

$$\int_r^s \frac{z^i}{(z-r_1)^k(z-r_2)^k} dz = 2\pi\sqrt{-1} \cdot (\gamma_Q \cdot (r,s)) \cdot A^{(i)}_{Q,1} + \int_{S_\rho} \frac{z^i}{(z-r_1)^k(z-r_2)^k} dz. \tag{34}$$

The theorem follows as one can check that

$$\int_{S_\rho} \frac{z^i}{(z-r_1)^k(z-r_2)^k} dz = \pi\sqrt{-1} \cdot (\gamma_Q \cdot (r,s)) \cdot A^{(i)}_{Q,1} + G^{Q,i}_{(r,s)}.$$

□

Lemma 6.2. *Given a binary quadratic form $Q = [a,b,c]$, let $\widetilde{Q} := [-a,b,-c]$. Then for any $l = 0,...,k$ and $i = 0,...,2k-2$ we have*

$$A^{(i)}_{\widetilde{Q},l} = (-1)^{l+i} A^{(i)}_{Q,l}, \qquad B^{(i)}_{\widetilde{Q},l} = (-1)^{l+i} B^{(i)}_{Q,l},$$

and

$$G^{Q,i}_{(r,s)} = (-1)^{i+1} G^{\widetilde{Q},i}_{(-r,-s)}.$$

Proof. Note that if r_1, r_2 are the first and second root of Q, then $-r_1$ and $-r_2$ are the first and second root of $\widetilde{Q}$, respectively. The computation of $A^{(i)}_{Q,l}, B^{(i)}_{Q,l}$ is simply a residue computation, indeed

$$A^{(i)}_{Q,l} = \mathrm{Res}_{r_1}\left(\frac{z^i(z-r_1)^{l-1}}{(z-r_1)^k(z-r_2)^k}\right) = \sum_{j=0}^{i} \binom{i}{j} r_1^{i-j}\,\mathrm{Res}_{r_1}\left(\frac{(z-r_1)^{j-k+l-1}}{(z-r_2)^k}\right).$$

But

$$\frac{(z-r_1)^{j-k+l-1}}{(z-r_2)^k} = \frac{(z-r_1)^{j-k+l-1}}{(r_1-r_2)(k-1)!} \sum_{t=k-1}^{\infty} \frac{t(t-1)...(t-k+2)(z-r_1)^{t-k+1}}{(r_2-r_1)^t},$$

hence

$$\mathrm{Res}_{r_1}\left(\frac{(z-r_1)^{j-k+l-1}}{(z-r_2)^k}\right) = \binom{2k-j-l-1}{k-1} \frac{(-1)}{(r_2-r_1)^{2k-j-l}}.$$

It follows that

$$A^{(i)}_{Q,l} = \sum_{j=0}^{i} \binom{i}{j} r_1^{i-j} \binom{2k-j-l-1}{k-1} \frac{(-1)}{(r_2-r_1)^{2k-j-l}}.$$

Now $A^{(i)}_{\widetilde{Q},l}$ can be found from the formula for $A^{(i)}_{Q,l}$ simply by substititing $-r_1$ and $-r_2$ in place of r_1, r_2. Then it is clear that $A^{(i)}_{\widetilde{Q},l} = (-1)^{l+i} A^{(i)}_{Q,l}$. The relationship for $B^{(i)}_{Q,l}$, $B^{(i)}_{\widetilde{Q},l}$ can be found in a similar way and the Lemma follows immediately. $\square$

7 A Shimura-Shintani correspondence for rigid analytic cocycles of higher weight

The aim of this section is to construct a cusp form $\hat{\Omega}_k(q)$ of weight $k+1/2$ and level $4p^2$ with coefficients in $\mathrm{MS}^\Gamma(\mathcal{A}_{2k})$, for $k \geq 3$ odd. For this section only, we will assume that $p \equiv 3 \ (mod\, 4)$. More precisely, $\hat{\Omega}_k(q)$ should be an element of $S^{(\bar{\mathbb{Q}})}_{k+1/2}(\Gamma(4p^2)) \otimes \mathrm{MS}^\Gamma(\mathcal{A}_{2k}) \subset \bar{\mathbb{Q}}[[q]] \otimes \mathrm{MS}^\Gamma(\mathcal{A}_{2k})$, where $S^{(\bar{\mathbb{Q}})}_{k+1/2}(\Gamma(4p^2))$ are the weight $k+1/2$ cusp forms of level $4p^2$ whose Fourier coefficients are in $\bar{\mathbb{Q}}$.

