Irregular Weight One Point with Dihedral Image

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April 15 2017

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Master of Science

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my supervisor, Professor Henri Darmon for his guidance and for always being available to provide any form of assistance and advice I may need. I would like to thank the Department of Mathematics and Statistics, as well as the National Science and Engineering Research Council of Canada (NSERC) for their financial support. I thank the many fellow students and post-docs that I have encountered. Their drive and passion for mathematics have greatly motivated me to work just as hard. I would especially like to thank Michele Fornea, Alexandre Maksoud, Alice Pozzi, Nicolas Simard and Jan Vonk for sharing their vast wisdom and extremely valuable insight in various topics critical to my studies. Last but not least, I would like to thank my amazing family for their unconditional love and support.

ABSTRACT

Darmon, Lauder and Rotger conjectured that at a classical, ordinary, irregular weight one point of the eigencurve, the relative tangent space is of dimensional two. Conjecturally, we can explicitly describe the Fourier coefficients of the normalized generalized eigenforms that span the space in terms of p-adic logarithms of algebraic numbers. In this thesis, we will present a proof of this conjecure in the following special case. The weight one point is the intersection of two Hida families consisting of theta series attached to Hecke characters on two imaginary quadratic fields that cut out a D_4-extension (the dihedral group of order 8).

ABRÉGÉ

Darmon, Lauder et Rotger ont conjecturé que en un point classique, ordinaire et irrégulier de poids un de la courbe de Hecke, l'espace tangent relatif est de dimension deux. Conjecturalement, nous pouvons décrire explicitement les coefficients de Fourier des fonctions propres normalisées généralisées surconvergentes qui engendrent l'espace, en termes de logarithmes p-adiques de nombres algébriques. Dans cette thèse, nous allons présenter une preuve de cette conjecture dans le cas particulier suivant. Le point de poids un est l'intersection de deux familles de Hida composées de fonctions thêta attachées à deux caractères de Hecke sur des corps quadratiques imaginaires qui découpent une D_4-extension (le groupe diédral d'ordre 8).

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Introduction

Fix a prime number p. In [20], Hida showed that at the classical ordinary points of weight at least two, the eigencurve is smooth and étale over the weight space. This led Bellaïche and Dimitrov [2] to study the eigencurve at classical ordinary points of weight one. Let $f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z}$ be a classical cuspidal newform of weight one, level N and nebentypus χ. By the works of Deligne and Serre [15], there exists an odd continuous irreducible Artin representation

$$\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$$

associated to f. Suppose $\alpha, \beta \in \overline{\mathbb{Q}}_p$ are the roots of the p-th Hecke polynomial $x^2 - a_p x + \chi(p)$. Suppose α and β are roots of unity, then p-stabilizations of f, defined by

$$f_\alpha(z) = f(z) - \beta f(pz) \text{ and } f_\beta(z) = f(z) - \alpha f(pz),$$

are ordinary at p. We say f is regular at p if $\alpha \neq \beta$ and f is irregular otherwise.

Bellaïche and Dimitrov showed that if the modular form f is regular at p, then the eigencurve is smooth at f_α. Additionally, the weight map is étale if and only if f is not the theta series attached to a finite order character on a real quadratic field in which p splits. In the latter case, Cho and Vatsal [4] showed that the weight map is not étale. In [9], Darmon, Lauder and Rotger introduced a one dimensional space consisting of overconvergent generalized eigenforms, which can be naturally identified as the relative tangent space of the eigencurve at f_α. Furthermore, they were able to explicitly describe the ℓ-th Fourier coefficients of a natural basis element in terms of logarithms of ℓ-units. Darmon, Lauder and Rotger conjectured that in the case where f is irregular at p, the tangent space is of dimension 2, and we can describe the Fourier coefficients as logarithms of ℓ-units in a similar way. These results and conjectures will be explained in greater detail in chapter 4.

The main goal of this thesis is to consider the case where f_α is the intersection of two Hida families consisting of theta series attached to two Hecke characters over distinct imaginary quadratic fields that cut out a dihedral group of order eight extension. In chapter 5 of this thesis, we will present an original proof of the aforementioned conjecture of Darmon, Lauder and Rotger in this scenario. The first three chapters will be dedicated to developing the theory necessary to explain the conjecture and the proof.

Chapter 1
Modular Forms

In this chapter, we will give an introduction to the theory of classical modular forms. Most importantly, we will introduce Hecke operators, Hecke eigenforms and Galois representations associated to eigenforms. This theory will be crucial towards motivating p-adic modular forms in later chapters. We will discuss an important class of modular forms, theta series attached to Hecke characters, which we will be working with frequently in future chapters. Finally, the last section will be dedicated towards introducing modular curves and discussing some basic results. The main reference for this chapter is [16].

1.1 Classical Modular Forms

In this section, we will first introduce congruence subgroups and their action on the complex upper half plane, followed by the definition of classical modular forms.

Definition 1.1.1. Let $N \geq 1$ be an integer. Define the group $\Gamma(N)$ to be

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

It is called the *principal congruence subgroup* of level N. More generally, a subgroup Γ of $SL_2(\mathbb{Z})$ is called a *congruence subgroup* of level N if Γ contains the group $\Gamma(N)$.

Example 1.1.2. There are two special congruence subgroups

$$\begin{aligned} \Gamma_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}, \\ \Gamma_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}. \end{aligned}$$

Consider the map

$$\Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z} \quad \text{given by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \mod N.$$

It is easy to check that this is a surjection with kernel $\Gamma(N)$. Similarly, the map

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^* \quad \text{given by} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod N$$

is a surjection with kernel $\Gamma_1(N)$.

There is an action of $SL_2(\mathbb{Z})$ on the complex upper half plane

$$\mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{im}(z) > 0\}$$

given by Möbius transformation. More specifically, given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $z \in \mathfrak{H}$, define the action of γ on τ to be

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

Furthermore, γ acts on $\mathbb{Q} \cup \{\infty\}$ by

$$\gamma\left(\frac{s}{t}\right) = \frac{as + bt}{cs + dt},$$

where ∞ is taken to be $\frac{1}{0}$.

The fundamental domain of the action of $SL_2(\mathbb{Z})$ on the upper half-plane is "almost" the set

$$\mathcal{D} = \left\{\tau \in \mathfrak{H} : |\operatorname{Re}(\tau)| \leq \frac{1}{2}, |\tau| \geq 1\right\}.$$

See Figure 1–1 for a pictural representation of this region. The sketch of the proof is the following. The group $SL_2(\mathbb{Z})$ is generated by the matrices $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The transformation T acts on the complex upper half plane by sending τ to $\tau + 1$. By applying T or T^{-1} a sufficient number of times, τ can be mapped to the region $|\operatorname{Re}(\tau)| \leq \frac{1}{2}$. If the imaginary part of the result is too small, we can apply the transformation S, which sends τ to $-\frac{1}{\tau}$. If $|\tau| < 1$, the transformation S will increase the imaginary part of τ. By applying T or T^{-1} again to land in the $|\operatorname{Re}(\tau)| \leq \frac{1}{2}$ region and repeating the whole process, we can eventually send τ to $\mathcal{D}$. For more details, see Lemma 2.3.1 of [16].

Remark 1.1.3. The region $\mathcal{D}$ is not quite the fundamental domain of the action of the group $SL_2(\mathbb{Z})$ on $\mathfrak{H}$. However, we can show that if $\tau_1, \tau_2 \in \mathcal{D}$ are in the same orbit, then either $\operatorname{Re}(\tau_1) = \pm\frac{1}{2}$ and $\tau_2 = \tau_1 \mp 1$; or $|\tau_1| = 1$ and $\tau_2 = -\frac{1}{\tau_1}$. In other words, with some boundary identifications, $\mathcal{D}$ is the fundamental domain. See Lemma 2.3.2 of [16] for more details.

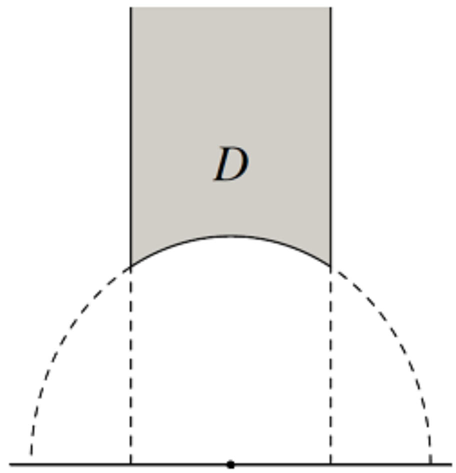

Figure 1–1: Fundamental Domain, Figure 2.3 of [16]

Remark 1.1.4. Let Γ be a congruence subgroup. The set of Γ-equivalent classes of $\mathbb{Q}\cup\{\infty\}$ are called the *cusps*. These cusps have significant geometric meaning. Let Y_Γ be the set of orbits $\Gamma\backslash\mathfrak{H}$. We have already seen that we can take $\mathcal{D} = Y_{SL_2(\mathbb{Z})}$. This is commonly called the *modular curve*. By giving $Y(\Gamma)$ the quotient topology induced from the map $\pi : \mathfrak{H} \to \Gamma\backslash\mathfrak{H} = Y(\Gamma)$, the modular curves are in fact Riemann surfaces. However, these curves are not compact. To compactify them, we need to adjoin the cusps to obtain the Riemann surface $X_\Gamma = \Gamma\backslash(\mathfrak{H}\cup\mathbb{Q}\cup\{\infty\})$. In the case where $\Gamma = SL_2(\mathbb{Z})$, the modular curve $X_{SL_2(\mathbb{Z})}$ can identified as $\mathcal{D}^* = \mathcal{D}\cup\{\infty\}$. See Chapter 2 of [16] for more details. We will discuss modular curves in greater details in the last section of this chapter.

Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and an integer $k \geq 0$, define a weight-k operator $[\gamma]_k$ acting on functions $f : \mathfrak{H} \to \mathbb{C}$ in the following way

$$(f\,[\gamma]_k)(\tau) = (c\tau + d)^{-k} f\left(\gamma(\tau)\right).$$

Definition 1.1.5. Let $N \geq 1$ and $k \geq 0$ be integers. Suppose Γ is a congruence subgroup of level N. A holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is a *modular form* of weight k and level N with respect to Γ, if

$$f(\gamma\tau) = (c\tau + d)^{-k} f(\tau) \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \text{ and } \tau \in \mathfrak{H}.$$

In addition, the function f has to be holomorphic at infinity. More specifically, $(f\,[\gamma]_k)(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$ for all $\gamma \in SL_2(\mathbb{Z})$.

Denote the set of modular forms of weight k with respect to the congruence subgroup Γ by $M_k(\Gamma)$. By holomorphicity, f has a Fourier series expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n \quad \text{where} \quad q = e^{2\pi i \tau}.$$

If in addition, $a_0\left(f\,[\gamma]_k\right) = 0$ for all $\gamma \in SL_2(\mathbb{Z})$, then it is called a *cusp form*. Denote the space of cusp forms of weight k with respect to the congruence subgroup Γ by $S_k(\Gamma)$.

1.2 Hecke Operators

The space of modular forms of fixed weight and level is a vector space. It is endowed with some very special endomorphisms called Hecke operators. In this section, we will define them and show some of their basic properties.

Definition 1.2.1. Let N be a positive integer and suppose $d \in (\mathbb{Z}/N\mathbb{Z})^*$. Define the action of the *diamond operator* $\langle d \rangle$ on $M_k\left(\Gamma_1(N)\right)$ by

$$\langle d \rangle\, f = f[\alpha]_k \text{ for any } \alpha = \left(\begin{smallmatrix} a & b \\ c & \delta \end{smallmatrix}\right) \in \Gamma_0(N) \ \text{ satisfying } \ \delta \equiv d \mod N.$$

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a Dirichlet character and define

$$M_k\left(N, \chi\right) = \left\{ f \in M_k\left(\Gamma_1(N)\right) : \langle d \rangle\, f = \chi(d) f \text{ for all } \ d \in (\mathbb{Z}/N\mathbb{Z})^* \right\}.$$

Then we have the following decomposition of eigenspaces of $\langle d \rangle$:

$$M_k\left(\Gamma_1(N)\right) = \bigoplus_{\chi} M_k\left(N, \chi\right).$$

For a modular form $f \in M_k\left(N, \chi\right)$, χ is called the *nebentypus character* of f. This decomposition is not important for this thesis, but it is very important for the study of the space $M_k\left(\Gamma_1(N)\right)$. We will see that the Hecke operators T_ℓ commute with the diamond operators, which implies that the Hecke operators will preserve this decomposition.

The definition of the diamond operators can be extended to all natural numbers in the following way. If $\gcd\left(d, N\right) = 1$, then define $\langle d \rangle = \langle d \mod N \rangle$. If $\gcd\left(d, N\right) > 1$, define

$\langle d \rangle = 0$. For subsequent chapters, it will be useful to instead use the notation $S_\ell = \ell^{k-2} \langle \ell \rangle$ for $\ell \nmid N$ to denote the operator acting on $M_k(\Gamma_1(N))$.

Definition 1.2.2. Let $N \geq 1$ be an integer and p be a prime number. For each modular form $f \in M_k(\Gamma_1(N))$ define the *Hecke operator* T_p to be

$$T_p f = \begin{cases} \sum_{j=0}^{p-1} f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right]_k & \text{if } p \mid N \\ \sum_{j=0}^{p-1} f\left[\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\right]_k + f\left[\begin{pmatrix} m & n \\ N & p \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k & \text{if } p \nmid N, \text{ where } mp - nN = 1. \end{cases}$$

Remark 1.2.3. Thus far, we have defined the diamond operator $\langle d \rangle$ and the Hecke operator T_p to be some maps sending modular forms to some formal power series expansion. However, their image are in fact still modular forms. That is, they are well-defined endomorphisms on the space of modular forms $M_k(\Gamma_1(N))$, thus justifying the name "operators". One can alternatively define the Hecke operator T_n for any natural number n via some double coset construction (see Chapter 5 of [16]). In that case, it will be immediately clear that they are indeed operators. Furthermore, this construction can be used to define Hecke operators for general congruence subgroups.

Theorem 1.2.4. *Suppose $p \nmid N$, then in terms of q-expansions,*

$$(T_p f) = \sum_{n=0}^{\infty} a_{np}(f) q^n + p^{k-1} \sum_{n=0}^{\infty} a_n(\langle p \rangle f) q^{np}.$$

Proof. This is just a direct calculation. Alternatively, see Proposition 5.3.1 of [16]. $\square$

Proposition 1.2.5. *Let $N \geq 1$ be a positive integer, $c, d \in (\mathbb{Z}/N\mathbb{Z})^*$ and p, ℓ be prime numbers. Then the diamond and Hecke operators satisfy the following identities:*

1. $\langle d \rangle T_p = T_p \langle d \rangle$
2. $\langle c \rangle \langle d \rangle = \langle cd \rangle = \langle d \rangle \langle c \rangle$
3. $T_p T_\ell = T_\ell T_p$.

Proof. See Proposition 5.2.4 of [16]. $\square$

Definition 1.2.6. For $r \geq 1$, recursively define

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}} \text{ for } \; r \geq 2.$$

For a natural number $n \in \mathbb{N}$ with prime factorization $n = \prod p_i^{e_i}$, define $T_n = \prod T_{p_i^{e_i}}$. In light of part three of Proposition 1.2.5, this is well-defined.

1.3 Petersson Inner Product

The goal of this section is to introduce an inner product acting on the vector space of cusp forms $S_k(\Gamma_1(N))$. Most importantly, it will be shown that the Hecke operators are self-adjoint with respect to this inner product. Only a sketch of the construction will be given. The reader should consult Section 5.4 and 5.5 of [16] for more details.

Definition 1.3.1. For each $\tau \in \mathfrak{H}$, write $\tau = x + iy$ where $x, y \in \mathbb{R}$. Define the *hyperbolic measure* to be

$$d\mu(\tau) = \frac{dx\, dy}{y^2}.$$

It can easily be checked that $d\mu(\gamma(\tau)) = d\mu(\tau)$ for all $\gamma \in SL_2(\mathbb{Z})$. In fact, this is true for all $\gamma \in GL_2^+(\mathbb{R})$, the group of invertible matrices with positive determinant. Fix a congruence subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$. Since $\mathbb{Q} \cup \{\infty\}$ is a countable discrete set when viewed as a subset of $\mathbb{C} \cup \{\infty\}$ with the usual topology, it is of measure zero. Therefore, $d\mu$ naturally extends to the set $\mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ and is well-defined on the modular curve X_Γ.

Let $\{\gamma_i\} \subseteq SL_2(\mathbb{Z})$ be some chosen set of coset representatives of $\pm\Gamma \backslash SL_2(\mathbb{Z})$. Up to some boundary identification, the modular curve X_Γ can be represented by the disjoint union

$$\bigsqcup_i \gamma_i(\mathcal{D}^*).$$

For any continuous bounded function $\varphi : \mathfrak{H} \to \mathbb{C}$, and any $\gamma \in SL_2(\mathbb{Z})$, we can check that $\int_{\mathcal{D}^*} \varphi(\gamma(\tau))\, d\mu(\tau)$ converges. Define

$$\int_{X_\Gamma} \varphi(\tau) d\mu(\tau) = \sum_i \int_{\gamma_i(\mathcal{D}^*)} \varphi(\tau)\, d\mu(\tau) = \sum_i \int_{\mathcal{D}^*} \varphi(\gamma_i(\tau))\, d\mu(\tau),$$

which converges as well and is well-defined.

Proposition 1.3.2. *Suppose $f, g \in S_k(\Gamma)$. Then $f(\tau)\overline{g(\tau)}\,(Im(\tau))^k$ is bounded on $\mathfrak{H}$ and is invariant under the action of Γ.*

Sketch of the Proof. The first part can be proven by considering the q-expansion of f and g and noticing that as $\mathrm{Im}(\tau) \to \infty$, the growth rate is $\mathcal{O}(q^{1/h})$ for some positive integer h. The Γ-invariance can be checked directly. $\square$

Definition 1.3.3. The *Petersson inner product* is the Hermitian form $\langle \cdot, \cdot \rangle : S_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$ defined by

$$\langle f, g \rangle_\Gamma = \frac{1}{V_\Gamma} \int_{X_\Gamma} f(\tau)\overline{g(\tau)}\,(\mathrm{Im}(\tau))^k \, d\mu(\tau),$$

where V_Γ is the volume of the fundamental domain $\int_{X_\Gamma} d\mu(\tau)$.

Theorem 1.3.4. *As operators on the vector space $S_k(\Gamma_1(N))$, the adjoints of the operators $\langle \ell \rangle$ and T_ℓ for $\ell \nmid N$ are*

$$\langle \ell \rangle^* = \langle \ell \rangle^{-1} \quad \textit{and} \quad T_\ell^* = \langle \ell \rangle^{-1} T_\ell.$$

Proof. See Section 5.5 and Theorem 5.5.3 of [16]. $\square$

1.4 Eigenforms and Newforms

By Theorem 1.3.4, the Hecke operators commute with their adjoints and so they are normal operators. By the spectral theorem in linear algebra, they are diagonalizable. Since the operators commute by Proposition 1.2.5, the operators are simultaneously diagonalizable. Thus, there is a basis of $S_k(\Gamma_1(N))$ consisting of simultaneous eigenvectors for $\langle \ell \rangle$ and T_ℓ for all $\ell \in \mathbb{N}$, $\gcd(\ell, N) = 1$. These eigenvectors are called *Hecke eigenforms*. If f is an eigenform with $a_1(f) = 1$, we say it is *normalized*.

Proposition 1.4.1. *Suppose Γ is a congruence subgroup of level N. Let $\mathbb{T}_k(\Gamma)$ be the $\mathbb{Z}$-subalgebra of $End_{\mathbb{Z}}(S_k(\Gamma))$ generated by T_ℓ for $\gcd(\ell, N) = 1$ and $\langle d \rangle$ for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then the map given by*

$$\begin{aligned} \mathbb{T}_k(\Gamma) \times S_k(\Gamma) &\to \mathbb{C} \\ (T, f) &\mapsto a_1(Tf) \end{aligned}$$

is a perfect $\mathbb{C}$*-bilinear pairing, which is also* $\mathbb{T}_k(\Gamma)$*-equivariant. Thus, it induces an isomorphism of vector spaces* $S_k(\Gamma) \cong (\mathbb{T}_k(\Gamma))^\vee$*.*

Proof. See Section 6.6 of [16] for the proof of the weight $k = 2$ case. The same proof will also work for general weight k. $\square$

This shows that any algebra homomorphism $\lambda : \mathbb{T}_k(\Gamma_1(N)) \to \mathbb{C}$ is the system of Hecke eigenvalues attached to some eigenform. In fact, they take values in $\overline{\mathbb{Z}}$, because the Fourier coefficients of normalized eigenforms are algebraic integers (see Section 6.5 of [16]). By fixing an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, the Fourier coefficients can be seen to have values in $\overline{\mathbb{Q}}_p$, which will be the approach taken in later chapters. Unfortunately, these systems of eigenvalues do not necessarily correspond to unique eigenvectors. Before we attempt to fix this, we will state a corollary.