We will now describe how one can get a correspondence

$$S^{(\bar{\mathbb{Q}})}_{k+1/2}(\Gamma(4p^2)) \xrightarrow{\mathcal{C}} \mathrm{MS}^\Gamma(\mathcal{A}_{2k})$$

via $\hat{\Omega}_k(q)$. Let $\{g_1, ..., g_t\}$ be a basis of eigenforms for $S^{(\bar{\mathbb{Q}})}_{k+1/2}(\Gamma(4p^2))$. Then we can write

$$\hat{\Omega}_k(q) = \sum_{i=1}^{t} g_i \otimes m_i, \qquad \text{for some } m_i \in \mathrm{MS}^\Gamma(\mathcal{A}_{2k}).$$

Now let $g \in S^{(\bar{\mathbb{Q}})}_{k+1/2}(\Gamma(4p^2))$ and write $g = \sum_{i=1}^{t} \alpha_i g_i$ for some $\alpha_i \in \bar{\mathbb{Q}}$. Then the correspondence $\mathcal{C}$ is defined by letting

$$\mathcal{C}:\ g \mapsto \sum_{i=1}^{t} \alpha_i m_i.$$

As mentioned in the introduction, the rigid analytic cocycle $J_{k,D}$ should play for the correspondence that we aim to construct a role analogous to the role played by the Zagier form $f_{k,D}$ for the classical Shimura-Shintani correspondence. In particular, the series $\hat{\Omega}_k(q)$ should have an expression of the form $\hat{\Omega}_k(q) = \sum_{D>0} D^{k-1/2} J_{k,D} \cdot q^D$, which mimics the formula for the holomorphic kernel function for the Shimura-Shintani correspondence $\Omega_k(q) = \sum_{D>0} D^{k-1/2} f_{k,D} \cdot q^D$. However, if D is a square then $J_{k,D}$ is not defined, so our main result is slightly different.

Theorem 7.1. *Let $k \geq 3$ be odd. If D is not a square and $\left(\frac{D}{p}\right) = 1$, then $D^{k-1/2}J_{k,D}$ is the D-th coefficient of a weight $k+1/2$ cusp form $\hat{\Omega}_k(q)$ of level $4p^2$ with coefficients in $\mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$. The D-th coefficient of $\hat{\Omega}_k(q)$ vanishes if $\left(\frac{D}{p}\right) \neq 1$.*

Proof. The proof consists of two steps.

Step 1.

We will at first construct a level $4p^2$ cusp form $\bar{\Omega}_k(q) = \sum_{D>0} c_D \cdot q^D$ of weight $k+1/2$ with coefficients $c_D \in \mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}})$, where $\mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}}) \subset S_{2k}(\Gamma_0(p))$ is a certain vector space over $\bar{\mathbb{Q}}$. More precisely, $\bar{\Omega}_k(q)$ will be an element of $S_{k+1/2}^{(\bar{\mathbb{Q}})}(\Gamma(4p^2)) \otimes \mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}}) \subset \bar{\mathbb{Q}}[[q]] \otimes \mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}})$. The D-th coefficient of $\bar{\Omega}_k(q)$ vanishes if $\left(\frac{D}{p}\right) \neq 1$. This construction will be carried out in Theorem 7.2.

Step 2.

We will then construct a $\bar{\mathbb{Q}}$-linear map $\iota : \mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}}) \to \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$ and show that $c_D \mapsto D^{k-1/2}J_{k,D}$ if D is not a square. By definition, the resulting generating series $\hat{\Omega}_k(q) = \sum_{D>0} \iota(c_D) \cdot q^D$ is a weight $k+1/2$ cusp form of level $\Gamma(4p^2)$ with coefficients in $\mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$, i.e. an element of an element of $S_{k+1/2}^{(\bar{\mathbb{Q}})}(\Gamma(4p^2)) \otimes \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k}) \subset \bar{\mathbb{Q}}[[q]] \otimes \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$. This will be done in Theorem 7.3. $\square$

Fix an embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$ and let $\mathcal{P}_{2k-2}(\bar{\mathbb{Q}}) \subset \mathcal{P}_{2k-2}$ be the polynomials with coefficients in $\bar{\mathbb{Q}}$ and degree at most $2k-2$. Let $\mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}}) \subset S_{2k}(\Gamma_0(p))$ be the subset of forms f such that $(3\pi\sqrt{-1})^{-1} \cdot \bar{\kappa}_f \in \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2}(\bar{\mathbb{Q}}))$. This is a vector space over $\bar{\mathbb{Q}}$ containing $f_{k,D}^{(p)}$.