Corollary 1.4.2. *A cusp form* f *of weight* k *is a normalized eigenform if and only if its Fourier coefficients satisfy the following:*

1. $a_1(f) = 1$.
2. *For all primes* p*, and integers* $n \geq 2$*, we have* $a_{p^n}(f) = a_p(f)a_{p^{n-1}}(f) - \chi(p)p^{k-1}a_{p^{n-2}}(f)$*.*
3. *For all integers* m, n *satisfying* $\gcd(m,n) = 1$*, we have* $a_{mn}(f) = a_m(f)a_n(f)$*.*

Proof. This essentially follows from the fact that the Hecke operators satisfy these properties. For a more detailed and convincing argument, see Proposition 5.8.5 of [16]. $\square$

Suppose N is a positive integer and suppose $d \mid N$. Let

$$\alpha_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}.$$

Define the map

$$\begin{aligned} i_d : S_k\left(\Gamma_1(Nd^{-1})\right) \times S_k\left(\Gamma_1(Nd^{-1})\right) &\to S_k\left(\Gamma_1(N)\right) \\ (f,g) &\mapsto f + g\left[\alpha_d\right]_k. \end{aligned}$$

By checking the necessary properties, it is easy to see that there is a natural inclusion $S_k\left(\Gamma_1\left(Nd^{-1}\right)\right) \subseteq S_k\left(\Gamma_1(N)\right)$. It is also an easy exercise to show that $g\left[\alpha_d\right]_k \in S_k\left(\Gamma_1(N)\right)$. These show that i_d is well-defined.

Definition 1.4.3. The space of oldforms is the subspace of $S_k\left(\Gamma_1(N)\right)$ given by

$$S_k^{old}\left(\Gamma_1(N)\right) = \sum_{primes\, p|N} i_p\left(S_k\left(\Gamma_1(Np^{-1})\right)^2\right),$$

Let

$$S_k^{new}\left(\Gamma_1(N)\right) = S_k^{old}\left(\Gamma_1(N)\right)^{\perp}$$

be the orthogonal complement of the space of oldforms with respect to the Petersson inner product.

By Atkin-Lehner Theory (see [1] and Sections 5.6-5.8 of [16] for more details), we have the following important theorem.

Theorem 1.4.4.

1. *The spaces $S_k^{new}\left(\Gamma_1(N)\right)$ and $S_k^{old}\left(\Gamma_1(N)\right)$ are stable under the Hecke operators T_n and $\langle n\rangle$ for all $n \in \mathbb{N}$.*
2. *The spaces both have an orthogonal basis of Hecke eigenforms for the Hecke operators T_n and $\langle n\rangle$ for all $n \in \mathbb{N}$ with $\gcd(n, N) = 1$.*
3. *Suppose $f, f' \in S_k^{new}\left(\Gamma_1(N)\right)$ are non-zero eigenforms for the Hecke operators T_n and $\langle n\rangle$, for all $n \in \mathbb{N}$ with $\gcd(n, N) = 1$. If f and f' have the same system of eigenvalues then f, f' differ by some constant scalar.*

Proof. See Proposition 5.6.2, Corollary 5.6.3 and Theorem 5.8.2 of [16] for the proofs of part $1, 2$, and 3 respectively. □

Definition 1.4.5. A normalized eigenform in $S_k^{new}\left(\Gamma_1(N)\right)$ is called a *newform*.

1.5 Galois Representations

Now we are ready to introduce one of the most important results in the theory of modular forms. Every eigenform has an associated Galois representation that satisfies some special properties. The study of these Galois representation is a very important and active part of number theory. Additionally, the basis of the proof of Fermat's Last Theorem, was determining when a certain class of Galois representations (coming from elliptic curves) are modular (coming from modular forms).

Theorem 1.5.1. *Let f be an eigenform of weight k, level N and nebentypus χ. Suppose p is a prime number and fix a finite extension K of $\mathbb{Q}_p$. Let K_f denote the K-algebra over $\mathbb{Q}_p$ generated by all the Fourier coefficients $a_n(f)$ and the values of χ.*

1. *Suppose $k \geq 2$. Then there exists an irreducible Galois representation*
$$\rho_f : G_{\mathbb{Q}} \to GL_2(K_f),$$
such that for all primes $\ell \nmid Np$, the representation ρ_f is unramified at ℓ and the characteristic polynomial of $\rho_f\,(Frob_\ell)$ is $x^2 - a_\ell(f)x + \chi(\ell)\ell^{k-1}$.

2. *Suppose $k = 1$. Then there exists an irreducible Artin representation*
$$\rho_f : G_{\mathbb{Q}} \to GL_2(\mathbb{C}),$$
of conductor N, such that for all $\ell \nmid N$, the characteristic polynomial of $\rho_f\,(Frob_\ell)$ is $x^2 - a_\ell(f)x + \chi(\ell)$.

Proof. For the original proofs, see [13] for part 1 and [15] for part 2. Other good references, but only for the case where the weight is 2, are Chapter 1 of [7] and Chapter 9 of [16]. □

1.6 Hecke Character and Theta Series

In this section, we will introduce Hecke characters. From these characters, we can define and study a special class of modular forms called Hecke theta series. References for these topics include Chapter 8 of [3], Chapter 5 of [25], Section 3.2 of [30], Lecture 1 of [32], [37] and [39].

1.6.1 Hecke Character

Let $K/\mathbb{Q}$ be a finite field extension of degree n. Let $n = r_1 + 2r_2$, where r_1 is the number of real embeddings $K \hookrightarrow \mathbb{R}$ and r_2 is the number of complex embeddings $K \hookrightarrow \mathbb{C}$, up to complex conjugation. Let $\mathcal{O}_K$ be its ring of integer, I_K be the group of idèles of K, and P_K the group of principal idèles. Let $C_K = I_K/P_K$ denote the idèle class group of K.

Definition 1.6.1. A continuous character $\chi : C_K \to \mathbb{C}^\times$ is called a *Hecke character.*

Suppose $a = (a_\mathfrak{p}) \in I_K$ is an idèle, then it determines an ideal $A = \prod_{\mathfrak{p}\nmid\infty} \mathfrak{p}^{v_\mathfrak{p}(\alpha_\mathfrak{p})}$. This assignment determines a surjective group homomorphism $C_K \to Cl_K$ of the idèle class group to the ideal class group. There is an absolute norm on idèles given by

$$\mathfrak{N}(a) = \prod_{\mathfrak{p}} |a_\mathfrak{p}|_\mathfrak{p},$$

where if $\mathfrak{p}$ is an infinite place corresponding to an embedding $i : K \to \mathbb{C}$, then

$$|a_\mathfrak{p}|_\mathfrak{p} = \begin{cases} |i(a_\mathfrak{p})| & \text{if } i \text{ is a real embedding.} \\ |i(a_\mathfrak{p})|^2 & \text{if } i \text{ is a complex embedding.} \end{cases}$$

By the product formula, $\mathfrak{N}$ is trivial on P_K, and so $\mathfrak{N}$ is a well-defined map on C_K (see Chapter 3, Proposition 1.3 of [31]). Given a Hecke character χ, there exists a character $\chi_1 : C_K \to \mathbb{C}^\times$, such that for all $a \in C_K$, $\chi(a) = \chi_1(a)\mathfrak{N}(a)^\sigma$ for some $\sigma \in \mathbb{R}$, where χ_1 takes values in the unit circle of $\mathbb{C}$ (see Proposition 1.1 of [32] for the proof). Such a Hecke character χ_1 is called *unitary.*

Recall that a basis of open sets of 1 of the idèle class group is

$$\prod_{\mathfrak{p}\in S} U_\mathfrak{p} \times \prod_{\mathfrak{p}\notin S} \mathcal{O}_\mathfrak{p}^\times,$$

where S is a finite set of places that includes all the infinite ones, and $U_\mathfrak{p}$ is some basic open set of $K_\mathfrak{p}^\times$. Since χ is required to be continuous, its kernel is a closed subgroup. It is then immediate that the kernel must contain $\mathcal{O}_\mathfrak{p}^\times$ for all but finitely many $\mathfrak{p} \nmid \infty$. Fix such a place $\mathfrak{p}$, and consider the induced character $\chi_\mathfrak{p} : K_\mathfrak{p}^\times \to \mathbb{C}^\times$. Since $\mathcal{O}_\mathfrak{p}^\times \subseteq \ker \chi_\mathfrak{p}$, $\chi_\mathfrak{p}$ is uniquely determined by its value on a uniformizer. This analysis shows that χ_p is given by $\chi_\mathfrak{p}(\alpha) = |\alpha|_\mathfrak{p}^u$ for some $u \in \mathbb{C}$. We say $\chi_\mathfrak{p}$ is *unramified* in this scenario, and its conductor is $\mathcal{O}_\mathfrak{p}$.

Now, consider the ramified case ($\mathfrak{p} \nmid \infty$ such that $\mathcal{O}_{\mathfrak{p}}^{\times}$ is not contained in the kernel of χ). Take a small neighbourhood $U \subseteq \mathbb{C}^{\times}$ of 1 that contains no nontrivial subgroups. Since the sets $1+\mathfrak{p}^n$ form a basic neighbourhood of 1 in $K_{\mathfrak{p}}^{\times}$, $\chi_{\mathfrak{p}}^{-1}(U)$ must contain $1+p^e$ for some $e \in \mathbb{N}$. Then $\chi_{\mathfrak{p}}(1+p^e)$ must be a subgroup of U, but there is none by our choice of U. Hence, $1+\mathfrak{p}^e \subseteq \ker \chi_{\mathfrak{p}}$. Let $e_{\mathfrak{p}}$ be the smallest such number. Then the local conductor of χ at $\mathfrak{p}$ is defined to be $\mathfrak{p}^{e_{\mathfrak{p}}}$. Piecing all the local conductors together, we say the conductor of χ is $\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}}}$ (only finitely many of these terms are not $\mathcal{O}_{\mathfrak{p}}$).

Now, we will introduce the definition of a classical Hecke character. Let I be the group of fractional ideals and P the group of principal ideals. Fix a non-zero ideal $\mathfrak{m}$ of $\mathcal{O}_K$ and denote

$$\begin{aligned} I(\mathfrak{m}) &= \{I \in J : \gcd(I, \mathfrak{m}) = 1\}, \\ P(\mathfrak{m}) &= P \cap I(\mathfrak{m}), \\ K_{\mathfrak{m}} &= \left\{\alpha \in K^{\times} : \alpha \equiv 1 \mod{}^{\times}\mathfrak{m}\right\}, \\ P_{\mathfrak{m}} &= \{\alpha\mathcal{O}_K : \alpha \in K_{\mathfrak{m}}\}, \end{aligned}$$

where $\mod{}^{\times}$ denotes multiplicative congruence. That is, for all prime ideals $\mathfrak{p} \mid \mathfrak{m}$, $\mathrm{ord}_{\mathfrak{p}}(a-1) \geq \mathrm{ord}_{\mathfrak{p}}(\mathfrak{m})$.

Definition 1.6.2. A *Hecke character of modulus* $\mathfrak{m}$ is a group homomorphism $\chi : I(\mathfrak{m}) \to \mathbb{C}^{\times}$, such that there exists a continuous group homomorphism,

$$\chi_{\infty} : (\mathbb{R} \otimes_{\mathbb{Q}} K)^{\times} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \to \mathbb{C}^{\times},$$

such that if $\alpha \in K_{\mathfrak{m}}$, then $\chi((\alpha)) = \chi_{\infty}^{-1}(1 \otimes \alpha) = \chi_{\infty}^{-1}(\alpha)$. The character χ_{∞} is called the *infinity type character* of χ.

Proposition 1.6.3. *Let μ_n denote the n-th roots of unity of $\mathbb{C}$. Then there exists $n \geq 1$ and a finite order character $\epsilon : (\mathcal{O}_K/\mathfrak{m})^{\times} \to \mu_n$ such that*

$$\chi((\alpha)) = \epsilon(\alpha)\chi_{\infty}^{-1}(1 \otimes \alpha) \textit{ for all } \alpha \in K^{\times} \textit{ prime to } \mathfrak{m}.$$

Proof. See Proposition 1.2 of [32]. □

Now, we will like to show how to obtain a classical Hecke character from an idèlic Hecke character χ. Define $\tilde{\chi} : I(\mathfrak{m}) \to \mathbb{C}^\times$ in the following way. For each place $\mathfrak{p}$, let $\pi_\mathfrak{p}$ denote a choice of uniformizer. For each ideal $A \in I(\mathfrak{m})$, define

$$\tilde{\chi}(A) = \prod_{\mathfrak{p}} \chi_\mathfrak{p}(\pi_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p} A}).$$

The goal now is to determine the infinity type character of $\tilde{\chi}$. Suppose $a \in K_\mathfrak{m}$ and suppose at each prime p, $a_\mathfrak{p} = \pi_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p} a} u_\mathfrak{p}$ for some unit $u_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times$. By the definition of the conductor, for all primes $\mathfrak{p} \nmid \mathfrak{m}$, we have $\chi_\mathfrak{p}(u_\mathfrak{p}) = 1$. It follows that $\chi_\mathfrak{p}(a_\mathfrak{p}) = \chi_\mathfrak{p}(\pi_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p} a})$. For the primes $p \mid \mathfrak{m}$, by the definition of $a \equiv 1 \mod {}^\times\mathfrak{m}$, it must be that $a_\mathfrak{p} \in \ker \chi_\mathfrak{p}$. Combining everything together, we obtain the identity

$$1 = \chi(a) = \prod_{\mathfrak{p}} \chi_\mathfrak{p}(a_\mathfrak{p}) = \prod_{\mathfrak{p} \nmid \mathfrak{m}} \chi_\mathfrak{p}(\pi_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p} a}) \prod_{\mathfrak{p} \mid \infty} \chi_\mathfrak{p}(a_\mathfrak{p}) = \tilde{\chi}\left((a)\right) \prod_{\mathfrak{p} \mid \infty} \chi_\mathfrak{p}(a_\mathfrak{p}).$$

Define $\tilde{\chi}_\infty(a) = \prod_{p \mid \infty} \chi_\mathfrak{p}(a_\mathfrak{p})$. From the identity above, we can deduce that $\tilde{\chi}\left((a)\right) = \tilde{\chi}_\infty^{-1}(a)$ for all $a \in K_\mathfrak{m}$ as required. See [37] and [39] for more details and the converse (constructing an idèlic Hecke character from a classical Hecke character).

Up to scaling by a power of the norm map, we can assume that $\tilde{\chi}_\infty$ is unitary. Then, for each $1 \leq \mathfrak{p} \leq r_1 + r_2$, there exists $u_\mathfrak{p}, v_\mathfrak{p} \in \mathbb{R}$ such that for each principal ideal $(a) \in P_\mathfrak{m}$

$$\tilde{\chi}_\infty\left((a)\right) = \prod_{\mathfrak{p}=1}^{r_1+r_2} \left(\frac{a_\mathfrak{p}}{|a_\mathfrak{p}|}\right)^{u_\mathfrak{p}} |a_\mathfrak{p}|^{i v_\mathfrak{p}},$$

where

$$\begin{cases} u_\mathfrak{p} \in \{0,1\} & \text{if } \mathfrak{p} \leq r_1 \\ u_\mathfrak{p} \in \mathbb{Z} & \text{if } \mathfrak{p} > r_1 \end{cases} \qquad \text{and} \qquad \sum_{\mathfrak{p}=1}^{r_1+r_2} v_\mathfrak{p} = 0.$$

If $u_\mathfrak{p} = 0$ for all complex embeddings ($r_1 < \mathfrak{p} \leq r_1 + r_2$) and $v_\mathfrak{p} = 0$ for all infinite places, then we call $\tilde{\chi}_\infty$ a *class character*. It is easy to see that class characters have finite orders.

1.6.2 Hecke Theta Series

This subsection will be dedicated to introducing theta series attached to Hecke characters. The main reference for this material is Chapter 5 of [25] and Section 4.8 of [30].

First, consider the case where $K = \mathbb{Q}(\sqrt{d})$ is a quadratic imaginary field, so that $r_1 = 0$ and $r_2 = 1$. Let D be the discriminant of K. Suppose χ is a unitary Hecke character with modulus $\mathfrak{m}$. Then χ is given by

$$\chi_\infty((a)) = \left(\frac{a}{|a|}\right)^u \qquad \text{for all } (a) \in P(\mathfrak{m}),$$

for some $u \in \mathbb{Z}_{\geq 0}$. We can extend χ to all fractional ideals, by declaring $\chi(A) = 0$ for all $A \notin I(\mathfrak{m})$.

Definition 1.6.4. Let k be a positive integer. Define the weight k *Hecke theta series* associated to the character χ to be

$$\theta_k(\chi, z) = \sum_{I \trianglelefteq \mathcal{O}_K} \chi(I) N(I)^{\frac{k-1}{2}} q^{N(I)},$$

where N is the absolute norm.

Theorem 1.6.5. *The Hecke theta series $\theta_k(\chi, z)$ is a modular form of weight k and level $N' = |D|\, N(\mathfrak{m})$ with nebentypus $\chi_f \cdot \left(\frac{D}{\cdot}\right)$. Here, $\left(\frac{D}{\cdot}\right)$ is the Kronecker symbol. The character χ_f is defined by $\chi_f(n) = \frac{\chi((n))}{\chi_\infty(n)}$ for all integers $n \in (\mathbb{Z}/N'\mathbb{Z})^\times$. The theta series is a cusp form, unless $k = 1$ and $\chi = \psi \circ N$, where ψ is some Dirichlet character. If χ is primitive (its conductor is $\mathfrak{m}$), then θ_k is a newform.*

Proof. See Theorem 4.8.2 of [30]. □

Suppose $K = \mathbb{Q}(\sqrt{d})$ is a real quadratic field of discriminant D. In this case, $r_1 = 2$ and $r_2 = 0$. Let χ be a unitary Hecke character of modulus $\mathfrak{m}$. Suppose that $v_1 = v_2 = 0$, and $(u_1, u_2) = (1, 0)$ or $(0, 1)$. That is, if a' is the conjugate of some $a \in K$, then

$$\chi((a)) = \frac{a}{|a|} = sign(a) \quad \text{or} \quad \frac{a'}{|a'|} = sign(a') \quad \text{for all } a \in P(\mathfrak{m}).$$

Such a character χ is said to be of *mixed signature*, because it is even at one of the archimedean places and odd at the other. The abelian extension F of K corresponding to the kernel of the idèlic character χ will have both real and complex places.

Definition 1.6.6. Define the *Hecke theta function* associated to χ to be

$$\theta(\chi, z) = \sum_{I \trianglelefteq \mathcal{O}_K} \chi(I) q^{N(I)}.$$

Theorem 1.6.7. *The theta series $\theta(\chi, z)$ is a cusp form of weight 1 of level $D \cdot N(\mathfrak{m})$, with nebentypus character $\psi(t) = \left(\frac{D}{t}\right) \chi((t))$. If χ is primitive (if $\mathfrak{m}$ is its conductor) then θ is a newform.*

Proof. See Theorem 4.8.3 of [30]. ☐

The importance of these theta series is the equality between the L-function attached to its corresponding Artin representation (via Theorem 1.5.1) and the L-function attached to the Hecke character. We will briefly explain this result.

Let χ be a unitary Hecke character. Define the *Hecke L-function* attached to χ to be

$$L(\chi, s) = \sum_{I \trianglelefteq \mathcal{O}_K} \chi(I) N(I)^{-s} = \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-s}\right)^{-1}.$$

It converges absolutely and uniformly for $Re(s) \geq 1 + \delta$ for any $\delta > 0$.

Let ρ be a finite dimensional representation on some Galois group G. Then the *Artin L-function* associated to ρ is defined to be

$$L(\rho, s) = \prod_{\mathfrak{p}} \frac{1}{\det\left(1 - N(\mathfrak{p})^{-s} \rho^{I_{\mathfrak{p}}}\left(\mathrm{Frob}_{\mathfrak{p}}\right)\right)}.$$

It is the product of the inverses of the characteristic polynomials of Frobenius elements evaluated at $N(\mathfrak{p})^{-s}$. Here, $I_{\mathfrak{p}}$ is the inertia subgroup of $\mathfrak{p}$, and $\rho^{I_{\mathfrak{p}}}$ is the restriction of ρ to the subspace of elements fixed by $I_{\mathfrak{p}}$. This construction is made so that it does not matter which choice of the Frobenius element is picked, even if $\mathfrak{p}$ is ramified.

Let K be a quadratic field (real or imaginary) and suppose χ is a class character. By global class field theory, we can view χ as a character on $\mathrm{Gal}\left(K^{ab}/K\right)$ where K^{ab} is the maximal abelian extension of K (inside some separable closure). Since χ is a finite order character, it factors through some finite field extension L/K. Let $\rho = \mathrm{Ind}\chi$ be the induced representation of χ to $\mathrm{Gal}(L/\mathbb{Q})$. Then the Galois representation ρ is in fact the associated Artin representation of the Hecke theta series given by Theorem 1.5.1. Furthermore, there is an equality of L-functions,

$$L(\chi, s) = L(\rho, s).$$

See Section 4.8 of [30] for more details.