Theorem 7.2. *Let $k \geq 3$ be odd. Consider the series $\bar{\Omega}_k(q) = \sum_{D>0} D^{k-1/2} f_{k,D}^{(p)} \cdot q^D$, where D ranges over discriminants with $(\frac{D}{p}) = 1$. Then $\bar{\Omega}_k$ is a weight $k+1/2$ cusp form of level $4p^2$ with coefficients in $\mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}})$.*

Proof. We will proceed similarly to [KZ1] in their Theorem 2, in particular we will use a theorem of [Vig] which was generalized by Stopple ([St]). At first, note that the square root of D appearing in the definition of $\bar{\Omega}_k$ is the positive one in $\mathbb{R}$, and that $\bar{\Omega}_k$ is well defined as $p \equiv 3$ $(mod\, 4)$. Indeed, for such primes there is a canonical choice of $\sqrt{D} \in \mathbb{Q}_p$ as D varies, and this implies that there is no ambiguity in the choice of the terms $(\gamma_Q \cdot e_0)$ appearing in the definition of $f_{k,D}^{(p)}$. For $\left(\frac{D}{p}\right) = 1$, let $s, -s$ be the square roots of D $(mod\, p)$ and let

$$\mathcal{F}_D^{(p),s} := \{Q = [a,b,c] \in \mathcal{F}_D^{(p)} \text{ such that } b \equiv s \ (mod\, p)\}.$$

Then note that

$$f_{k,D}^{(p)}(z) = \sum_{Q \in \mathcal{F}_D^{(p)}(\mathbb{Z})} \frac{(\gamma_Q \cdot e_0)}{Q(z,1)^k} = \pm 2\Bigg(\sum_{Q \in \mathcal{F}_D^{(p),s}(\mathbb{Z})} \frac{1}{Q(z,1)^k} \Bigg),$$

where the sign depends on the choice of a square root of D in $\mathbb{Q}_p$. Let $\mathbb{F}_p^+$ be a set of representatives for $\mathbb{F}_p^\times/\{\pm 1\}$. There is a canonical choice for these representatives, as $p \equiv 3$ $(mod\, 4)$ and hence -1 is not a square modulo p. By letting $q = e^{2\pi i \tau}$ we can see $\bar{\Omega}_k$ as a function of two variables $z, \tau \in \mathcal{H}$ as

$$\bar{\Omega}_k(z,\tau) = \sum_{D>0} D^{k-1/2} f_{k,D}^{(p)}(z) e^{2\pi i D \tau} = \sum_{\alpha \in \mathbb{F}_p^+} \bar{\Omega}_{k,\alpha}(z,\tau), \tag{35}$$

where

$$\bar{\Omega}_{k,\alpha}(z,\tau) := \sum_{\substack{[a,b,c] \in \mathfrak{H}_\alpha \\ b^2-4ac>0}} \frac{(b^2-4ac)^{k-\frac{1}{2}}}{(az^2+bz+c)^k} \cdot e^{2\pi i (b^2-4ac)\tau} \tag{36}$$

and $\mathfrak{H}_\alpha$ is the coset of $[0,\alpha,0]$ in L/L' with $L := \{[a,b,c] \in \mathbb{Z}^3$ such that $p|a\}$ and $L' := \{[a,b,c] \in L$ such that $p|b\}$. Now we apply Theorem 1 of [St], which is a generalization of the theorem on p. 228 of [Vig] (Stopple considers only the case of lattices of even dimension, while Vigneras considers also odd dimensions. The result of Stopple still holds in our case as we are working over $\mathbb{Q}$, see [St] for more details). This shows that the functions $\bar{\Omega}_{k,\alpha}(z,\tau)$

are weight $k+1/2$ modular forms in τ whose level is the same as the level of the lattice L'. By computing the dual of L', we see that the level is $4p^2$. Since $\bar{\Omega}_k(z,\tau)$ has no constant term, the theorem follows. $\square$

Remark 7.1. Note that the functions $\bar{\Omega}_{k,\alpha}$ defined in the proof of Theorem 7.2 are, in general, well defined only up to sign. In our case there is no ambiguity in their definition as $p \equiv 3 \ (mod\, 4)$.

Theorem 7.3. *If D is not a square, there exists a $\bar{\mathbb{Q}}$-linear map $\iota : \mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}}) \to \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k})$ such that $\iota\big(f_{k,D}^{(p)}\big) = J_{k,D}$.*

Proof. The map ι will be the composite $\iota = ST \circ \mathfrak{p}$ of the two maps

$$\mathfrak{S}_{2k}^{(p)}(\bar{\mathbb{Q}}) \xrightarrow{\mathfrak{p}} \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2}(\bar{\mathbb{Q}})) \subset \mathrm{MS}^{\Gamma_0(p)}(\mathcal{P}_{2k-2}) \xrightarrow{\mathrm{ST}} \mathrm{MS}^{\Gamma}(\mathcal{A}_{2k}).$$

Here $\mathfrak{p}$ is the $\bar{\mathbb{Q}}$-linear map defined as

$$\mathfrak{p}(f) := \frac{1}{3\pi\sqrt{-1}} \cdot \bar{\kappa}_f$$

and ST is the Schneider-Teitelbaum lift for rigid analytic cocycles defined in Section 4. Theorem 6.1 and Theorem 5.1 imply that $\mathfrak{p}(f_{k,D}^{(p)}) = \kappa_{k,D}$, where $\kappa_{k,D}$ was defined in proposition 5.3.