1.7 Modular Curve

In this section, we will return to discuss the theory of modular curves. Another construction of these objects will be given when we introduce representable functors in the next chapter.

Definition 1.7.1. Let Γ be a congruence subgroup. Define the *modular curve* to be $Y(\Gamma) = \Gamma\backslash\mathfrak{H}$. For the following special congruence subgroups, we will use the following special notations:

$$Y(1) = Y(SL_2(\mathbb{Z})),\ Y_0(N) = Y(\Gamma_0(N)),\ Y_1(N) = Y(\Gamma_1(N)),\ \text{and } Y(N) = Y(\Gamma(N)).$$

Theorem 1.7.2. *We have the following bijections of the complex points of the modular curves.*

1. *$Y_0(N)(\mathbb{C})$ is in bijection with the set of isomorphism classes of pairs (E, C), where E is an elliptic curve over $\mathbb{C}$ and C is an order N cyclic subgroup of E.*
2. *$Y_1(N)(\mathbb{C})$ is in bijection with the set of isomorphism classes of pairs (E, P), where E is an elliptic curve over $\mathbb{C}$ and $P \in E(\mathbb{C})$ is a point of exact order N.*
3. *$Y(N)(\mathbb{C})$ is in bijection with the set of equivalent classes of triples (E, P, Q), where E is an elliptic curve over $\mathbb{C}$ and $P, Q \in E(\mathbb{C})$ generate $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$, the N-torsion subgroup of E. We also require that $e_N(P, Q) = e^{2\pi i/N}$, where e_N is the Weil pairing.*

Proof. See Section 1.5 and most importantly, Theorem 1.5.1 of [16]. $\square$

Corollary 1.7.3. *Since $Y_0(1) = Y(1)$, the complex points of the curve $Y(1)$ is in bijection with the isomorphism classes of elliptic curves over $\mathbb{C}$.*

In the next chapter, we will see that these results are easy corollaries, because these modular curves are coarse moduli spaces that represent some functor classifying isomorphism classes of elliptic curves with some level N structure.

Chapter 2
Deformation Theory

Let n be a positive integer and let

$$\overline{\rho} : \, G_{\mathbb{Q}} \to GL_n(\mathbb{F}_p)$$

be a Galois representation. The goal of this chapter is to study the possible lifts of $\overline{\rho}$ to a Galois representation

$$\rho : \, G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_p),$$

with the property that $\overline{\rho} = \rho \mod p$. Later, we will apply this theory to our study of representations associated to modular forms. In this chapter, we will introduce representable functors and criterions for obtaining representability. Finally, we will discuss tangent spaces of a deformation functor and several important additional conditions that can be imposed on representable functors and still remain representable. The main references for this theory are the works of Mazur from [28] and [29]. Another good reference is the 2009-2010 seminar at Stanford University on modularity lifting from [26] and [27].

2.1 Representable Functors

Definition 2.1.1. Let $\mathcal{C}$ be a locally small category. That is, for all $a, b \in \mathit{Object}(\mathcal{C})$, $\mathrm{Hom}\,(a, b)$ is a set. A functor $\mathcal{F} : \mathcal{C} \to \mathit{Sets}$ is called *representable* if there exists an object $\mathcal{R} \in \mathit{Object}(\mathcal{C})$ and a natural isomorphism $\Phi : \mathrm{Hom}\,(\mathcal{R}, \cdot) \to \mathcal{F}$. Such an $\mathcal{R}$ is called *universal*. If $\mathcal{F}$ is contravariant, then the natural isomorphism is with $\mathrm{Hom}\,(\cdot, \mathcal{R})$.

By Yoneda's Lemma, natural transformations $\Phi : \mathrm{Hom}\,(\mathcal{R}, \cdot) \to \mathcal{F}$ are in one to one correspondence with elements in $\mathcal{F}(\mathcal{R})$. That is, given a natural transformation Φ, we get a special element in $\mathcal{F}(\mathcal{R})$ which is $\Phi\,(Id_{\mathcal{R}})$. This shows that if $\mathcal{F}$ is representable by $(\mathcal{R}, \Phi)$, then there is an *universal element* $c \in \mathcal{F}(\mathcal{R})$ which uniquely determines Φ.

We will encounter two types of representable functors in this thesis.

1. In the first case, the functor $\mathcal{F}$ maps certain subcategories of rings to isomorphism classes of framed elliptic curves. That is, every element of $\mathcal{F}(A)$ is some isomorphism

class of elliptic curves with some additional structure over some base ring A. The universal space $\operatorname{Spec}\mathcal{R}$ is called the *moduli space* and the universal element c is called the *universal elliptic curve*.

2. In the next scenario, $\mathcal{F}(A)$ will be the set of deformations of some given base representation to the ring A, up to strict equivalence. The object $\mathcal{R}$ is called the *universal deformation ring*, and c is the *universal representation*.

Let $A, B \in \mathit{Object}\,(\mathcal{C})$. Since Φ is a natrual isomorphism, it induces an isomorphism of sets $\Phi_A : \operatorname{Hom}(\mathcal{R}, A) \to \mathcal{F}(A)$. Additionally, for all $f : A \to B$, we have the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Hom}(\mathcal{R}, A) \arrow[r, "f"] \arrow[d, leftrightarrow, "\Phi_B"] & \operatorname{Hom}(\mathcal{R}, B) \arrow[d, leftrightarrow, "\Phi_A"] \\
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(B)
\end{tikzcd}$$

Suppose $B = \mathcal{R}$. Let $a \in \mathcal{F}(A)$ and let $f = \Phi_A(c) \in \operatorname{Hom}(\mathcal{R}, A)$. Since $\Phi_{\mathcal{R}}^{-1}(Id_{\mathcal{R}}) = c$ is the universal element, our commutative diagram maps the following elements to one another.

$$\begin{tikzcd}
Id \in \operatorname{Hom}(\mathcal{R}, \mathcal{R}) \arrow[r, "f"] \arrow[d, leftrightarrow, "\Phi_{\mathcal{R}}"] & \operatorname{Hom}(\mathcal{R}, A) \ni f \arrow[d, leftrightarrow, "\Phi_A"] \\
c \in \mathcal{F}(\mathcal{R}) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(A) \ni a
\end{tikzcd}$$

The interpretation of this result in scenario one is that every framed elliptic curves on any base space $\operatorname{Spec} A$ comes about as a pull-back of the universal framed elliptic curve in a unique way. Similarly, in the second scenario, we can also say that any deformation arises from the universal deformation in a unique way.

2.2 Modular Curve Revisited

In this section, we will introduce some moduli problems of classifying isomorphism classes of framed elliptic curves. The standard references for this section are [14] and [24].

Let Z be a ring, and let $\mathcal{C}_Z$ be the set of Z-algebras. Often times, we will just take $Z = \mathbb{Z}$, but this theory applies in positive characteristics as well.

Definition 2.2.1. Define the functor $\mathcal{F}_Z : \mathcal{C}_Z \to Sets$ by

$$\mathcal{F}_Z(A) = \{\text{isomorphism classes of elliptic curves over } \operatorname{Spec} A\}.$$

An *elliptic curve* E over a scheme S, is a scheme E with a smooth proper morphism $p : E \to S$, whose geometric fibres are smooth curves of genus one, together with a section $e : S \to E$ called the zero section. The zero section e identifies the point at infinity of each fibre in the classical definition of elliptic curves.

Given a Z-algebra homomorphism from $f : A \to B$, the morphism $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ is given by base extension. That is, given an elliptic curve E over A,

$$\mathcal{F}(f)(E) = E \times_{\operatorname{Spec} A} \operatorname{Spec} B.$$

Remark 2.2.2. If we view an elliptic curve as a variety defined by a Weierstrass equation in the classical sense, $\mathcal{F}_Z(A)$ is the set of families of elliptic curves over a base space Spec$A$. Instead of considering the category of elliptic curves over a ring A, it is also possible to consider the category of elliptic curves over a general scheme S.

This functor is not representable in general. The problem is that there are elliptic curves E_1 and E_2 that are not isomorphic over some ring, say a field K, but they may be isomorphic over some base extension, such as an algebraic closure $\bar{K}$. To salvage this, we can add additional structures in the definition of our functor, which we will do. Another approach is to settle with a *coarse moduli space*, which is a unique universal space whose points are in bijection with the isomorphism classes of elliptic curves, but does not have an universal element. See [24] for more details on the theory of coarse moduli spaces.

Definition 2.2.3. For an elliptic curve E over a scheme S, with smooth proper morphism $p : E \to S$, let $\underline{\omega}_{E/S}$ denote the invertible sheaf $p_*\left(\Omega^1_{E/S}\right)$ on S.

Consider the functor

$$\mathcal{F}_Z(A) = \{\text{isomorphism classes of } (E, \omega) \text{ over } \operatorname{Spec} A\},$$

where E is an elliptic curve over $\operatorname{Spec} A$, and ω is a basis of $\underline{\omega}_{E/A}$. Here, ω can be thought of as a nowhere vanishing section of $\Omega^1_{E/A}$ on E. This functor is then representable (see [24] for more details).

Definition 2.2.4. A framed elliptic curve over a ring A with level N-structure is a triple (E, ω, α_N), where α_N is of one of the following types.

1. Type 0, α_N is a subgroup scheme of order N on $E_{/A}$.
2. Type 1, $\alpha_N : \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]_{/A}$, where $\xi(1)$ is a section of order N on $E_{/A}$.
3. Type 2, $\alpha_N = (P, Q)$ where (P, Q) is a $\mathbb{Z}/NZ$ basis for $E[N]$.

If N is invertible in A, then the functor $\mathcal{F}_Z$ parametrizing isomorphism classes of framed elliptic curves over A is representable, for any of the above types (see [24] for more details). Suppose $\mathcal{R}$ is the universal Z-algebra and denote $\mathcal{M} = \operatorname{Spec} \mathcal{R}$, which is called the moduli space. By representability, every framed elliptic curve $(E, \omega, \alpha_N) \in \mathcal{F}_Z(A)$ for some $A \in \mathcal{C}_Z$ corresponds to a morphism in $\operatorname{Hom}(\mathcal{R}, A)$, which is the same as a morphism $\operatorname{Hom}(\operatorname{Spec} A, \mathcal{M})$. This says that the A-valued points on this moduli space are exactly the isomorphism classes of framed elliptic curves. The similarity of this result and Theorem 1.7.2 is due to the fact that the modular curves $Y(\Gamma)$ we defined in Chapter 1, are coarse moduli spaces that represent the functor of framed elliptic curves (E, α_N). The moduli space $\mathcal{M}$ is naturally a cover of $Y(\Gamma)$.

Remark 2.2.5. It is possible to consider another moduli problem so that the resulting moduli space corresponds to the compact modular curves $X(\Gamma)$. To do so we will need to replace elliptic curves with generalized elliptic curves and replace the level N-structures with Drinfeld structures. See [6] for more details.

Remark 2.2.6. The coarse moduli spaces $Y(\Gamma)$ are fine moduli spaces (they are universal spaces that represent the functor) in the following cases:

1. the congruence subgroup Γ is $\Gamma(N)$ for $N \geq 3$.
2. the congruence subgroup Γ is $\Gamma_1(N)$ for $N \geq 4$.

Additionally, when $\Gamma = \Gamma_0(N)$, the moduli space is never fine. See [17] for a deeper discussion and see [14] for the proofs of these facts.

2.3 Deformation Functors

For the rest of the chapter, let G be a profinite group and k be a finite field of characteristic p. Let $k[\epsilon]/\epsilon^2$ denote the ring of dual numbers over k. Let Λ be a complete discrete valuation ring with residue field k, such as the ring of Witt vectors $W(k)$.

Let $\mathcal{C}_\Lambda$ be the category of local Artinian Λ-algebras with residue field k, along with local morphisms between the objects. Let $\hat{\mathcal{C}}_\Lambda$ denote the category of complete local Noetherian Λ-algebras with residue field k.

Definition 2.3.1. The group G is said to satisfy the *p-finiteness condition* Φ_p if for every open subgroup G_0 of finite index, it satisfies one of the following equivalent definitions (see [28] for more details).

1. The pro-p completion of G_0 is topologically finitely generated.
2. The abelianization of the pro-p completion of G_0 is of finite type over $\mathbb{Z}_p$.
3. There are only a finite number of continuous homomorphisms $G_0 \to \mathbb{F}_p$.

Example 2.3.2. Let K be a number field and S be a finite set of primes of K. Then $G_{K,S}$, the Galois group associated to the maximal extension of K unramified outside S, satisfies the p-finiteness condition.

Example 2.3.3. Suppose L is some finite extension of $\mathbb{Q}_p$. Then $G_L = \mathrm{Gal}(\bar{L}/L)$ also satisfies Φ_p.

Suppose $\overline{\rho} : G \to GL_n(k)$ is a continuous representation, which we will refer to as a residual representation. Suppose $A \in \hat{\mathcal{C}}_\Lambda$. We say a continuous representation $\rho : G \to GL_n(A)$ is a lift or a deformation of $\overline{\rho}$ if $\rho \mod \mathfrak{m}_A = \rho$, where $\mathfrak{m}_A$ is the maximal ideal of A. Two lifts ρ, γ are strictly equivalent, denoted by $\sim$, if they differ by an element of $\ker(GL_n(A) \to GL_n(k))$ by conjugation. Now consider the following two functors:

- $\mathrm{Def}^{\square}(\bar{\rho})(A) = \{\rho : G \to GL_n(A) : \rho \mod \mathfrak{m}_A = \bar{\rho}\}$.

- $\mathrm{Def}\left(\overline{\rho}\right)(A) = \mathrm{Def}^{\square}\left(\overline{\rho}\right)(A)/\sim$.

The former is called the *framed deformation functor*, while the latter is called the *deformation functor*. In essence, we have "framed" some basis of $\overline{\rho}$ and the deformation is considering the set of pairs (ρ, β) where ρ is a lift of $\overline{\rho}$ and β is a lift of the chosen basis. There is a canonical forgetful functor $\mathrm{Def}^{\square} \to \mathrm{Def}$ defined by forgetting the chosen basis.

We can prove that for each $A \in \hat{\mathcal{C}}_{\Lambda}$,

$$\mathrm{Def}^{\square}\left(\overline{\rho}\right)(A) = \varprojlim_{i} \mathrm{Def}^{\square}\left(\overline{\rho}\right)\left(A/m_A^i\right).$$

The same result holds for the deformation functor under strict equivalence. This shows that for a lot of results, we only need to study the deformations on the category $\mathcal{C}_{\Lambda}$. In particular, if the functors are representable over $\mathcal{C}_{\Lambda}$, then they are pro-representable over $\hat{\mathcal{C}}_{\Lambda}$.

2.4 Schlessinger's Criterion and Representability

Our first goal is to show that these functors are representable (or pro-representable). To do this we need the Schlessinger's Criterion.

Definition 2.4.1. Suppose $A, B \in \mathcal{C}_{\Lambda}$. A morphism $f : A \to B$ is called *small* if it is surjective and its kernel is principal and annihilated by $\mathfrak{m}_A$.

Theorem 2.4.2. *(Schlessinger's Criterion) Suppose $D : \mathcal{C}_{\Lambda} \to Sets$ is a covariant functor, with the property that $D(k)$ is a point. Suppose $A, B, C \in \mathcal{C}_{\Lambda}$, and we have morphisms $A \to C$ and $B \to C$. Canonically, we have a map*

$$\varphi :\ D\left(A \times_C B\right) \to D(A) \times_{D(C)} D(B).$$

Suppose D satisfies the following properties.

1. *φ is a surjection whenever $B \to C$ is a small morphism.*
2. *φ is a bijection when $C = k$ and $B = k\left[\epsilon\right]/\epsilon^2$.*
3. *$D\left(k\left[\epsilon\right]/\epsilon^2\right)$ is finite dimensional.*
4. *φ is a bijection whenever $A \to C$ and $B \to C$ are equal and small.*

Then D is representable.

Proof. See [33] for the proof of this result. ☐

Theorem 2.4.3. *Suppose G satisfies Φ_p then $Def^{\square}(\overline{\rho})$ is representable in $\hat{\mathcal{C}}_\Lambda$. If, in addition, $End_G(\overline{\rho}) = k$, then $Def(\overline{\rho})$ is representable as well.*

Proof. See Proposition 1.6 of [26]. ☐

The additional condition, $End_G(\overline{\rho}) = k$ can be satisfied if $\overline{\rho}$ is absolutely irreducible, or if $n = 2$ and $\overline{\rho}$ is a non-split extension of distinct characters. Note, these are not the only scenarios in which the additional condition can be satisfied, but they will suffice for this thesis. For the rest of the chapter, we will assume that these conditions are satisfied so that $Def(\bar{\rho})$ is representable.

2.5 Tangent Space

Definition 2.5.1. For a functor $D : \hat{\mathcal{C}}_\Lambda \to Sets$, define its *tangent space* to be $t_D = D(k[\epsilon]/\epsilon^2)$.

Notice that given $\rho \in t_D$ and $g \in G$, we have $\rho(g) = \bar{\rho}(g)(1 + \epsilon c(g))$ for some $c \in \mathrm{Ad}(\rho)$. Since $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$, by applying the previous identity, we find that

$$c(gh) = \bar{\rho}^{-1}(h)c(g)\bar{\rho}(g) + c(h).$$

This shows that c is a 1-cocycle and so $c \in Z^1(G, \mathrm{Ad}\overline{\rho})$. By considering ρ up to strict equivalence, it can be deduced that $c \in H^1(G, \mathrm{Ad}\overline{\rho})$. Hence, $t_D = H^1(G, \mathrm{Ad}\overline{\rho})$.

In the case where D is representable, by a Λ-algebra $\mathcal{R}$, we have the following isomorphism of k-vector spaces

$$t_D \cong t_{\mathcal{R}/D} = \mathrm{Hom}_k\left(\Omega_{R/\Lambda} \otimes_R k, k\right) = \mathrm{Hom}_k\left(\mathfrak{m}_{\mathcal{R}}/\left(\mathfrak{m}_{\mathcal{R}}^2 + \mathfrak{m}_\Lambda \mathcal{R}\right), k\right),$$

which justifies the name tangent space. For a proof of this fact, see Section 17 of [29].

2.6 Deformation Conditions

In this section, we will discuss some subfunctors of $\mathrm{Def}(\bar{\rho})$ obtained by imposing some conditions on the possible lifts and explore their representability. The main reference for this material is Section 23 of [29].

Let $F_n = F_n(\Lambda, G)$ be a category defined in the following way. The objects of F_n are pairs (A, V), where $A \in \mathcal{C}_\Lambda$ and V is a free A-module of rank n with a continuous A-linear action of G on V. For example, we can take V to be the representation space of $\rho : G \to GL_n(A)$. A morphism $(A, V) \to (A', V')$ consists of a morphism $A \to A'$ in the category $\mathcal{C}_\Lambda$ and a A-module morphism $V \to V'$ that induces a G-compatible isomorphism $V \otimes_A A' \cong V'$.

Definition 2.6.1. Let $\bar{V}$ be a representable space of $\bar{\rho}$ (say $\bar{V} = k^n$ viewed as a G-module by the action of $\bar{\rho}$). A *deformation condition* $\mathcal{D}$ for $\bar{\rho}$ is a full subcategory $\mathcal{D}F_n$ that contains $\big(k, \bar{V}\big)$, and satisfies:

1. Suppose there is a morphism between two objects of F_N, $(A, V) \to (A', V')$. If (A, V) is an object of $\mathcal{D}F_n$, then so is (A, V).
2. With the same set up, if $A \to A'$ is injective and (A', V') is an object of $\mathcal{D}F_n$ then so is (A, V).
3. Suppose we have morphisms

$$\begin{tikzcd} A \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd}$$

for some $A, B, C \in \mathcal{C}_\Lambda$. Let

$$p_A : A \times_C B \to A \qquad \text{and} \qquad p_B : A \times_C B \to B$$

be the natural projections. Suppose $(A \times_C B, V) \in \mathit{Object}(F_n)$. Let $V_A = V \otimes_{p_A} A$ and $V_B = V \otimes_{P_B} B$. Then

$$(A \times_C B, V) \in \mathit{Object}(DF_n) \text{ if and only if } (A, V_A)\,, (B, V_B) \in \mathit{Object}(DF_n).$$

Given a deformation condition $\mathcal{D}$, define a subfunctor $\mathrm{Def}_\mathcal{D}(\bar{\rho})$ of $\mathrm{Def}(\bar{\rho})$ in the following way. For all $A \in \mathcal{C}_\Lambda$, define $\mathrm{Def}_\mathcal{D}(\bar{\rho})(A)$ to be the subset of $\mathrm{Def}(\bar{\rho})(A)$ that are in the category

$\mathcal{D}F_n$, up to strict equivalence. By extending $\mathrm{Def}_{\mathcal{D}}(\bar{\rho})$ through limits, we can also view it as a functor on $\hat{\mathcal{C}}_\Lambda$.

Theorem 2.6.2. *If $\mathcal{D}$ is a deformation condition, then $\mathrm{Def}_{\mathcal{D}}(\bar{\rho})$ is also representable on $\mathcal{C}_\Lambda$ and pro-representable on $\hat{\mathcal{C}}_\Lambda$.*

Proof. Check that it still satisfies Schlessinger's criterion. See Section 23 of [29] for more details. □

Suppose $\mathrm{Def}(\bar{\rho})$ is represented by $\mathcal{R}$ and $\mathrm{Def}_{\mathcal{D}}(\bar{\rho})$ is represented by $\mathcal{R}_{\mathcal{D}}$. Since $\mathcal{R}$ is universal, there is a natural morphism $\mathcal{R} \to \mathcal{R}_{\mathcal{D}}$. This morphism is in fact surjective, because the induced map on cotangent spaces coming from $\mathrm{Def}_{\mathcal{D}}(\bar{\rho})\left(k[\epsilon]/\epsilon^2\right) \subseteq \mathrm{Def}(\bar{\rho})\left(k[\epsilon]/\epsilon^2\right)$ is injective. This tells us that by picking the conditions nice enough, the ring $\mathcal{R}_{\mathcal{D}}$ is a quotient of $\mathcal{R}$. Geometrically, the space $\mathrm{Spec}\,\mathcal{R}_{\mathcal{D}}$ is a closed subscheme of $\mathrm{Spec}\,\mathcal{R}$.

We will now consider three very important examples of deformation conditions.

Example 2.6.3. Let $\mathrm{Def}_{\mathcal{D}}(\bar{\rho})(A)$ be the set of $\rho : G \to GL_n(A)$ where $\det \rho = \det \bar{\rho}$. It is not hard to show that the tangent space $t_{\mathcal{D}} = \mathrm{Def}_{\mathcal{D}}(\bar{\rho})(k[\epsilon]/\epsilon^2)$ is isomorphic to

$$t_{\mathcal{D}} \cong H^1\left(G, Ad^0\bar{\rho}\right),$$

where $Ad^0\bar{\rho}$ is the trace zero adjoint representation.

Let $\delta^{univ} : G \to \mathcal{R}$ be the determinant of the universal representation of the full deformation functor. Let $\bar{\delta}$ be the determinant of $\bar{\rho}$, and let

$$\delta : G \xrightarrow{\bar{\delta}} \Lambda^* \to \mathcal{R}^*,$$

where the last morphism is just the one giving $\mathcal{R}$ the structure of an Λ-algebra. Then $\mathcal{R}_{\mathcal{D}} = \mathcal{R}/I$ where I is the ideal generated by $\delta(g) - \delta^{univ}(g) \in \mathcal{R}$ for all $g \in G$. For more details, see Section 25 of [29].

Example 2.6.4. Suppose K is a global field and S is a finite set of primes of K. Let $G_{K,S}$ be the Galois group of the maximal extension of K unramified outside S. Recall that $G_{K,S}$ still satisfies the p-finiteness condition Φ_p. If $\bar{\rho}$ is unramified at some $\mathfrak{p} \in S$, then the set of lifts of $\bar{\rho}$ that are still unramified at $\mathfrak{p}$ is a deformation condition.

Example 2.6.5. Let K be a local field with residue field characteristic p. Let $I_K \subseteq G_K$ be its inertia subgroup. A representation $\rho : G_K \to GL_n(A)$ is *ordinary* if

$$\rho\mid_{I_K} \cong \begin{pmatrix} \psi_1 & * & * \\ 0 & ... & * \\ 0 & 0 & \psi_n \end{pmatrix},$$

where $\psi_n : G_K \to A^\times$ is unramified. Then the set of ordinary lifts of $\bar{\rho}$ is a deformation condition. See Section 30 of [29] for a proof of this fact. The ordinary condition will be very important for this thesis, as the Galois representations associated to modular forms that we will encounter are ordinary.

The field K does not necessarily have to be a local field. By picking an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, there is a canonical embedding $\mathrm{Gal}\left(\bar{\mathbb{Q}}_p/\mathbb{Q}_p\right) \hookrightarrow \mathrm{Gal}\left(\bar{\mathbb{Q}}/\mathbb{Q}\right)$ that identifies $\mathrm{Gal}\left(\bar{\mathbb{Q}}_p/\mathbb{Q}_p\right)$ as the decomposition group of $\mathrm{Gal}\left(\bar{\mathbb{Q}}/\mathbb{Q}\right)$ at p. Then we say a representation $\rho : G_{\mathbb{Q}} \to GL_n(A)$ is ordinary at p if $\rho\mid_{G_{\mathbb{Q}_p}}$ is ordinary. Similarly, we can extend this definition to Galois groups of number fields.

Chapter 3
p-adic Modular Forms

In this chapter, we will survey several important topics in the theory of p-adic modular forms in order to set up for the rest of this thesis. We will start with Serre's original definition [34] to motivate p-adic modular forms. Our next goal will be to define Katz's overconvergent modular forms [22]. To do so, we will give an alternate definition of classical modular forms in terms of a moduli problem, and introduce the Hasse invariant. Furthermore, we will introduce generalised p-adic modular functions as defined by Katz in [23] from the point of view of Emerton [18]. The advantage of taking Emerton's point of view is that we can naturally introduce the eigencurve. Finally, we will introduce Hida families and give an important example of a family of theta series.

3.1 Serre's p-adic Modular Forms

We will first introduce Serre's p-adic modular forms as defined in [34]. Fix a prime p. Let v_p denote the standard p-adic valuation on $\mathbb{Q}_p$, normalized so that $v_p(p) = 1$. For a formal power series $f(q) = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Q}_p[\![q]\!]$, we define the valuation of f to be

$$v_p(f) = \inf_n v_p(a_n).$$

Say that a sequence of power series $f_i \in \mathbb{Q}_p[\![q]\!]$ converges to $f \in \mathbb{Q}_p[\![q]\!]$ if $v_p(f_i - f) \to \infty$ as $i \to \infty$.

Let $m \geq 1$ be a positive integer ($m \geq 2$ if $p = 2$). Let

$$X_m = \begin{cases} \mathbb{Z}/(p-1)p^{m-1}\mathbb{Z} \cong \mathbb{Z}/p^{m-1}\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} & \text{if } p \neq 2 \\ \mathbb{Z}/2^{m-2}\mathbb{Z} & \text{if } p = 2 \end{cases}$$

Let

$$X = \varprojlim X_m = \begin{cases} \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z} & \text{if } p \neq 2 \\ \mathbb{Z}_2 & \text{if } p = 2 \end{cases}$$

There is a natural injection $\mathbb{Z} \hookrightarrow X$ whose image will be a dense subgroup of X.

Definition 3.1.1. A *p-adic modular form* is a formal power series

$$f(q)=\sum_{n=0}^{\infty} a_n q^n \in \mathbb{Q}_p[\![q]\!],$$

such that there exists a sequence f_i consisting of classical modular forms of weight k_i with coefficients in $\mathbb{Q}$ such that $\lim f_i = f$.

Theorem 3.1.2. *Suppose f is a non-zero p-adic modular form, which is the limit of a sequence of classical modular forms f_i of weight k_i with rational coefficients. Then the sequence of k_i's converge in X. The limit is called the weight of f and does not depend on the choice of the f_i's.*

Proof. See Theorem 2 of Chapter 1 of [34] for the proof. □

3.2 Alternate Definition of Classical Modular Forms

In the next few sections, we will explore an important class of p-adic modular forms called overconvergent modular forms. However, before we define these objects, we will need to introduce an alternative way to define classical modular forms in order to motivate the construction of overconvergent modular forms. The main references for this section are [14] and [24].

Let $Z = \mathbb{Z}\left[\frac{1}{6}\right]$ and let $\mathcal{C}$ be the category of Z-algebras. Here, we can also let $\mathcal{C}$ be the category of R-algebra for some fixed ring R where 6 is invertible, or the category of S-schemes, for some fixed scheme S. Everything below can be defined in the exact same way, by replacing Z with R or S. Let $\mathcal{F} : \mathcal{C} \to Sets$ be the functor sending $A \in Object(\mathcal{C})$ to the set $\mathcal{F}(A)$ of framed elliptic curves (E, ω, α_N) over A, up to isomorphism. Here, ω is a basis element of $\underline{\omega}_{E/A}$ and α_N is some level N structure (ignore α_N if $N = 1$). Recall that this functor is representable, by some Z-algebra $\mathcal{R}$.

Definition 3.2.1. A *weakly holomorphic modular form* f over Z is an element of $\mathcal{R}$. Alternatively, a modular form f over Z is a rule, that sends a framed elliptic curve $(E, \omega, \alpha_N)_{/A}$ over a Z-algebra A to an element $f(E, \omega, \alpha_N) \in A$, satisfying:

1. f only depends on the isomorphism class of $(E, \omega, \alpha_N)_{/A}$.
2. f commutes with base change. More specifically, given a Z-algebra homomorphism $g : A \to B$, we have a map $\mathcal{F}(g) : \mathcal{F}(A) \to \mathcal{F}(B)$ given by base change of the framed elliptic curves. Then

$$f\left(\mathcal{F}(g)\left((E, \omega, \alpha_N)_{/A}\right)\right) = g\left(f\left((E, \omega, \alpha_N)_{/A}\right)\right).$$

Proposition 3.2.2. *These definitions are equivalent.*

Sketch of the Proof. First, let's view f as an element $a \in \mathcal{R}$. Suppose $(E, \omega, \alpha_N)_{/A}$ is some framed elliptic curve over a Z-algebra A. By representability, this gives rise to a morphism $\phi \in \operatorname{Hom}(\mathcal{R}, A)$. Then $\phi(a)$ gives rise to an element in A, which we define to be $f(E, \omega, \alpha_N)$. It is not hard to check that f satisfies the above properties.

Conversely, suppose we view f as a rule described above. Let $a = f(E_{univ}, \omega_{univ}, \alpha_{N,\,univ}) \in \mathcal{R}$ be the value of f evaluated at the universal framed elliptic curve. □

Definition 3.2.3. A weakly holomorphic modular form f is of weight k if it satisfies the additional property that

$$f\left((E, \lambda\omega, \alpha_N)_{/A}\right) = \lambda^{-k} f\left((E, \omega, \alpha_N)_{/A}\right) \qquad \text{for all } \lambda \in A^\times.$$

To get the "q-expansion" of f, we will draw some intuition from the case of complex numbers. Recall that all elliptic curves over $\mathbb{C}$ is isomorphic to $\mathbb{C}$ modulo a lattice. More specifically, the lattice can be chosen to be of the form $\langle 1, \tau \rangle = \mathbb{Z} + \mathbb{Z}\tau$ for some $\tau \in \mathfrak{H}$. The exponential map gives us an identification $\mathbb{C}/\langle 1, \tau \rangle \overset{\exp}{\to} \mathbb{C}^\times/q^{\mathbb{Z}}$ where $q = e^{2\pi i \tau}$, and the map is $z \mapsto e^{2\pi i z} = t$. A canonical choice of differential for this elliptic curve is given by $\frac{dt}{t} = 2\pi i dz$. Given a modular form f, we can define $f(q) = f\left(\mathbb{C}^\times/q^{\mathbb{Z}}, \frac{dt}{t}\right) \in \mathbb{C}[\![q]\!]$. It can checked that the modular property of the classical modular form definition follows directly from the properties defined by viewing f as a rule.

Definition 3.2.4. For all $x \in \mathbb{C}$, define the *divisor function* to be

$$\sigma_x(n) = \sum_{d|n} d^x.$$

For a positive integer k, define the series $s_k(q)$ to be

$$s_k(q) = \sum_{n\geq 1} \sigma_k(n) q^n.$$

Furthermore, define

$$a_4(q) = -5s_3(q), \text{ and } a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12}.$$

The series $a_4(q)$ and $a_6(q)$ converges for all $q \in K^\times$ and $|q| < 1$, where K is the field of complex numbers $\mathbb{C}$ or K is a p-adically complete field. Otherwise, we can just view these series as formal power series.

Theorem 3.2.5. *Let E_q be the elliptic curve defined by the Weierstrass equation $y^2 + xy = x^3 + a_4(q)x + a_6(q)$ defined over $\mathbb{Z}((q))$. Then $\left(E_q, \frac{dx}{2y+x}\right)$ is isomorphic to $\left(\mathbb{C}^\times/q^{\mathbb{Z}}, \frac{dt}{t}\right)$ over $\mathbb{C}$ and $\left(\mathbb{Q}_p^\times/q^{\mathbb{Z}}, \frac{dt}{t}\right)$ over $\mathbb{Q}_p$.*

Proof. See Theorem 1.1 and Theorem 3.1 in Chapter 5 of [38]. □

The elliptic curve E_q is called the Tate curve and $\frac{dx}{2y+x}$ is called its canonical invariant differential. Since $a_4(q)$ and $a_6(q)$ have integral q-expansions, the Tate curve can be defined over fields of positive characteristics as well. However, it only has good reduction if $p \geq 5$.

For a weakly holomorphic modular form f, the finite tailed Laurent series $f\left(E_q, \omega_{can}\right) \in Z((q))$ is called the q-expansion of f when the level is 1. For a general Z-algebra A, we define the q-expansion of f over A to be the series $f\left((E_q, \omega_{can})_{/A}\right) \in Z((q)) \otimes_Z A$.

For general level N, the level N structure α_N is defined over $Z[\zeta_N]((q^{\frac{1}{N}}))$. However, it is not unique, but there are only finitely many of them, called the *cusps*. For each α_N structure, we have a finite tailed Laurent series

$$f\left((E_q, \omega_{can}, \alpha_N)_{/A}\right) \in Z\left[\zeta_N\right]((q^{\frac{1}{N}})) \otimes_Z A.$$

The cusp that we canonically took to get a classical q-expansion is the cusp at infinity. A weakly holomorphic modular form f is called a modular form if all of its q-expansions at all cusps are power series in $Z[\zeta_N][\![q^{\frac{1}{N}}]\!]$. A modular form f is called a cusp form if its q expansions at all cusps are in $q^{1/N} Z[\zeta_N][\![q^{\frac{1}{N}}]\!]$.

3.3 Hasse Invariant

Fix a prime $p \neq 2, 3$, and suppose R is a $\mathbb{F}_p$-algebra. For an elliptic curve E/R, we have the p-th power absolute Frobenius map F_{abs}. More specifically, since every open affine subset $U = \operatorname{Spec} A$ of E is an $\mathbb{F}_p$-algebra, we have a Frobenius endomorphism on U. By piecing these together, we obtain the absolute Frobenius F_{abs}. This endomorphism also induces a p-linear endomorphism of $H^1(E, \mathcal{O}_E)$, where $\mathcal{O}_E$ is the structure sheaf. By picking a base ω of $\underline{\omega}_{E/R}$, we have also picked a dual base η of $H^1(E, \mathcal{O}_E)$. Then

$$F_{abs}^*(\eta) = A(E, \omega)\,\eta$$

for some $A(E, \omega) \in R$. We also have the following identity,

$$F_{abs}^*\left(\lambda^{-1}\eta\right) = A(E, \lambda\omega)\,\lambda^{-1}\eta \text{ for all } \lambda \in R^\times.$$

On the other hand,

$$F_{abs}^*\left(\lambda^{-1}\eta\right) = \lambda^{-p}F_{abs}^*(\eta) = \lambda^{-p}A(E, \omega)\,\eta.$$

Combining the two identities, we find that

$$A(E, \lambda\omega) = \lambda^{1-p}A(E, \omega).$$

This shows that A is a weakly holomorphic modular form mod p of weight $p-1$ and level 1. In fact, it is a modular form, with q-expansion $A\left((E_q, \omega_{can})_{/\mathbb{F}_p}\right) = 1 \in \mathbb{F}_p[\![q]\!]$ (see Section 2.0 of [22]). We call A the *Hasse invariant*.

Since this construction is not very enlightening to what we want to do next, we will present another construction of the Hasse invariant, following the works of Serre [36]. The motivation here, is to study modular forms over $\mathbb{F}_p$. Let $Z = \mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at (p), and define modular forms over Z of level 1 the same way we have in this chapter.

Definition 3.3.1. Define the *Ramanujan delta function* to be

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} \in q\mathbb{Z}[\![q]\!],$$

which is a weight 12 cusp form. For a positive integer k, define the *Eisenstein series* of weight $2k$ to be

$$E_{2k}(q) = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n,$$

where B_{2k} is the $2k$-th Bernoulli number.

Proposition 3.3.2. *Let $M_k(Z)$ denote the ring of modular forms over Z of weight k. Then M_k is free over Z, and generated by $\left\{E_4^a E_6^b : 4a+6b=k\right\}$.*

Proof. This is a standard result. See Section 1.1 of [16]. $\square$

Corollary 3.3.3. *$M_k(\mathbb{F}_p) = M_k(Z) \otimes \mathbb{F}_p$ is then a $\mathbb{F}_p$-vector space, generated by $E_4^a E_6^b$.*

By mapping each modular form to its corresponding q-expansion, we get a map $M_k(\mathbb{F}_p) \to \mathbb{F}_p[\![q]\!]$.

Proposition 3.3.4. *The above map is injective.*

Proof. Suppose $\sum_{4a+6b=k} \lambda_{a,b} E_4^a E_6^b(q) = 0$ for $\lambda_{a,b} \in \mathbb{F}_p$. For each a, b, pick $\tilde{\lambda}_{a,b} \in Z$ to be a lift of $\lambda_{a,b}$. Then we must have

$$\sum_{4a+6b=k} \tilde{\lambda}_{a,b} E_4^a E_6^b(q) \in pZ[\![q]\!].$$

Since $M_k(Z)$ is free with basis $E_4^a E_6^b$, $p \mid \tilde{\lambda}_{a,b}$ and so $\lambda_{a,b} = 0$ in $\mathbb{F}_p$. $\square$

While the weight k q-expansion map is injective, the map $M(\mathbb{F}_p) = \oplus_k M_k(\mathbb{F}_p) \to \mathbb{F}_p[\![q]\!]$ is not. There can be modular forms of different weights with the same q-expansion over $\mathbb{F}_p$. Let $\bar{M}(\mathbb{F}_p)$ denote the image of $M(\mathbb{F}_p)$ in $\mathbb{F}_p[\![q]\!]$, whose elements we call *modular forms mod p* . To understand this space, we can study the kernel,

$$I = \ker\left(\mathbb{F}_p[E_4, E_6] \to \mathbb{F}_p[\![q]\!]\right). \tag{3.1}$$

It turns out to be a principal ideal, generated by $A(x,y) - 1 \in \mathbb{F}_p[x,y]$, where A is an irreducible homogenous polynomial of degree $p-1$ satisfying the identity $A(E_4, E_6) = E_{p-1}$. By showing that $E_{p-1} \mod p = 1$ (has q expansion 1 in $\mathbb{F}_p[\![q]\!]$), A is exactly the Hasse invariant we had previously defined. The construction we have described only works if $p \neq 2, 3$. However, when $p = 2$ or 3, there are other methods of lifting the Hasse invariant. See [36] for more details.

Theorem 3.3.5. *An elliptic curve $E_{/\bar{\mathbb{F}}_p}$ is supersingular if and only if $A(E, \omega) = 0$ for all ω.*

3.4 Overconvergent Modular Form

Now, we will define overconvergent modular forms following the definition of Katz [22]. The main idea is that we cut out a disk of radius r around each supersingular elliptic curve from the modular curve, and define modular forms on the resultant space.

Definition 3.4.1. Let R_0 be a p-adically complete ring. Let $r \in R_0$, $N \geq 1$ be an integer prime to p. A *weakly holomorphic p-adic modular form* f of level N, weight k and growth r is a rule that to any quadruples $(E/R, \omega, \alpha_N, Y)$, gives an element $f\left(E/R, \omega, \alpha_N, Y\right) \in R$, subjecting to the requirements that

- R is an R_0-algebra where p is nilpotent ($p^n = 0$ for some $n > 0$).
- E/R is an elliptic curve.
- ω is a base of $\underline{\omega}_{E/R}$.
- α_N is some level N-structure.
- $Y \in R$ satisfies the property that $Y \cdot E_{p-1}\left(E, \omega\right) = r$.