In Theorem 5.1 we proved that $\mathrm{Res}_0(J_{k,D}) = \kappa_{k,D}$. By Corollary 2.3.4. and Theorem 4.5.2. of [DT], the map Res defined in Section 5 is injective, and by Lemma 4.1 the map Res_0 is also injective. Hence $\mathrm{ST}(\kappa_{k,D}) = J_{k,D}$ and the theorem follows. $\square$

Remark 7.2. In Theorem 7.3 we are implicitly associating a square root of D in $\mathbb{Q}_p$ to the positive real square root of D. As $p \equiv 3 \ (mod\, 4)$, there is a canonical choice for $\sqrt{D} \in \mathbb{Q}_p$, uniformly in D.

7.1 Future research

It would be interesting to relate our work to the theory of the theta correspondence by writing the series $\bar{\Omega}_k(q)$ as the theta series attached to a Schwartz function $\varphi \in \mathcal{S}(\mathbb{Q}^3 \otimes \mathbb{A}_{\mathbb{Q}})$. Seeing our construction as part of this more general framework would offer several future research directions as the theta correspondence can be specialized to many different settings. For instance, the Zagier form $f_k(D, z)$ defined in the introduction is the restriction of a certain Hilbert modular form which plays a key role in the Doi-Naganuma correspondence between cusp forms and Hilbert modular forms ([Za]). We would like to see if one can define an analogue of this correspondence for rigid meromorphic or analytic cocycles. We would also like to see if this can be done for more general correspondences which were studied by Oda who also generalized the form $f_k(D, z)$ ([Oda]). In general, it would be interesting to learn more about the Kudla program and the theta correspondence, with the intention to explore related p-adic aspects, particularly in the framework of the nascent p-adic version of the program.

In an ongoing work, Darmon, Gehrmann and Lipnowski are laying the foundations for the study of rigid meromorphic cocycles attached to quadratic spaces over $\mathbb{Q}$. Let (V, q) be such a quadratic space of real signature (r, s), and let $V_{\mathbb{Z}}$ be a lattice in V where q has integer values. Let $\Gamma := O(V_{\mathbb{Z}}[1/p])$. In this setting, rigid meromorphic cocycles are classes in $\mathrm{H}^s(\Gamma, \mathcal{M}^\times(X_p))$, where $\mathcal{M}^\times(X_p)$ denotes the multiplicative group of non zero rigid meromorphic functions on a certain rigid analytic space X_p over $\mathbb{Q}_p$, which depends on q. In the case of signature $(2, 1)$, one recovers the rigid meromorphic cocycles defined in this thesis. In [Neg], we studied certain p-adic theta functions that can be seen as rigid meromorphic cocycles for a quadratic space of signature $(3, 0)$. The next step would be to consider the case of signature $(4, 0)$, and in general other signatures, both from a theoretical and computational point of view. In general it would be interesting to understand rigid meromorphic cocycles in broader settings and to study questions in the case of general orthogonal groups similar to the ones currently studied for the case of signature $(2, 1)$. One could for example ask what the space V looks like in other settings, as well as try to define some examples of cocycles in these settings, and eventually classify them. It would be interesting to seek correspondences similar to the one constructed in this thesis in this more general setting. Some work in the direction of extending rigid meromorphic cocycles has already been done by Guitart, Masdeu and Xarles, who in [GMX] considered a quaternionic case from a theoretical and computational point of view.

It would also be interesting to evaluate at RM points the rigid analytic cocycles arising from the correspondence constructed in this thesis to see if they satisfy patterns.

Moreover, the RM values of rigid analytic cocycles should be related to the p-units that arise in the Gross-Stark conjecture and the work of Dasgupta and Kakde, and clarifying this relationship is one of our goals.

In Section 2 we classified certain rigid meromorphic cocycles of higher weight, and we intend to do the same for rigid *analytic* cocycles of higher weight.

It would also be interesting to define the cocycle $J_{k,D}$ in the case when D is a square. This could possibly be done by studying the *winding cocycle* defined in [DPV1], and in particular its logarithmic derivative. This is an object of weight 2 and one can ask what happens for higher weight.

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