Similar to the definition of classical modular forms, we also require that f only depends on the isomorphisms classes of the quadruple, and commutes with base change. Additionally, f satisfies the identity

$$f\left(E/R, \lambda\omega, \alpha_N, \lambda^{p-1}Y\right) = \lambda^{-k} f\left(E/R, \omega, \alpha_n, Y\right) \quad \text{for all } \lambda \in R^{\times}.$$

A weakly holomorphic modular form f is *holomorphic* if at all level N structures α_N, the value of f at

$$\left(E_q, \omega_{can}, \alpha_N, r\left(E_{p-1}(E_q, \omega_{can})\right)^{-1}\right)$$

is in $\mathbb{Z}[\![q]\!] \otimes \left(R_0/p^N R_0\right)[\zeta_N]$. Similarly, f is a cusp form if its value at the same quadruple is in $q\mathbb{Z}[\![q]\!] \otimes \left(R_0/p^N R_0\right)[\zeta_N]$. Denote by $M_k\left(R_0, N, r\right)$, the space of holomorphic p-adic modular forms over R_0 of weight k, level N and growth r. Similarly, denote the space of p-adic cusp forms of weight k, level N and growth r by $S_k\left(R_0, N, r\right)$. If r is not a p-adic unit, we call these modular forms *overconvergent*.

We will now give some motivation for this construction. To define p-adic modular forms, we would like the congruences of the q-expansions of modular forms to be congruences themselves. For example, since

$$E_{p-1}(q) \equiv 1 \mod p,$$

we would like to to have the identity $E_{p-1} - 1 = pf$ for some modular form f defined over R. However, this will not hold for the following reason. Let $(E, \omega)_{/R}$ be a supersingular elliptic curve. Since E_{p-1} is a lift of the Hasse invariant, by Theorem 3.3.5, we must have $E_{p-1}(E, \omega) = 0 \mod p$.

Let $M(N)$ denote the modular curve associated to $\Gamma_1(N)$-type framed elliptic curves over $\mathbb{Z}_p$, and $\bar{M}(N)$ its compactification. Another definition of classical modular forms, is that they are sections of bundles on the modular curve. Let $\bar{M}(N)_{\geq r}$ denote the resulting modular curve after we have removed a disk of radius r around every elliptic curve with supersingular reduction. In essence, we still allow elliptic curves that are not "too" supersingular. By defining p-adic modular forms to be the sections of some bundles on $\bar{M}(N)_{\geq r}$, we will have the desired congruence property (see [19] for a deeper discussion of this fact). The earlier definition of overconvergent modular forms is equivalent to this construction. If there exists $Y \in R$ such that $Y \cdot E_{p-1}(E, \omega) = r$, then $|E_{p-1}(E, \omega)|_p \geq |r|_p$. This shows that the elliptic curve (E, ω) is at least $|r|_p$ away from supersingular elliptic curves. When r is a p-adic unit, the resultant modular curve consists only of ordinary elliptic curves.

Another extremely important motivation to studying the space of overconvergent modular forms is that the U_p operator is better behaved on this space. We will now briefly describe this result. We can define the Hecke operators T_ℓ and diamond operators $\langle d \rangle$ acting on the space $M_k(R_0, N, r)$. Their action on q-expansions will coincide with their action in the classical case. Since this result suffices for the purposes of this thesis, we will take this to be the definition. See Chapter II, Section 2.1 of [19] for more details. By examining the acting of the Frobenius, we can define another important continuous operator on the space of overconvergent modular forms, called the U_p operator. On q-expansions, U_p is given by the map

$$U_p : \sum a_n q^n \mapsto \sum a_{np} q^n,$$

which we will take to be the definition. The space of overconvergent modular forms $M_k(R_0, N, r)$ is a p-adic Banach space (see Section 2.6 of [22]). The action of U_p on this space has the following nice property.

Theorem 3.4.2. *Let K denote the field of fractions of R_0. Suppose $N \geq 3$ and $p \nmid N$. Assume that $k \neq 1$ or $N \leq 11$. Then the map*

$$U_p : M_k(R_0, N, r) \otimes K \to M_k(R_0, N, r^p) \otimes K$$

is a bounded homomorphism of p-adic Banach spaces. In other words, U_p improves overconvergence.

Proof. See Proposition II.3.6 and Corollary II.3.7 of [19]. ☐

Corollary 3.4.3. *With the same assumptions as the previous theorem. Assume in addition that R_0 is a p-adically complete discrete evaluation ring such that R_0/pR_0 is finite and $0 < \mathrm{ord}_p(r) < \frac{p}{p+1}$. Then U_p is a completely continuous operator on the space $M_k(R_0, N, r)$ in the sense of [35].*

Proof. See Proposition II.3.15 of [19]. ☐

This corollary allows us to study the spectral theory of U_p on the space of overconvergent modular forms. In particular, for all $\alpha \geq 0$, the set of eigenvalues λ of U_p satisfying $\mathrm{ord}_p(\lambda) = \alpha$ is finite and its generalized eigenspace is finite dimensional (see Section II.3 of [19]). Meanwhile, the kernel of U_p is infinite dimensional.

3.5 The Hecke Algebra

In this section, we will introduce the space of generalized p-adic modular functions as defined by Katz [23]. This space contains the space of overconvergent modular forms defined in the previous section, and still satisfy nice congruence properties between modular forms of different weights. Similar to the classical case, we also have a duality between the space of generalized p-adic modular functions with the Hecke algebra acting on it (technically, only true for the parabolic case and with slight modifications for the non-parabolic functions, see

Chaper 3, Section 1.2, 1.3 of [19]). In light of this duality, we will instead study the Hecke algebra following the approach of [18] instead. An advantage of this is the ability introduce the eigencurve more easily later in this chapter.

Let N be a positive integer. For each $k \geq 1$, let $\mathbb{T}_k(N)$ be the $\mathbb{Z}$-subalgebra of $\operatorname{End}(M_k(\Gamma_1(N)))$ generated by the Hecke operators T_ℓ and $S_\ell = \ell^{k-2}\langle \ell \rangle$, where $\ell \nmid N$ are primes. By definition, we will also have T_n for all $n \in \mathbb{N}$ with $\gcd(n, N) = 1$. If the level N is understood, we will denote $\mathbb{T}_k(N)$ by just $\mathbb{T}_k$.

Recall that a Hecke eigenform can be viewed as an algebra homomorphism $\lambda : \mathbb{T}_k \to \mathbb{C}$, where λ is the system of eigenvalues attached to f, given by $Tf = \lambda(T)f$ for all $T \in \mathbb{T}_k$. Additionally, the values of λ are in fact algebraic integers, ie. in $\bar{\mathbb{Z}}$. This means that by fixing a prime $p \nmid N$ and an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, λ can be viewed as taking values in $\overline{\mathbb{Q}}_p$, still denoted by λ. By reducing modulo the maximal ideal of $\bar{\mathbb{Z}}_p$, we get a homomorphism $\overline{\lambda} : \mathbb{T}_k \to \overline{\mathbb{F}}_p$.

Definition 3.5.1. Let $\mathbb{T}_k^{(p)}$ denote the subalgebra of $\mathbb{T}_k$ generated by T_ℓ and S_ℓ for primes $\ell \nmid Np$. The restriction of some system of eigenvalue λ to $\mathbb{T}_k^{(p)}$ will be denoted $\lambda^{(p)}$ and called a *p-deprived system of eigenvalues.*

Proposition 3.5.2. *Suppose λ_1, $\lambda_2 : \mathbb{T}_k \to \mathbb{C}$ or $\overline{\mathbb{Q}}_p$ agree on T_ℓ for all but finitely many $\ell \nmid N$, then they must be equal. Hence, $\mathbb{T}_k^{(p)}$ has finite index inside $\mathbb{T}_k$.*

Proof. See Proposition 1.26 and Corollary 1.27 of [18]. □

Definition 3.5.3. Let $\mathbb{T}_{\leq k}^{(p)}(N)$, or $\mathbb{T}_{\leq k}^{(p)}$ if the level is understood, be the $\mathbb{Z}$-subalgebra of the endomorphisms of $\bigoplus_{i=1}^{k} M_i(\Gamma_1(N))$ generated by T_ℓ and S_ℓ for primes $\ell \nmid Np$. Here, the action of T_ℓ and S_ℓ is the diagonal action on each direct summand by the Hecke operators of the same name.

Suppose $k' \geq k$. Then we have a surjection

$$\mathbb{T}_{\leq k'}^{(p)} \to \mathbb{T}_{\leq k}^{(p)}$$

obtained by restriction on the spaces in which the Hecke operators act. By tensoring, we get another surjection

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_{\leq k'} \to \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_{\leq k}.$$

By taking projective limits with respective to the above surjections, we obtain the *p-adic Hecke algebra*

$$\mathbb{T} = \mathbb{T}(N) = \lim_{\substack{\leftarrow \\ k}} \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_{\leq k}.$$

The space of modular forms that $\mathbb{T}$ acts on is called the space of *generalized p-adic modular functions*, defined by Katz [23]. Furthermore, there is a canonical injection $\mathbb{T}^{(p)}_{\leq k} \hookrightarrow \prod_{i=1}^{k} \mathbb{T}_i$. By passing through limits, we obtain an injection

$$\mathbb{T} \hookrightarrow \prod_{k \geq 1} \left(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_k\right).$$

Theorem 3.5.4. *$\mathbb{T}$ is a product of finitely many complete Noetherian local $\mathbb{Z}_p$-algebras.*

Proof. See Theorem 2.7 of [18]. $\square$

Definition 3.5.5. A $\mathbb{Z}_p$-algebra homomorphism $\xi : \mathbb{T} \to \overline{\mathbb{Z}}_p$ is called a *p-adic system of Hecke eigenvalues.*

We can canonically construct a p-adic system from a classical system of eigenvalues $\lambda^{(p)} : \mathbb{T}^{(p)}_k \to \overline{\mathbb{Z}}_p$ in the following way. The p-deprived system of eigenvalues $\lambda^{(p)}$ extends to a $\mathbb{Z}_p$-algebra homomorphism $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_k \to \overline{\mathbb{Z}}_p$. By precomposing with the surjection $\mathbb{T} \twoheadrightarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}^{(p)}_k$, we have obtained the desired homomorphism.

Theorem 3.5.6. *Let $\xi : \mathbb{T} \to \overline{\mathbb{Z}}_p$ be a p-adic system of Hecke eigenvalues. Then there is a continuous, semisimple representation*

$$\rho_\xi : G_{\mathbb{Q}} \to GL_2\left(\overline{\mathbb{Q}}_p\right),$$

that is unramified at all primes $\ell \nmid Np$. For such a prime ℓ, the characteristic polynomial of $Frob_\ell$ is of the form $x^2 - \xi(T_\ell)x + \xi(S_\ell)$.

Sketch of the Proof. If ξ is classical, originating from some classical system of eigenvalues λ, then set $\rho_\xi = \rho_\lambda$ coming from Theorem 1.5.1. The rest follows from the fact that classical systems are “dense”. See Theorem 2.11 of [18] for more details. $\square$

3.6 Weight Space

Our goal is to construct a “weight” associated to p-adic system of Hecke eigenvalues that generalizes the classical notion. That is, if $\xi : \mathbb{T}_k \to \overline{\mathbb{Z}}_p$ is a p-adic system of Hecke eigenvalues coming from a classical system of weight k, then ξ should have weight k.

Let $q = 4$ if $p = 2$ and $q = p$ otherwise. Denote the group $1 + q\mathbb{Z}_p$ by Γ. Define $\mathfrak{L}$ to be the set

$$\mathfrak{L} = \{\ell \text{ prime} : \ell \equiv 1 \mod Nq\} \subseteq \Gamma.$$

We claim that $\mathfrak{L}$ is dense in Γ. Suppose $\alpha = 1 + qa \in \Gamma$ for some $a \in \mathbb{Z}_p$. Let n be a positive integer. Since $p \nmid N$, we can let $c = \frac{a}{N} \mod p^n$. Since $1 + Nqc$ and Nqp^n are trivially coprime, Dirichlet's theorem on primes in arithmetic progression tell us that there are infinitely many positive integers m such that $1 + Nqc + Nqp^n m$ is a prime. Suppose $\ell = 1 + Nqc + Nqp^n m \in \mathfrak{L}$ is such a prime. From the identity

$$\ell - \alpha = q\,(Nc - a + p^n m) \equiv 0 \mod p^n,$$

we can deduce that $\ell - \alpha$ has p-adic valuation greater than n as required.

Proposition 3.6.1. *Let $\mathbb{Z}_p[\![\Gamma]\!] = \varprojlim_n \mathbb{Z}_p\left[\Gamma/\Gamma^{p^n}\right]$ denote the completed group ring of Γ over $\mathbb{Z}_p$. The map $\mathfrak{L} \to \mathbb{T}^\times$ given by $\ell \mapsto S_\ell$, extends to a continuous $\mathbb{Z}_p$-algebra homomorphism*

$$\omega : \mathbb{Z}_p[\![\Gamma]\!] \to \mathbb{T}^\times.$$

Proof. Suppose λ is a classical system of eigenvalues of weight k. Since $\ell \equiv 1 \mod Nq$, then trivially $\ell \equiv 1 \mod N$. This means $\langle \ell \rangle = \langle 1 \rangle$ acts trivially on modular forms and so $\lambda(\ell) = \ell^{k-2} \langle \ell \rangle = \ell^{k-2}$. Since the map

$$\mathfrak{L} \to \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_k^{(p)} \text{ given by } x \mapsto x^{k-2}$$

is continuous on Γ, it naturally extends to a continuous map $\Gamma \to \left(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{T}_k^{(p)}\right)^\times$. By passing through inverse limits, we get a continuous map $\Gamma \to \mathbb{T}^\times$, which uniquely extends to a $\mathbb{Z}_p$-algebra homomorphism as required. □

A $\overline{\mathbb{Z}}_p$-valued point of $\mathbb{Z}_p[\![\Gamma]\!]$ is a morphism $\operatorname{Spec} \overline{\mathbb{Z}}_p \to \operatorname{Spec} \mathbb{Z}_p[\![\Gamma]\!]$, which is equivalent to a character $\chi : \Gamma \to \overline{\mathbb{Z}}_p^\times$. In this way, $\operatorname{Spec} \mathbb{Z}_p[\![\Gamma]\!]$ can be viewed as “the space of characters of Γ". Let $k \in \mathbb{N}$ be a positive integer. Consider the character $\chi_k : \Gamma \to \overline{\mathbb{Z}}_p^\times$ given by $\chi_k(x) = x^{k-2}$. The $\overline{\mathbb{Z}}_p$-point χ_k is called the *point of weight* k on $\operatorname{Spec}\mathbb{Z}_p[\![\Gamma]\!]$. The set $\{\chi_k\}_{k\geq 1}$ is Zariski dense in $\operatorname{Spec}\mathbb{Z}_p[\![\Gamma]\!]$. In this way, $\operatorname{Spec}\mathbb{Z}_p[\![\Gamma]\!]$ can be viewed as the "weight space".

To justify these names, suppose $\lambda : k \to \overline{\mathbb{Z}}_p$ is a classical system of Hecke eigenvalue, and $\xi : \mathbb{T} \to \overline{\mathbb{Z}}_p$ is its associated p-adic system. Notice that for all $\ell \in \mathfrak{L}$, the map $\xi \circ \omega$ is given by

$$\xi\left(\omega(\ell)\right) = \xi(S_\ell) = \ell^{k-2}.$$

Since $\mathfrak{L}$ is dense in Γ, the map $\xi \circ \omega$ is given by $x \mapsto x^{k-2}$ on all of Γ. That is, $\xi \circ \omega = \chi_k$ is the point of weight k. Hence, the map ω, can be thought of as sending a p-adic system of eigenvalue to its corresponding weight.

Remark 3.6.2. Recall that in the definition of Serre's p-adic modular form, the weight space is $p-1$ number copies of Γ. Here, we are just taking one of these disks.

Remark 3.6.3. There is a natural isomorphism $\mathbb{Z}_p[\![T]\!] \cong \mathbb{Z}_p[\![\Gamma]\!]$ by sending $1+T$ to the topological generator $[1+q]$. This is commonly called the Iwasawa algebra of the group Γ.

3.7 The Eigencurve

Let $\mathbb{G}_m$ denote the group scheme $\operatorname{Spec}\mathbb{Z}_p\left[T, T^{-1}\right]$. Suppose f is a classical eigenform of level N and weight k with system of eigenvalues λ_f. We can associate a $\overline{\mathbb{Q}}_p$-valued point of $\operatorname{Spec}\mathbb{T} \times_{\mathbb{Z}_p} \mathbb{G}_m$ to f in the following way. The first coordinate will be $\xi : \operatorname{Spec}\mathbb{T} \to \bar{\mathbb{Z}}_p$, the associated p-adic system of eigenvalues that we have constructed in the last section, and the second coordinate is α, a root of the p-th Hecke polynomial of f,

$$x^2 - \lambda_f(T_p)x + p^{k-1}\lambda_f\left(\langle p\rangle\right) = x^2 - \lambda_f(T_p)x + p\lambda_f(S_p).$$

Let $\mathcal{X}$ be the set of such $\bar{\mathbb{Q}}_p$-valued points. Let $\mathcal{X}^{\text{ord}}$ be the subset of $\mathcal{X}$ whose $\bar{\mathbb{Q}}_p$-valued points (ξ, α) satisfies the additional constraint that $\alpha \in \overline{\mathbb{Z}}_p^\times$.

Definition 3.7.1. Let $\mathcal{C}$ be the rigid analytic Zariski closure of $\mathcal{X}$ inside $\left(\operatorname{Spec}\mathbb{T} \times_{\mathbb{Z}_p} \mathbb{G}_m\right)^{an}$. It is called the *eigencurve* of tame level N. Let $\mathcal{C}^{\text{ord}}$ be the Zariski closure of $\mathcal{X}^{\text{ord}}$ inside

$\operatorname{Spec}\mathbb{T} \times_{\mathbb{Z}_p} \mathbb{G}_m$. Denote the analytification of $\mathcal{C}^{\mathrm{ord}}$ the same way. It is called the ordinary part of the eigencurve.

Theorem 3.7.2. *The eigencurve $\mathcal{C}$ is one-dimensional, thus justifying the name eigencurve. More specifically, the map*

$$\mathcal{C} \hookrightarrow \left(\operatorname{Spec}\mathbb{T} \times_{\mathbb{Z}_p} \mathbb{G}_m\right)^{an} \to \left(\operatorname{Spec}\mathbb{Z}_p[\![\Gamma]\!]\right)^{an}$$

is flat with discrete fibres. The last map is given by projection onto $\operatorname{Spec}\mathbb{T}$ followed by the weight map ω.

Proof. See [5] for more details. $\square$

Let $\mathbb{T}^*$ be the quotient of $\mathbb{T}[U_p]$ that acts faithfully on the space of generalized p-adic modular functions. Let f be a classical modular form, and suppose α and β are the roots of its p-th Hecke polynomial. Then the p-stabilizations of f, defined to be

$$f_\alpha(\tau) = f(\tau) - \beta f(p\tau), \text{ or } f_\beta(\tau) = f(\tau) - \alpha f(p\tau)$$

are U_p eigenvectors with eigenvalues α and β respectively. Therefore, $\operatorname{Spec}\mathbb{T}^*$ contains $\mathcal{X}$ and its Zariski closure. However, due to the infinite dimensional kernel of U_p as remarked earlier in the chapter, $\operatorname{Spec}\mathbb{T}^*$ is much bigger than $\mathcal{C}$ and $\mathcal{C}^{\mathrm{ord}}$.

Define the *ordinary projector* to be $e = \lim_{n\to\infty} U_p^{n!}$, which exists and is well-defined (see Section 7.2 of [21]). Suppose f is an eigenvector of U_p with eigenvalue α, then

$$e(f) = \begin{cases} f & \text{if } |\alpha|_p = 1 \\ 0 & \text{if } |\alpha|_p < 1 \end{cases}.$$

This shows that e projects the space of generalized p-adic modular functions to the space of ordinary generalized p-adic modular functions. Let $\mathbb{T}^{\mathrm{ord}}$ be the quotient of $\mathbb{T}^*$ that acts faithfully on the space of ordinary functions. Then, in fact, $\mathcal{C}^{\mathrm{ord}} = \operatorname{Spec}\mathbb{T}^{\mathrm{ord}}$ and is finite over $\mathbb{Z}_p[\![\Gamma]\!]$. This shows that we can view $\mathcal{C}^{\mathrm{ord}}$ as the union of many families of modular forms parametrized by the weight. It is the union of Hida families, which we will introduce in the subsequent section. See [18] for a deeper discussion into this result.

3.8 Hida Families

In this section, we will introduce Hida families. They can be viewed as a family of Galois representations, parametrized by the weight. Instead of taking the geometric approach to constructing these families, as Hida originally did, we will define them as Λ-adic forms following Chapter 7 of [21].

Fix a prime p. Recall that $\mathbb{Z}_p^\times = (1+q\mathbb{Z}_p) \times \mu = \Gamma \times \mu$ where $q = 4$ if $p = 2$ and $q = p$ otherwise; and μ is the group of $(p-1)$-ths roots of unity for p odd and $\{\pm 1\}$ for $p = 2$. We will now define the two natural projections of $\mathbb{Z}_p^\times$ in this decomposition.

Definition 3.8.1. Let ω be the *Teichmuller character* $\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ defined by

$$\omega(x) = \lim_{n\to\infty} x^{p^n}.$$

The image of $\omega(x)$ is $\mu \subseteq \mathbb{Z}_p^\times$. Let $\langle\cdot\rangle : \mathbb{Z}_p^\times \to \Gamma$ be the map given by $\langle x\rangle = \omega(x)^{-1}x$.

Let $u = 1+q \in \Gamma$. For each $x \in \Gamma$, define $s(x) = \frac{\log_p(x)}{\log_p(u)} \in \mathbb{Z}_p$. The map s induces an isomorphism $s : \Gamma \cong \mathbb{Z}_p$. Hence, we can write $x = u^{s(x)}$.

Definition 3.8.2. Suppose $K/\mathbb{Q}_p$ is a finite field extension, and $\mathcal{O}_K$ its ring of integers. Let $\Lambda = \mathcal{O}_K[\![X]\!]$. Fix a character χ with modulus $p^\alpha q$ for some $\alpha \in \mathbb{Z}$ (not necessarily primitive). Say that a formal power series

$$F(X,q) = \sum_{n=0}^{\infty} a_n(X)q^n \in \Lambda[\![q]\!]$$

with $a_n(X) \in \Lambda$ is a Λ-*adic form* of character χ, if for all but finitely many positive integers k,

$$F(u^k-1,q) = \sum_{n=0}^{\infty} a_n(u^k-1)q^n \in \mathcal{O}_K[\![q]\!]$$

is the q-expansion of a classical modular form in $M_k\left(\Gamma_0(p^\alpha q), \chi\omega^{-k}, \mathcal{O}_K\right)$. If almost all of the $F\left(u^k-1,q\right)$'s are cusp forms (resp. ordinary), then we say F is a Λ-adic cusp (resp. ordinary) form.

Since $u^k - 1 = (1+q)^k - 1$, we get that $\left|u^k-1\right|_p < 1$. Hence, the evaluations $a_n(u^k - 1)$ do converge for all $n \geq 0$. From the isomorphism $\mathbb{Z}_p[\![\Gamma]\!] \cong \mathbb{Z}_p[\![X]\!]$, we see that the parametrization by Λ agrees with the parametrization over the weight space from the previous chapter.

Let $M(\chi, \Lambda)$ and $M^{\text{ord}}(\chi, \Lambda)$ denote the space of Λ-adic forms and ordinary Λ-adic forms respectively. Similarly, let $S(\chi, \Lambda)$ and $S^{\text{ord}}(\chi, \Lambda)$ denote the subspace of cuspidal Λ-adic forms.

3.8.1 Hecke Operators on Λ-adic Forms

In this section, we will define Hecke operators acting on the space of Λ-adic forms. Naturally, we can demand that the action fo the Hecke operators on Λ-adic forms should agree with the action of the classical Hecke operators after taking the weight k specialization map. Suppose κ is the character

$$\kappa : \Gamma \to \Lambda^{\times} \qquad \text{given by} \qquad x \mapsto (1+X)^{s(x)}.$$

By applying the weight k specialization, we obtain the identity

$$\left(1 + (u^k - 1)\right)^{s(x)} = u^{ks(x)} = x^k.$$

In particular, for an integer n coprime to p,

$$\kappa(\langle n \rangle)(u^k - 1) = \langle n \rangle^k = \omega^{-k}(n) n^k.$$

Definition 3.8.3. Let F be a Λ-adic form with character χ, with coefficients $a_\ell(F)(X) \in \Lambda$ for all $\ell \geq 0$. For all integers n coprime to p, define

$$a_\ell(T_n F)(X) = \sum_{b \mid \gcd(\ell, n)} \kappa(\langle b \rangle)(X) \chi(b) b^{-1} a_{\frac{\ell n}{b^2}}(F)(X),$$

where the summation is over all divisors of $\gcd(\ell, n)$ that is also prime to p.

By computation, we find that at any weight k specialization,

$$\begin{aligned} a_\ell(T_n F)(u^k - 1) &= \sum_{b \mid \gcd(\ell, n)} \omega^{-k}(b) \chi(b) b^{k-1} a_{\frac{\ell n}{b^2}}(F)(u^k - 1) \\ &= a_\ell\left(T_n\left(F(u^k - 1)\right)\right). \end{aligned}$$

This is exactly our desired property for the Hecke operator on Λ-adic forms.

Let $H^{\text{ord}}(\chi, \Lambda)$ and $h^{\text{ord}}(\chi, \Lambda)$ be the subalgebra of the endomorphisms of $M^{\text{ord}}(\chi, \Lambda)$ and $S^{\text{ord}}(\chi, \Lambda)$ respectively, generated by T_n for all positive integers n. The reason why we are restricting ourselves to the ordinary case, is beacause of the following theorem.

Theorem 3.8.4. *There is a perfect Λ-bilinear pairing*

$$\langle \cdot, \cdot \rangle : H^{\text{ord}}(\chi, \Lambda) \times M^{\text{ord}}(\chi, \lambda) \to \Lambda$$

and similarly for the cuspidal forms.

Proof. See Theorem 5 of Section 7.3 of [21]. □

This shows that by considering the ordinary forms, we have a duality between the Hecke algebra and the Λ-adic forms, just like in the classical case. In fact, both of these algebras are free of finite rank over Λ as well (see Theorem 1 of Section 7.3 of [21]). Similar to the classical case, there is a basis of $M^{\text{ord}}(\chi, \Lambda)$ and $S^{\text{ord}}(\chi, \Lambda)$ consisting of eigenforms for T_n for all positive n (see Theorem 6 of the same section in [21]). In fact, F is an eigenform iff almost all of its specializations are. Finally, we also have Galois representations associated to normalized eigenforms.

Theorem 3.8.5. *Let F be a Λ-adic normalized eigenform in $S^{\text{ord}}(\chi, \Lambda)$, and let L be the quotient field of Λ. Suppose $\lambda : h^{\text{ord}}(\chi, \Lambda) \to \Lambda$ is its corresponding Λ-algebra homomorphism. Then there exists a unique Galois representation*

$$\rho : G_{\mathbb{Q}} \to GL_2(L),$$

such that ρ is continuous, absolutely irreducible and unramified oustide p. Additionally, for all primes $\ell \neq p$, the ℓ-th Hecke polynomial is given by

$$\det \left(T - \rho\left(Frob_\ell\right)\right) = T^2 - \lambda(T_\ell)T + \chi(q)\kappa\left(\langle q \rangle\right) q^{-1}.$$

Sketch of the Proof. Idea: for almost all the weight k specializations, we have a Galois representation associated to the classical eigenforms. Glue these representations together by going through the theory of pseudo-representations. See Theorem 1 of Section 7.5 of [21] for more details. □

Note that the image of λ and κ are in Λ, and so this representation is truly one that parametrizes the family of Galois representation associated to classical eigenforms.

Theorem 3.8.6. *(Hida's Control Theorem) Let $k \geq 1$ be a positive integer, and let ϵ be a primitive finite order character of Γ/Γ^{p^α}. Then for all classical modular form f in $M_k^{\text{ord}}\left(\Gamma_0(p^\alpha p), \epsilon\chi\omega^{-k}; \mathcal{O}_K[\epsilon]\right)$, there exists a Λ-adic form F of character χ such that $F\left(\epsilon(u)u^k - 1\right) = f$. In the case where $k \geq 2$, the map $F \mapsto F\left(\epsilon(u)u^k - 1\right)$ induces isomorphisms*

$$\begin{aligned} M^{\text{ord}}(\chi, \Lambda) / P_{k,\epsilon} M^{\text{ord}}(\chi, \Lambda) &\cong M_k^{\text{ord}}\left(\Gamma_0(p^\alpha p), \epsilon\chi\omega^{-k}; \mathcal{O}_K[\epsilon]\right) \\ S^{\text{ord}}(\chi, \Lambda) / P_{k,\epsilon} S^{\text{ord}}(\chi, \Lambda) &\cong S_k^{\text{ord}}\left(\Gamma_0(p^\alpha p), \epsilon\chi\omega^{-k}; \mathcal{O}_K[\epsilon]\right) \end{aligned}$$

where $P_{k,\epsilon}$ is the ideal generated by $X - \left(\epsilon(u)u^k - 1\right)$.

Proof. See [20] or Theorem 3 of Section 7.3 of [21]. $\square$

3.9 The Hida Family of Theta series

In this section, we will compute an explicit example of a Hida family, which will be a family of theta series. This example will be very important for the actual thesis problem. The presentation of this section will follow the example in Section 7.6 of [21].

Let $K/\mathbb{Q}$ be a quadratic imaginary field with ring of integers $\mathcal{O}_K$. Fix a prime p that splits in K into distinct primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. By fixing an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, it determines a prime ideal $\mathfrak{p}$ above p in K. Let $\mathcal{O}_{K,\mathfrak{p}}$ denote the completion of $\mathcal{O}_K$ at $\mathfrak{p}$. Suppose ψ is a Hecke character of K with conductor $\mathfrak{c}\mathfrak{p}$ where $\mathfrak{c}$ is an ideal prime to $\mathfrak{p}$, satisfying

$$\psi((\alpha)) = \alpha \quad \text{for all } \alpha \equiv 1 \mod \mathfrak{c}\mathfrak{p}.$$

Let $M = \mathbb{Q}_p(\psi)$ and let $\mathcal{O}_M$ be its ring of integers. Write $\mathcal{O}_M^\times = W_M \times \mu_M$ where W_M is $\mathbb{Z}_p$-free and μ_M is a finite group. Similar to the notations used early, we will denote the projections of $\mathcal{O}_M^\times$ to W_M and μ_M by $\langle\cdot\rangle$ and ω respectively.

Let U be the subgroup of W_M topologically generated by $\langle\psi(I)\rangle$ for all ideals I coprime to $\mathfrak{p}$. The natural inclusion of $\Gamma = 1 + q\mathbb{Z}_p$ into $M = \mathbb{Q}_p(\psi)$ is given in the following way.

The group Γ is naturally a subgroup of $\mathcal{O}_{K,\mathfrak{p}}$ and the natural inclusion into M is given by the map $z \mapsto \langle\psi(z)\rangle$. From the construction, we see that we have a natural injection

$$\begin{array}{ccc} \Gamma & \hookrightarrow & U \\ z & \mapsto & \langle\psi(z)\rangle . \end{array}$$

Suppose $[U : \Gamma] = p^{\gamma} < 0$. Since p splits, we can pick a generator λ of U such that $\lambda^{p^{\gamma}} = u = 1 + q$.

Let $\mathbb{I} = \mathcal{O}_M[\![Y]\!] \supseteq \Lambda = \mathcal{O}_M[\![X]\!]$ where Y satisfies the identity $(1+Y)^{p^{\gamma}} = (1+X)$. Consider the $\mathbb{I}$-adic form

$$F_{\psi}(Y,q) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) \left\langle\psi(a)\right\rangle^{-2} (1+Y)^{s(\mathfrak{a})}\, q^{N(\mathfrak{a})},$$

where the summation is over all ideals $\mathfrak{a}$ of K prime to $\mathfrak{p}$. By $s\left(\mathfrak{a}\right)$, we mean $\frac{\log(\langle\psi(\mathfrak{a})\rangle)}{\log\lambda}$, so that $\lambda^{s(a)} = \langle\psi(\mathfrak{a})\rangle$. By definition,

$$\left(1 + (\lambda^k - 1)\right)^{s(a)} \;\; = \;\; \lambda^{ks(\mathfrak{a})} = \left\langle\psi(\mathfrak{a})\right\rangle^{k}.$$

This calculation shows that the weight k specializations of F_{ψ} are

$$F_{\psi}\left(\lambda^k - 1, q\right) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) \left\langle\psi(\mathfrak{a})\right\rangle^{k-2} q^{N(\mathfrak{a})}.$$

They are precisely the theta series attached to the character $\psi \left\langle\psi\right\rangle^{k-2}$.

Remark 3.9.1. If we took our Hecke character ψ to be unitary, then a cuspidal family of theta series is given by

$$\sum_{\mathfrak{a}} \psi(\mathfrak{a})\chi(\mathfrak{a})^{k-1} q^{N(\mathfrak{a})}$$

where χ is a character induced from composing a p-adic embedding with a Hecke character χ_{∞} on the idèle class group of K satisfying

$$\chi_{\infty}\left((\alpha)\right) = \pm\alpha$$

for all $\alpha \equiv 1$ modulo the conductor of χ_{∞}. This χ_{∞} called a *canonical Hecke character*.

Chapter 4
Classical Weight One Points on the Eigencurve

In this chapter, we will introduce the main conjecture tackled in this thesis, as well as known results that motivate the conjecture.

4.1 Known Results

Let $g(q) = \sum_{n\geq 1} a_n q^n$ be a newform of weight one, level $\Gamma_1(N)$ with nebentypus χ. By Theorem 1.5.1, there exists an odd continuous irreducible Artin representation

$$\rho_g : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$$

which is unramified outside N. Additionally, for all primes $\ell \nmid N$, $\rho_g(Frob_\ell)$ has characteristic polynomial $x^2 - a_\ell x + \chi(\ell)$.

Fix a prime number $p \nmid N$. Since the eigenvalues of g are in $\overline{\mathbb{Q}}$, by fixing an embedding $i_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, we can assume that ρ_g takes values in $\overline{\mathbb{Q}}_p$. Suppose $\alpha_p, \beta_p \in \overline{\mathbb{Q}}_p$ are the roots of the p-th Hecke polynomial $x^2 - a_p x + \chi(p)$. The newform g is said to be *regular* at p if $\alpha_p \neq \beta_p$, and *irregular* at p if $\alpha_p = \beta_p$.

Let $\mathcal{C}$ be the eigencurve of tame level N induced by the Hecke operators U_p and T_ℓ with primes $\ell \nmid Np$ as described in Chapter 3. Suppose $\omega : \mathcal{C} \to W$ is the weight map.

Theorem 4.1.1. *In the case where g is regular at p, the eigencurve $\mathcal{C}$ is smooth at g_α. Additionally, ω is étale at g_α if and only if there does not exists a real quadratic field K in which p splits and $\rho\mid_{G_K}$ is reducible.*

Proof. See the work of Bellaïche and Dimitrov in [2]. □

Let K be a real quadratic field of discriminant $D > 0$, where $p = \mathfrak{p}\mathfrak{p}'$ splits. Suppose τ is the non-trivial involution of $\mathrm{Gal}(K/\mathbb{Q})$. Suppose

$$\psi : G_K = \mathrm{Gal}\left(\bar{K}/K\right) \to \mathbb{C}^\times$$

is a ray class character of mixed signature of order m with conductor $\mathfrak{f}_\psi$ and central character χ_ψ. Let $L = \mathbb{Q}(\mu_m)$, where μ_m is the set of m-ths roots of unity. Let ψ' denote the character

$\psi \circ \tau$ and $\hat{\psi}$ denote the character ψ/ψ'. Let H be the ring class field of K fixed by the kernel of $\hat{\psi}$.

Let $g = \theta_\psi$, the theta series associated to ψ. By Theorem 1.6.7, it is a newform of weight one, level N with nebetypus χ where $N = D \cdot N_{K/\mathbb{Q}}\mathfrak{f}_\psi$ and $\chi = \tau\chi_\psi$. Fix a prime p, and assume that g is regular at p. In this scenario, it is expected that the ramification index of the weight map ω at g_α is exactly 2, so ω is not étale. Darmon, Lauder and Rotger [9] described the Fourier coefficients of the generalized overconvergent modular forms that span the relative tangent space. We will now present the details of this result.

Let $S_1(Np,\chi)$ and $S_1^{(p)}(N,\chi)$ denote the space of classical and p-adic overconvergent modular form of weight k, level N with nebentypus character χ and coefficients in $\bar{\mathbb{Q}}_p$. Let $\mathbb{T}$ be the Hecke algebra of level Np spanned over $\mathbb{Q}$ by T_ℓ for primes $\ell \nmid Np$ and U_ℓ for primes $\ell \mid Np$. The p-stabilization g_α gives us an algebra homomorphism $\lambda_{g_\alpha} : \mathbb{T} \to L$. More specifically, λ_{g_α} satisfies

$$\lambda_{g_\alpha}(T_\ell) = a_\ell(g_\alpha) \text{ for } \ell \nmid Np \qquad \text{and} \qquad \lambda_{g_\alpha}(U_\ell) = \begin{cases} a_q(g_\alpha) & \text{if } \ell \mid N \\ \alpha & \text{if } \ell = p \end{cases}.$$

Definition 4.1.2. Let I_{g_α} denote the kernel of λ_{g_α}. Let

$$S_1(Np,\chi)[g_\alpha] = S_1(Np,\chi)[I_{g_\alpha}]$$

be the I_{g_α}-torsion elements of $S_1(Np,\chi)$. Similarly, define

$$S_1^{(p)}(N,\chi)[\![g_\alpha]\!] = S_1^{(p)}(N,\chi)\left[I_{g_\alpha}^2\right]$$

to be the $I_{g_\alpha}^2$-torsion elements of $S_1^{(p)}(N,\chi)$.

The elements of the latter space are called *overconvergent generalized eigenforms* attached to g_α. An element in this space is called *normalised* if its first coefficient is equal to 0. There is a natural inclusion

$$S_1(Np,\chi)[g_\alpha] \hookrightarrow S_1^{(p)}(N,\chi)[\![g_\alpha]\!].$$

In our scenario, $S_1^{(p)}(N,\chi)\llbracket g_\alpha \rrbracket$ is two dimensional, so it contains an eigenform that is not a multiple of g_α. Denote the unique normalised generalized eigenform relative to g_α by

$$g_\alpha^\flat = \sum_{n=2}^{\infty} a_n\left(g_\alpha^\flat\right) q^n.$$

The Hecke operators act on $g_\alpha^\flat$ in the following way:

$$\begin{aligned} T_\ell g_\alpha^\flat &= a_\ell\left(g_\alpha\right) g_\alpha^\flat + a_\ell\left(g_\alpha^\flat\right) g_\alpha \quad \text{for all primes } \ell \nmid Np, \\ U_q g_\alpha^\flat &= a_q\left(g_\alpha\right) g_\alpha^\flat + a_q\left(g_\alpha^\flat\right) g_\alpha \quad \text{for all primes } q \mid Np. \end{aligned}$$

Suppose $\ell \nmid N$ is a prime that is inert in $K/\mathbb{Q}$. Since H is a ring class field of conductor prime to ℓ, the prime ℓ splits completely in H/K. Let Σ_ℓ denote the set of these primes in H above ℓ, which is naturally a principal $\mathrm{Gal}(H/K)$-set. Fix $\lambda \in \Sigma_\ell$ and suppose $u_\lambda \in \mathcal{O}_H[1/\lambda]^\times \otimes \mathbb{Q}$ is a λ-unit of H where $\mathrm{ord}_\lambda u_\lambda = 1$. Let

$$u(\hat{\psi},\lambda) = \sum_{\sigma\in\mathrm{Gal}(H/K)} \hat{\psi}^{-1}(\sigma)\otimes \sigma u_\lambda \in L\otimes \mathcal{O}_H[1/\ell]^\times.$$

This construction is invariant of the choice of u_λ, but it does depend on the choice of $\lambda \in \Sigma_\ell$ (see [9] for details). In the article, the authors defined a special map $\eta : \Sigma_\ell \to \mu_m$. With it, we can define

$$u(\hat{\psi},\ell) = \eta(\lambda)\otimes u\left(\hat{\psi},\lambda\right) \in L\otimes \mathcal{O}_H[1/\ell]^\times,$$

which will no longer depend on the choice of $\lambda \in \Sigma_\ell$.

With the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we have a p-adic logarithm on $H^\times$, which we can extend to a map $\log_p : L\otimes H^\times \to \overline{\mathbb{Q}}_p$ by declaring it to be L-linear.

Theorem 4.1.3. *After some scaling, the Fourier coefficients of the normalized generalised eigenform $g_\alpha^\flat$ attached to g_α can be described in the following way. For all $\ell \nmid N$,*

$$a_\ell\left(g_\alpha^\flat\right) = \begin{cases} 0 & \text{if } \chi_K(\ell) = 1 \\ \log_p u(\hat{\psi},\ell) & \text{if } \chi_K(\ell) = -1 \end{cases}.$$

Then, for $n \geq 2$, $\gcd(n,N) = 1$,

$$a_n\left(g_\alpha^\flat\right) = \frac{1}{2}\sum_{\ell\mid n}\log_p u(\hat{\psi},\ell)\left(\mathrm{ord}_\ell n + 1\right) a_{n/\ell}\left(g_\alpha\right)$$

where the sum is over all primes that are inert in K.

Proof. See the work of Darmon, Lauder and Rotger in [9]. $\square$

4.2 Conjecture

Having now established some motivation and background, we are now able to state the main conjecture of this thesis project. The statement of the conjecture comes from [8]. We will restrict ourselves in the scenario where g is irregular at p, and so the p-stabilization g_α is unique. The conjecture is essentially an alternative version of Theorem 4.1.3 in the irregular case. It describes the Fourier coefficients of the normalized generalized overconvergent eigenforms in terms of logarithms of algebraic numbers.

Suppose $g \in S_1(N,\chi)$ is a newform, and let g_α be its unique p-stabilization. The Hecke operators T_ℓ for $\ell \nmid Np$ and U_q for $q \mid N$ act semi-simply on the space

$$S_1(Np,\chi)[g_\alpha] = \bar{\mathbb{Q}}_p g_\alpha \oplus \bar{\mathbb{Q}}_p \hat{g},$$

where $\hat{g}(q) = g(q^p)$. However, the action of U_p on this space is not semi-simple. By construction, $U_p g_\alpha = \alpha g_\alpha$. If $g(q) = \sum_{n\geq 1}^{\infty} a_n q^n$, then we can write

$$\hat{g}(q) = g(q^p) = \sum_{n\geq 1}^{\infty} a_n q^{np} = \sum_{\ell\geq 1}^{\infty} b_\ell q^\ell \quad \text{where } b_\ell = \begin{cases} a_n = a_{\ell/p} & \text{if } \ell = np \text{ for some } n\in\mathbb{N} \\ 0 & \text{else} \end{cases}.$$

By applying the U_p operator, we get

$$U_p\hat{g}(q) = \sum_{\ell\geq 1}^{\infty} b_{\ell p} q^\ell = \sum_{\ell\geq 1}^{\infty} a_\ell q^\ell = g(q) = g_\alpha(q) + \alpha\hat{g}(q).$$

Notice that the first and p-th Fourier coefficients of g and $\hat{g}$ are

$$\left(a_1(g_\alpha), a_p(g_\alpha)\right) = (1,\alpha) \quad \text{and} \quad \left(a_1(\hat{g}), a_p(\hat{g})\right) = (0,1).$$

This suggests that we can find a natural linear complement of $S_1(Np,\chi)[g_\alpha]$ in $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]$, consisting of generalized eigenforms $\tilde{g}$ with the property that $(a_1(\tilde{g}), a_p(\tilde{g})) = (0,0)$. Let $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ denote the space of these forms, and call the elements in this space *normalized.*

Let $\mathbb{T}$ be the Hecke algebra of level Np over $\bar{\mathbb{Q}}_p$ generated by T_ℓ for $\ell \nmid Np$ and U_ℓ for $\ell \mid Np$. By duality of eigenforms and Hecke algebras, as discussed in the previous chapters, we have identifications

$$S_1(Np,\chi)[g_\alpha] \cong \mathrm{Hom}\left(\mathbb{T}/I_{g_\alpha}, \bar{\mathbb{Q}}_p\right), \text{ and } S_1^{(p)}(N,\chi)[\![g_\alpha]\!] \cong \mathrm{Hom}\left(\mathbb{T}/I_{g_\alpha}^2, \bar{\mathbb{Q}}_p\right).$$

To get the first isomorphism, we also need $S_1(Np,\chi)[g_\alpha] = S_1^{(p)}(N,\chi)[g_\alpha]$. We can prove this by checking the q-expansions and finding that all the forms in $S_1^{(p)}(Np,\chi)[g_\alpha]$ are in fact classical. There is also a natural exact sequence

$$0 \to I_{g_\alpha}/I_{g_\alpha}^2 \to \mathbb{T}/I_{g_\alpha}^2 \to \mathbb{T}/I_{g_\alpha} \to 0.$$

By passing the above exact sequence through duality with the $\mathrm{Hom}\left(\cdot, \bar{\mathbb{Q}}_p\right)$ functor, we obtain the following proposition.

Proposition 4.2.1. *There is a natural isomorphism of vector spaces*

$$S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0 \cong \mathit{Hom}\left(I_{g_\alpha}/I_{g_\alpha}^2, \bar{\mathbb{Q}}_p\right). \tag{4.1}$$

The proposition shows that the space of normalized generalized eigenforms can be identified with the tangent space of the eigencurve at g_α, relative to the weight space. We can then study the space of normalized generalized eigenforms by studying the tangent space of modular deformations with constant determinant.

Suppose

$$\rho_g : G_{\mathbb{Q}} \to \mathrm{Aut}_{\bar{\mathbb{Q}}_p}(V_g) \cong GL_2\left(\bar{\mathbb{Q}}_p\right)$$

is the 2-dimensional Artin representation associated to g by Theorem 1.5.1. Since ρ_g is an Artin representation, it factors through a finite quotient $\mathrm{Gal}(L/\mathbb{Q})$. With the fixed embedding $i_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$, we can assume that $L \subseteq \bar{\mathbb{Q}}_p$. Let $\tilde{W}_g = \mathrm{Ad}\rho_g = \mathrm{End}(V_g)$ denote the space of endomorphisms of V_g, also called the adjoint representation of ρ_g. An element $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ acts on an element $w \in \tilde{W}_g$ by conjugation in the following way:

$$\sigma \star w = \rho_g(\sigma) \circ w \circ \rho_g^{-1}(\sigma).$$

Let $W_g = \mathrm{Ad}^0 \rho_g$ denote the subspace of $\tilde{W}_g$ consisting of trace zero endomorphisms. The adjoint representation also factors through a finite quotient $G = \mathrm{Gal}\,(H/\mathbb{Q})$. Since $\ker \rho_g \subseteq \ker \mathrm{Ad}\rho_g$, H is a subfield of L.

There is a canonical exact sequence of G-modules

$$0 \to W_g \to \tilde{W}_g \to \bar{\mathbb{Q}}_p \to 0,$$

with a canonical G-equivariant splitting given by

$$p: \tilde{W}_g \to W_g \quad \text{where} \quad p(A) = A - \frac{1}{2}\mathrm{Tr}\,(A) \cdot I.$$

By enlarging L, we can assume that $\rho_g\,(L[G_{\mathbb{Q}}]) \cong M_2\,(L)$ to get a two dimensional L-vector space V_g° and an identification $\iota: V_g^\circ \otimes_L \bar{\mathbb{Q}}_p \to V_g$. Similarly, we have G-stable L-vector spaces

$$\tilde{W}_g^\circ = \mathrm{Ad}V_g^\circ \quad \text{and} \quad W_g^\circ = \mathrm{Ad}^0 V_g^\circ$$

with identifications

$$\iota: \tilde{W}_g^\circ \otimes_L \bar{\mathbb{Q}}_p \to \tilde{W}_g \quad \text{and} \quad \iota: W_g^\circ \otimes_L \bar{\mathbb{Q}}_p \to W_g.$$

Let $[\cdot,\cdot]$ and $\langle \cdot,\cdot \rangle$ denote the Lie bracket and the symmetric non-degenerate pairing on the spaces $\tilde{W}_g, W_g, \tilde{W}_g^\circ, W_g^\circ$ given by

$$[A,B] = AB - BA, \text{ and } \langle A, B\rangle = \mathrm{Tr}\,(AB) \text{ respectively.}$$

It is easy to check that these maps are compatible with the G-action. That is, for all $\sigma \in G$,

$$[\sigma \star A, \sigma \star B] = \sigma \star [A,B]\,, \text{ and } \langle \sigma \star A, \sigma \star B\rangle = \langle A, B\rangle\,.$$

From Section 2.5, we recall that the tangent space associated to the deformation of ρ_g with constant determinant can be identified with the set $H^1\left(\mathbb{Q}, \mathrm{Ad}^0(V_g)\right) = H^1\left(\mathbb{Q}, W_g\right)$. Since $G = \mathrm{Gal}(H/\mathbb{Q}) = G_{\mathbb{Q}}/G_H$, the inflation-restriction exact sequence is given by

$$0 \to H^1\left(G, W_g^{G_H}\right) \to H^1\left(G_{\mathbb{Q}}, W_g\right) \to H^1\left(G_H, W_g\right)^G \to H^2\left(G, W_g^{G_H}\right).$$

Since G is a finite group, and $W_g^{G_H}$ is a characteristic zero representation, $H^1\left(G, W_g^{G_H}\right) = 0 = H^2\left(G, W_g^{G_H}\right)$. By the definition of H, the action of G_H on W_g is trivial. Thus, we have an isomorphism

$$H^1\left(G_{\mathbb{Q}}, W_g\right) \cong H^1\left(G_H, W_g\right)^G = \mathrm{Hom}_G\left(G_H, W_g\right).$$

Now we will describe elements in $\mathrm{Hom}_G\left(G_H, W_g\right)$. Since W_g is an abelian group, a homomorphism from G_H to W_g passes through G_H^{ab} the Galois group of the maximal abelian extension of H. By class field theory, these correspond to continuous homomoprhisms from the idèle class group. By respecting the p-adic topology, we see that

$$\begin{aligned} H^1\left(G_{\mathbb{Q}}, W_g\right) &\cong \mathrm{Hom}_G\left(\frac{\left(\mathcal{O}_H \otimes \mathbb{Z}_p\right)^\times}{\mathcal{O}_H^\times \otimes \mathbb{Z}_p}, W_g\right) \\ &\cong \ker\left(\mathrm{Hom}_G\left(\left(\mathcal{O}_H \otimes \mathbb{Z}_p\right)^\times, W_g\right) \xrightarrow{\mathrm{res}} \mathrm{Hom}_G\left(\mathcal{O}_H^\times \otimes \mathbb{Z}_p, W_g\right)\right). \end{aligned} \tag{4.2}$$

We will now study the space $\mathrm{Hom}_G\left(\left(\mathcal{O}_H \otimes \mathbb{Z}_p\right)^\times, W_g\right)$ following Section 3.1 of [2]. Let $\log_p :$ $\bar{\mathbb{Q}}_p^\times \to \bar{\mathbb{Q}}_p$ denote the standard logarithm sending p to 0. Notice that $\mathcal{O}_H \otimes \mathbb{Z}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_{H_\mathfrak{p}}$ where $H_\mathfrak{p}$ is the completion of H with respect to $\mathfrak{p}$. Every continuous homomorphism in $\mathrm{Hom}(\mathcal{O}_{H_\mathfrak{p}}^\times, \overline{\mathbb{Q}}_p)$ is given by

$$u \mapsto \sum_{\lambda_\mathfrak{p} \in J_\mathfrak{p}} a_{\lambda_\mathfrak{p}} \lambda_\mathfrak{p}\left(\log_p u\right) = \sum_{\lambda_\mathfrak{p} \in J_\mathfrak{p}} a_{\lambda_\mathfrak{p}}\left(\log_p\left(\lambda_\mathfrak{p} u\right)\right)$$

for some $a_{\lambda_\mathfrak{p}} \in \overline{\mathbb{Q}}_p$, where $J_\mathfrak{p}$ is the set of all embeddings $H_\mathfrak{p} \hookrightarrow \overline{\mathbb{Q}}_p$. From the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we have the following commutative diagram

$$\begin{tikzcd} \overline{\mathbb{Q}}^\times \arrow[r, hook, "i_p"] & \overline{\mathbb{Q}}_p^\times \\ H \arrow[u, "\lambda"] \arrow[r, hook] & H_\mathfrak{p} \arrow[u, "\lambda_p"'] \end{tikzcd}$$

which gives us a partition $G = \amalg_{\mathfrak{p}|p} J_\mathfrak{p}$. Hence, $\mathrm{Hom}\left((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, \overline{\mathbb{Q}}_p\right)$ is spanned by

$$\left(\log_p\left(\left(i_p \circ \sigma\right) \otimes 1\right)\right)_{\sigma \in G}$$

and the action of $\sigma' \in G$ on this basis is given by

$$\sigma'\left(\log_p\left(\left(i_p \circ \sigma\right) \otimes 1\right)\right)_{\sigma \in G} = \left(\log_p\left(\left(i_p \circ \sigma'\sigma\right) \otimes 1\right)\right)_{\sigma \in G}.$$

Therefore, we have an isomorphism of G-representations

$$\begin{aligned} \bar{\mathbb{Q}}_p[G] &\cong \mathrm{Hom}\left((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, \overline{\mathbb{Q}}_p\right) \\ \sum_{\sigma \in G} a_\sigma &\mapsto \left(u \otimes v \mapsto \sum_{\sigma \in G} a_\sigma \log_p\left(i_p(\sigma^{-1}(u))v\right)\right). \end{aligned}$$

By Schur's lemma we find that

$$\begin{aligned}\dim_{\bar{\mathbb{Q}}_p} \operatorname{Hom}_G\left((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, W_g\right) &= \dim_{\bar{\mathbb{Q}}_p}\left(\operatorname{Hom}\left((\mathcal{O}_H \otimes \mathbb{Z}_p)^\times, \bar{\mathbb{Q}}_p\right) \otimes W_g\right)^G \\ &= \dim_{\bar{\mathbb{Q}}_p}\left(\bar{\mathbb{Q}}_p\left[G\right] \otimes W_g\right)^G = 3.\end{aligned} \tag{4.3}$$

By Dirichlet's unit theorem (see Appendix), $\mathcal{O}_H^\times \otimes \mathbb{Z}_p \cong \operatorname{Ind}_{\langle c\rangle}^G \mathbb{Z}_p - 1_G$ where c is a complex conjugation, and 1_G is the trivial representation. By simple computation, we can show that complex conjugation acts on W_g with eigenvalues $1, -1$ and -1. By Frobenius reciprocity,

$$\dim_{\bar{\mathbb{Q}}_p} \operatorname{Hom}_G\left(\mathcal{O}_H^\times \otimes \mathbb{Z}_p, W_g\right) = 1. \tag{4.4}$$

To summarize, we have proved the following proposition.

Proposition 4.2.2. *The space $H^1(G_{\mathbb{Q}}, W_g)$ is two dimensional as a $\bar{\mathbb{Q}}_p$ vector space.*

Proof. Apply equation (4.3) and equation (4.4) to equation (4.2). ☐

By the irregularity of g, the prime p splits completely in H. With the fixed embedding $H \hookrightarrow \bar{\mathbb{Q}}_p$, and a chosen prime $\mathfrak{p}_0$ of H above p, let $\log_{\mathfrak{p}_0} : H_p \to \mathbb{Q}_p$ denote the $\mathfrak{p}_0$-adic logarithm that factors through $\log_p$. By equation (4.4),

$$\dim_L \left(\mathcal{O}_H^\times \otimes W_g^\circ\right)^G = 1.$$

For all $\mathfrak{u} \in \mathcal{O}_H^\times$, $\omega \in W_g^\circ$, let

$$\xi\left(\mathfrak{u}, \omega\right) = \frac{1}{|G|}\sum_{\sigma \in G} (\sigma \mathfrak{u}) \otimes (\sigma \star \omega) \in \left(\mathcal{O}_H^\times \otimes W_g\right)^G.$$

Then the elements

$$\xi_{\mathfrak{p}_0}\left(\mathfrak{u}, \omega\right) = \left(\log_{\mathfrak{p}_0} \otimes \mathrm{id}\right) \xi\left(\mathfrak{u}, \omega\right) = \frac{1}{|G|}\sum_{\sigma \in G} \log_{\mathfrak{p}_0}\left(\sigma \mathfrak{u}\right) \cdot (\sigma \star \omega) \in W_g$$

spans a 1-dimensional L-vector subspace of W_g. Let $\omega(1)$ be any generator of this space.

For each prime $\ell \nmid Np$, choose a prime λ in H above ℓ. Suppose h is the class number of H. Let $\tilde{\mathfrak{u}}_\lambda$ be a generator of the principal ideal λ^h, and let

$$\mathfrak{u}_\lambda = \tilde{\mathfrak{u}}_\lambda \otimes h^{-1} \in \left(\mathcal{O}_H\left[1/\ell\right]^\times\right) \otimes_{\mathbb{Z}} L.$$

Let $\sigma_\lambda \in G$ denote Frobenius map associated to λ. Set

$$\tilde{\omega}_\lambda = \rho_g(\sigma_\lambda) \in \tilde{W}_g^\circ \quad \text{and} \quad \omega_\lambda = p(\tilde{\omega}_\lambda) \in W_g^\circ.$$

With this notation, denote

$$\begin{aligned}\xi\left(\mathfrak{u}_\lambda,\omega_\lambda\right) &= \frac{1}{|G|}\sum_{\sigma\in G}\left(\sigma\mathfrak{u}_\lambda\right)\otimes\left(\sigma\star\omega_\lambda\right)\in\left(\mathcal{O}_H\left[1/\ell\right]^\times\otimes W_g^\circ\right)^G\\ \omega(\ell)=\xi_{\mathfrak{p}_0}\left(\mathfrak{u}_\lambda,\omega_\lambda\right) &= \frac{1}{|G|}\sum_{\sigma\in G}\log_{\mathfrak{p}_0}\left(\sigma\mathfrak{u}_\lambda\right)\cdot\left(\sigma\star\omega_\lambda\right)\in W_g.\end{aligned}$$

Finally, let

$$\mathfrak{M}(\ell)=[\omega(1),\omega(\ell)]\in W_g.$$

Since $\mathfrak{u}_\lambda$ is well-defined up to some non-zero scalar multiple of $\mathcal{O}_H^\times$, the above elements are well-defined up to an element of $\left(\mathcal{O}_H^\times\otimes W_g^\circ\right)^G$ and $L\cdot\omega(1)$, respectively. Additionally, the image of $\omega(\ell)$ inside $W_g/\left(L\cdot\omega(1)\right)$ does not depend on the choice of λ above ℓ. Hence, $\mathfrak{M}(\ell)$ is independent of all the choices made to define it.

We are now able to state the conjecture.

Conjecture 4.2.3. *(Theorem 5.3 of [8], whose proof depends on Conjecture 4.1) Given any $\omega\in W_g$, there is an associated $g_\omega^\flat\in S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ given by*

$$a_\ell\left(g_\omega^\flat\right)=\langle\omega,\mathfrak{M}(\ell)\rangle$$

for all primes $\ell\nmid Np$. The map $\omega\mapsto g_\omega^\flat$ induces an isomorphism between $W_g/\left(L\cdot\omega(1)\right)$ and $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$.

Corollary 4.2.4. *$S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ is two dimensional.*

4.3 Conjecture in the CM Case

We will discuss some special choices we can make in the case g is a CM form to simplify the conjecture. Suppose there exists an imaginary quadratic field $K/\mathbb{Q}$ and a finite order character

$$\psi_g:G_K=\mathrm{Gal}\left(\bar{K}/K\right)\to L^\times$$

such that $g=\theta_{\psi_g}$ the theta series attached to ψ_g. The associated Artin representation is then $V_g^\circ=\mathrm{Ind}_K^{\mathbb{Q}}\psi_g$. Fix $\tau\in G_{\mathbb{Q}}\backslash G_K$. Let ψ_g' be the character $\psi_g'(\sigma)=\psi_g(\tau\sigma\tau^{-1})$ for all $\sigma\in G_K$. Since ρ_g is irreducible, ψ_g and ψ_g' must be distinct. The action of G_K on V_g° and V_g decomposes into two G_K-stable lines

$$V_g^\circ=\mathfrak{L}_{\psi_g}^\circ\oplus\mathfrak{L}_{\psi_g'}^\circ\qquad\text{and}\qquad V_g=\mathfrak{L}_{\psi_g}\oplus\mathfrak{L}_{\psi_g'},$$

where G_K acts as ψ_g and ψ'_g respectively. Then $\tilde{W}_g^\circ$ and $\tilde{W}_g$ decomposed into four G_K-stable lines in the following way

$$\begin{aligned}\tilde{W}_g^\circ &= \left(\mathrm{Hom}(\mathfrak{L}^\circ_{\psi_g}, \mathfrak{L}^\circ_{\psi_g}) \oplus \mathrm{Hom}(\mathfrak{L}^\circ_{\psi'_g}, \mathfrak{L}^\circ_{\psi'_g})\right) \oplus \left(\mathrm{Hom}(\mathfrak{L}^\circ_{\psi_g}, \mathfrak{L}^\circ_{\psi'_g}) \oplus \mathrm{Hom}(\mathfrak{L}^\circ_{\psi'_g}, \mathfrak{L}^\circ_{\psi_g})\right), \\ \tilde{W}_g &= \left(\mathrm{Hom}(\mathfrak{L}_{\psi_g}, \mathfrak{L}_{\psi_g}) \oplus \mathrm{Hom}(\mathfrak{L}_{\psi'_g}, \mathfrak{L}_{\psi'_g})\right) \oplus \left(\mathrm{Hom}(\mathfrak{L}_{\psi_g}, \mathfrak{L}_{\psi'_g}) \oplus \mathrm{Hom}(\mathfrak{L}_{\psi'_g}, \mathfrak{L}_{\psi_g})\right),\end{aligned}$$

where the two dimensional spaces in the brackets are $G_{\mathbb{Q}}$-stable. They are isomorphic to $\mathrm{Ind}_K^{\mathbb{Q}} 1$ and $\mathrm{Ind}_K^{\mathbb{Q}} \psi$, where $\psi = \psi_g/\psi'_g$. To summarize, we have

$$W_g^\circ = L\left(\tau_K\right) \oplus \mathrm{Ind}_K^{\mathbb{Q}} \psi \quad \text{and} \quad W_g = \bar{\mathbb{Q}}_p\left(\tau_K\right) \oplus \mathrm{Ind}_K^{\mathbb{Q}} \psi.$$

Recall that the adjoint representation factors through a finite quotient $\mathrm{Gal}(H/K)$. By the analysis above, we see that H is the ring class field of K attached to ψ.

By choosing some appropriate basis (e_1, e_2) of $\mathfrak{L}^\circ_{\psi_g} \times \mathfrak{L}^\circ_{\psi'_g}$ which induces a natural basis $e_{11}, e_{12}, e_{21}, e_{22}$ of $\tilde{W}_g^\circ$, we can assume that $L\left(\chi_K\right)$ is identified with the space of diagonal matrices that have trace zero, and $\mathrm{Ind}_K^{\mathbb{Q}} \psi$ is the space of off-diagonal matrices. By scaling our basis, we can additionally assume that

$$\rho_g(\tau) = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix},$$

where the determinant $-t^2 = \chi(\tau)$ (recall that χ is the nebentypus character, and so this has to be satisfied).

Let $G_0 = \mathrm{Gal}\left(H/K\right)$ be the maximal abelian normal subgroup of the dihedral group $G = \mathrm{Gal}\left(H/\mathbb{Q}\right)$. Let

$$e_\psi = \frac{1}{|G|} \sum_{\sigma \in G_0} \psi^{-1}(\sigma)\sigma \in L\left[G_0\right].$$

Pick a unit $\mathfrak{u} \in \mathcal{O}_H^\times$, and define

$$\mathfrak{u}_\psi = e_\psi(\mathfrak{u}) \qquad \text{and} \qquad \tau u_\psi = e_{\psi'}(\tau \mathfrak{u}).$$

It is easy to see they are elements of $\mathcal{O}_H^\times \otimes L$ such that G_0 acts via the characters ψ and $\psi' = \psi^{-1}$. Let $\omega = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. By our choice of basis, for all $\sigma \in G_0$, $\rho_g(\sigma)$ is a diagonal matrix and of the form

$$\begin{pmatrix} \psi_g(\sigma) & 0 \\ 0 & \psi'_g(\sigma) \end{pmatrix}.$$

The action of conjugation then gives us

$$\sigma \star \omega = \begin{pmatrix} 0 & \psi(\sigma) \\ 0 & 0 \end{pmatrix}, \text{ and } (\sigma\tau) \star \omega = \begin{pmatrix} 0 & 0 \\ \psi^{-1}(\sigma) & 0 \end{pmatrix}.$$

Since G acts on $\mathcal{O}_H^\times \otimes W_g$ diagonally, we find that

$$\begin{aligned}
\xi(\mathfrak{u}, \omega) &= \frac{1}{|G|} \sum_{\sigma \in G} (\sigma \mathfrak{u}) \otimes (\sigma \star \omega) \\
&= \frac{1}{|G|} \left(\sum_{\sigma \in G_0} (\sigma \mathfrak{u}) \otimes (\sigma \star \omega) + \sum_{\delta \in G_0} ((\delta\tau)\mathfrak{u}) \otimes ((\delta\tau) \star \omega) \right) \\
&= \frac{1}{|G|} \left(\sum_{\sigma \in G_0} \psi(\sigma)(\sigma \mathfrak{u}) \otimes \omega + \sum_{\delta \in G_0} \psi^{-1}(\delta)\, (\delta(\tau \mathfrak{u})) \otimes (\tau \star \omega) \right) \\
&= \frac{1}{2} \begin{pmatrix} 0 & \tau \mathfrak{u}_\psi \\ \mathfrak{u}_\psi & 0 \end{pmatrix}.
\end{aligned}$$

Hence, we can naturally pick $w(1)$ to be

$$w(1) = \left(\log_{\mathfrak{p}_0} \otimes id\right)(2\xi(\mathfrak{u}, \omega)) = \begin{pmatrix} 0 & \log_{\mathfrak{p}_0}(\tau \mathfrak{u}_\psi) \\ \log_{\mathfrak{p}_0}(\mathfrak{u}_\psi) & 0 \end{pmatrix}.$$

Similarly, we can make some simplications to $\omega(\ell)$ and $\mathfrak{M}(\ell)$ as well. This will depend on whether ℓ is split or inert in K.

4.3.1 ℓ is split in K

Suppose that the prime $(\ell) = \lambda\lambda'$ is split in K. Assume that g is regular at ℓ, so that $\psi_g(\sigma_\lambda) \neq \psi'_g(\sigma_\lambda)$. By Dirichlet's Unit Theorem,

$$\mathcal{O}_K[1/\ell]^\times \otimes L \cong \mathrm{Ind}_{G_\lambda}^{\mathrm{Gal}(K/\mathbb{Q})} L \oplus \mathrm{Ind}_{\langle c \rangle}^{\mathrm{Gal}(K/\mathbb{Q})} L - 1_{\mathrm{Gal}(K/\mathbb{Q})},$$

where c is a complex conjugation and G_λ is a decomposition group of λ in $\mathrm{Gal}(K/\mathbb{Q})$. The group G_λ is trivial, because ℓ splits in K. Since $K/\mathbb{Q}$ is imaginary quadratic, $\langle c \rangle = \mathrm{Gal}\,(K/\mathbb{Q})$. Hence

$$\mathcal{O}_K[1/\ell]^\times \otimes L \cong \mathrm{Ind}_1^{\mathrm{Gal}(K/\mathbb{Q})} L - 1_{\mathrm{Gal}(K/\mathbb{Q})}.$$

Similarly, we have an isomorphism

$$\frac{\mathcal{O}_H[\frac{1}{\ell}]^\times}{\mathcal{O}_H^\times} \otimes L \cong \mathrm{Ind}_{G_\omega}^G L,$$

where ω is any fixed prime above ℓ in H. Since g is regular at ℓ, we can let α_ℓ and β_ℓ be the distinct roots of the characteristic polynomial of $Frob_\omega$. Then on the adjoint representation $Frob_\omega$ acts with eigenvalue $1, \frac{\alpha_\ell}{\beta_\ell}$ and $\frac{\beta_\ell}{\alpha_\ell}$. Since the roots are distinct, $\frac{\alpha_\ell}{\beta_\ell} \neq 1 \neq \frac{\beta_\ell}{\alpha_\ell}$. By Frobenius reciprocity, we find that

$$\dim_L \left(\mathcal{O}_K[1/\ell]^\times \otimes W_g^\circ\right)^G = 1 \quad \text{and} \quad \dim_L \left(\frac{\mathcal{O}_H[\frac{1}{\ell}]^\times}{\mathcal{O}_H^\times} \otimes W_g^\times\right)^G = 1.$$

Therefore, the following natural inclusion induces an isomorphism of L-vector spaces.

$$\left(\mathcal{O}_K[1/\ell]^\times \otimes W_g^\circ\right)^G \hookrightarrow \left(\frac{\mathcal{O}_H^\times[1/\ell]^\times}{\mathcal{O}_H^\times} \otimes W_g^\circ\right)^G$$

This suggests that we can construct $\mathfrak{u}_\lambda$ from $\mathcal{O}_K^\times[1/\ell] \otimes L$ instead.

Following the general construction, first pick $\tilde{\mathfrak{u}}_\lambda$ to be a generator of λ^h, where h is the class number of K. Let $\mathfrak{u}_\lambda = \tilde{\mathfrak{u}}_\lambda \otimes h^{-1}$. Let $\sigma_\lambda, \sigma_{\lambda'} \in G_0$ be the Frobenius elements associated to λ and λ'. With these choices, we compute and find that

$$\tilde{\omega}_\lambda = \rho_g(\sigma_\lambda) = \begin{pmatrix} \psi_g(\sigma_\lambda) & 0 \\ 0 & \psi_g(\sigma_{\lambda'}) \end{pmatrix}, \text{ and } \omega_\lambda = p(\tilde{\omega}_\lambda) = \frac{\psi_g(\lambda) - \psi_g(\lambda')}{2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $\rho_g(\sigma)$ is a diagonal matrix, it is easy to check that for all $\sigma \in G_0$

$$\sigma \star \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad (\sigma\tau) \star \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Combined this with the fact that for all $\sigma \in G_0 = \mathrm{Gal}\,(H/K)$, K is fixed,

$$\begin{aligned} \omega(\ell) &= \frac{1}{|G|} \sum_{\sigma \in G} \log_{\mathfrak{p}_0}(\sigma \mathfrak{u}_\lambda)\,(\sigma \star \omega_\lambda) \\ &= \frac{\psi_g(\lambda) - \psi_g(\lambda')}{2 \cdot |G|} \left(\sum_{\sigma \in G_0} \log_{\mathfrak{p}_0}(\sigma \mathfrak{u}_\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \sum_{\delta \in G_0} \log_{\mathfrak{p}_0}(\delta\tau \mathfrak{u}_\lambda) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\psi_g(\lambda)-\psi_g(\lambda')}{2\cdot|G|}\left(|G_0|\cdot\log_{\mathfrak{p}_0}(\mathfrak{u}_\lambda)\begin{pmatrix}1&0\\0&-1\end{pmatrix}+|G_0|\log_{\mathfrak{p}_0}(\tau\mathfrak{u}_\lambda)\begin{pmatrix}-1&0\\0&1\end{pmatrix}\right)\\
&= \frac{\psi_g(\lambda)-\psi_g(\lambda')}{4}\left(\log_{\mathfrak{p}_0}(\mathfrak{u}_\lambda)-\log_{\mathfrak{p}_0}(\tau\mathfrak{u}_\lambda)\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\\
&= \frac{\psi_g(\lambda)-\psi_g(\lambda')}{4}\log_{\mathfrak{p}_0}\left(\frac{\mathfrak{u}_\lambda}{\mathfrak{u}'_\lambda}\right)\begin{pmatrix}1&0\\0&-1\end{pmatrix}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathfrak{M}(\ell) &= [\omega(1),\omega(\ell)]\\
&= \frac{\psi_g(\lambda)-\psi_g(\lambda')}{4}\log_{\mathfrak{p}_0}\left(\frac{\mathfrak{u}_\lambda}{\mathfrak{u}'_\lambda}\right)\begin{pmatrix}0&-2\log_{\mathfrak{p}_0}(\tau\mathfrak{u}_\psi)\\2\log_{\mathfrak{p}_0}(\mathfrak{u}_\psi)&0\end{pmatrix}\\
&= \frac{\psi_g(\lambda)-\psi_g(\lambda')}{2}\log_{\mathfrak{p}_0}\left(\frac{\mathfrak{u}_\lambda}{\mathfrak{u}'_\lambda}\right)\begin{pmatrix}0&-\log_{\mathfrak{p}_0}(\tau\mathfrak{u}_\psi)\\\log_{\mathfrak{p}_0}(\mathfrak{u}_\psi)&0\end{pmatrix}.
\end{aligned}
$$

4.3.2 ℓ is inert in K

Now assume ℓ is inert in K. Since $\rho_g(\sigma_\ell)$ has trace zero, σ_ℓ has distinct eigenvalues, so g must be regular at ℓ. Choose a prime λ of H above ℓ, and suppose σ_λ is the Frobenius element associated to λ. Then with respect to our basis,

$$\tilde{\omega}_\lambda=\omega_\lambda=\rho_g(\sigma_\lambda)=\begin{pmatrix}0&b_\lambda\\c_\lambda&0\end{pmatrix}.$$

As usual, pick $\tilde{\mathfrak{u}}_\lambda$ to be a generator of the ideal λ^h, where h is the class number of H. Recall that $\mathcal{O}_H[1/\ell]^\times/\mathcal{O}_H^\times \cong \bigoplus_{\lambda|\ell}\mathbb{Z}\cdot\lambda$. Using the latter notation, pick $\hat{\mathfrak{u}}_\lambda \in \left(\mathcal{O}_H[1/\ell]^\times/\mathcal{O}_H^\times\right)\otimes L$ such that it has prime factorization

$$(\hat{\mathfrak{u}}_\lambda)=b_\lambda\lambda+c_\lambda(\tau\lambda).$$

Furthermore, let

$$\mathfrak{u}_\psi(\ell)=e_\psi(\hat{\mathfrak{u}}_\lambda)=\frac{1}{|G_0|}\sum_{\sigma\in G_0}\psi^{-1}(\sigma)\sigma(\hat{\mathfrak{u}}_\lambda).$$

Similar to the calculations in the previous section, for all $\sigma\in G_0$,

$$\sigma\star\omega_\lambda=\begin{pmatrix}0&\psi(\sigma)b_\lambda\\\psi^{-1}(\sigma)c_\lambda&0\end{pmatrix},\ \text{and}\ (\sigma\tau)\star\omega_\lambda=\begin{pmatrix}0&\psi(\sigma)c_\lambda\\\psi^{-1}(\sigma)b_\lambda&0\end{pmatrix}.$$

This can easily be seen by the fact that the off-diagonal matrices is given by $\mathrm{Ind}_K^{\mathbb{Q}}\psi$. With these calculations, we find that

$$\begin{aligned}
\omega(\ell) &= \frac{1}{|G|}\left(\sum_{\sigma\in G}\log_{\mathfrak{p}_0}(\sigma\mathfrak{u}_\lambda)\star(\sigma\star\omega_\lambda)\right)\\
&= \frac{1}{|G|}\left(\sum_{\sigma\in G_0}\log_{\mathfrak{p}_0}(\sigma\mathfrak{u}_\lambda)\begin{pmatrix} 0 & \psi(\sigma)b_\lambda \\ \psi^{-1}(\sigma)c_\lambda & 0\end{pmatrix} + \sum_{\delta\in G_0}\log_{\mathfrak{p}_0}(\delta\tau\mathfrak{u}_\lambda)\begin{pmatrix} 0 & \psi(\delta)c_\lambda \\ \psi^{-1}(\delta)b_\lambda & 0\end{pmatrix}\right)\\
&= \frac{1}{2}\begin{pmatrix} 0 & \log_{\mathfrak{p}_0}\mathfrak{u}_{\psi^{-1}}(\ell) \\ \log_{\mathfrak{p}_0}\tau\mathfrak{u}_\psi(\ell) & 0\end{pmatrix}.
\end{aligned}$$

Hence,

$$\mathfrak{M}(\ell) = \frac{1}{2}\begin{pmatrix} R_\psi(\ell) & 0 \\ 0 & -R_\psi(\ell)\end{pmatrix}$$

where

$$R_\psi(\ell) = \det\begin{pmatrix} \log_{\mathfrak{p}_0}\mathfrak{u}_\psi & \log_{\mathfrak{p}_0}\mathfrak{u}_\psi(\ell) \\ \log_{\mathfrak{p}_0}\tau\mathfrak{u}_\psi & \log_{\mathfrak{p}_0}\tau\mathfrak{u}_\psi(\ell)\end{pmatrix}.$$

Now, we are ready to state the specialized version of conjecture 4.2.3 in the CM case.

Conjecture 4.3.1. *There exists a canonical basis (g_1, g_2) of $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$ where*

1. *the Fourier coefficients $a_\ell(g_1)$ are supported on primes $\ell \nmid Np$ that are split in K. Additionally, if $(\ell) = \lambda\lambda'$ then*
$$a_\ell(g_1) = (\psi_g(\lambda) - \psi_g(\lambda'))\cdot\log_{\mathfrak{p}_0}(\mathfrak{u}_\lambda/\mathfrak{u}'_\lambda).$$
2. *the Fourier coefficients $a_\ell(g_2)$ are supported on primes $\ell \nmid Np$ that are inert in K. More specifically,*
$$a_\ell(g_2) = R_\psi(\ell).$$

If we assume the general case conjecture is correct, this is a simple corollary. Let

$$w_1 = 2\cdot\begin{pmatrix} 0 & a \\ b & 0\end{pmatrix}$$

where a, b satisfies $a\log_{\mathfrak{p}_0}(\mathfrak{u}_\psi) - b\log_{\mathfrak{p}_0}(\tau\mathfrak{u}_\psi) = 1$, and let

$$w_2 = \frac{1}{2}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}.$$

Then under the isomorphism of the general conjecture, w_1 gives rise to g_1 and w_2 gives rise to g_2. This is exactly Theorem 6.1 of [8].

Chapter 5
Special Case of the Conjecture

We will now give a proof of the conjecture in a special scenario, where the Artin representation associated to g has image isomorphic to the dihedral group of order 8.

5.1 Introduction

The special case we will describe in this section is same scenario that is considered in [11]. Let $F/\mathbb{Q}$ be a totally real field of degree $r+1$, with embeddings $v_0, ..., v_r$.

Definition 5.1.1. A quadratic extension F_1/F is an *almost totally real extension* of F if

$$F_1 \otimes_{F,v_0} \mathbb{R} \cong \mathbb{C}, \text{ and } F_1 \otimes_{F,v_j} \mathbb{R} \cong \mathbb{R} \oplus \mathbb{R} \text{ for all } 1 \le j \le r.$$

In particular, consider the case where $F = \mathbb{Q}(\sqrt{N})$ for some $N > 0$ is a real quadratic field. Since F_1 has two real places and one imaginary place, it is not Galois over $\mathbb{Q}$. Let L be its Galois closure. By construction, $L = F_1F_2$ where F_2 is the Galois conjugate of F_1 over $\mathbb{Q}$ and $\mathrm{Gal}(L/\mathbb{Q}) \cong D_4$ the dihedral group of order 8. Additionally, the Galois group $\mathrm{Gal}(L/F)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ the Klein 4-group. The field F_1 is of the form $\mathbb{Q}\left(\sqrt{a - b\sqrt{N}}\right)$ where $a, b \in \mathbb{Z}$ are such that $d = a^2 - Nb^2 < 0$. To summarize, we have the following diagram mapping out all the subextensions of $L/\mathbb{Q}$.

$$\begin{tikzcd}[arrows=dash]
 & & L \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
F_1 \arrow[dr] & F_2 \arrow[d] & H \arrow[d] & K_1 \arrow[d] & K_2 \arrow[dl] \\
 & F = \mathbb{Q}\left(\sqrt{N}\right) \arrow[dr] & M = \mathbb{Q}\left(\sqrt{Nd}\right) \arrow[d] & K = \mathbb{Q}\left(\sqrt{d}\right) \arrow[dl] & \\
 & & \mathbb{Q} & &
\end{tikzcd}$$

Here, every line indicates an extension of degree 2. Although, we can explicitly write down the generators of all these fields, it will not be necessary for the rest of the thesis. Moreover, there are additional properties that characters of these fields satisfy that may be of great use for future work, but not for the rest of the thesis, so we will not mention them. See Proposition 3.2 of [11] for more details.

Let

$$\psi_K : G_K \to \{\pm 1\}$$

be the Galois character of K that cuts out the extension K_1. That is, $\ker \psi_K = G_{K_1}$. Similarly, let

$$\psi_M : G_M \to \{\pm 1, \pm i\}$$

be the Galois character that cuts out the extension L.

By viewing these Galois characters as idèle class characters, they are Hecke characters of types we have seen in earlier chapters. Let θ_{ψ_K} and θ_{ψ_M} denote the weight one Hecke theta series associated to ψ_K and ψ_M. Let $\rho_K = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \psi_K$ denote the associated Artin representation attached to θ_{ψ_K}, and similarly for the field M. Since D_4 only has one irreducible two dimensional representation, $\rho_K \cong \rho_M$. This implies that $\theta_{\psi_K} = \theta_{\psi_M}$. By Theorem 3.8.6, we have two Hida families attached to each theta series and the families intersect at a classical weight one point corresponding to the p-stabilization of θ_{ψ_K}.

By the irregularity assumption, the prime p has to be split in both K and M. Therefore, we can apply the theory established in Section 3.8 to explicit describe the Hida families attached to θ_{ψ_K} and θ_{ψ_M}. In the next section, we will use these two explicit families to compute the tangent space of the eigencurve at this particular weight one point and prove Conjecture 4.2.3.

Remark 5.1.2. The fields L and H are named specifically to match the notations used to define the conjecture. That is, we can show that $G_L = \mathrm{Gal}(\bar{L}/L) = \ker \rho_F$ and $G_H = \mathrm{Gal}(\bar{H}/H) = \ker(\mathrm{Ad}\rho_F)$.

5.2 Explicit Fourier Coefficients

Let $g = \theta_{\chi_K} = \theta_{\chi_M}$. We know from Section 3.9 that the Galois representations associated to the cuspidal family of ordinary theta series at weight k is given by

$$\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \psi_K \nu_K^{k-1},$$

where ν_K is a canonical Hecke character. By letting k be a "weight variable" that we can manipulate analytically,

$$\frac{d}{dk}\bigg|_{k=1} \nu_K^{k-1} = \log_{\mathfrak{p}} \nu_K.$$

We find that the ℓ-th Fourier coefficient of the generalized eigenform is

$$Tr\left(\left(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\psi_K \log_{\mathfrak{p}} \nu_K)\right)(\sigma_\lambda)\right)$$

where λ is a prime of K above ℓ, and σ_λ is a Frobenius of λ. When ℓ is inert, the representation has trace 0, so the eigenform is supported on split primes.

Suppose ℓ is split in K and let h be the class number of K. Suppose $\tilde{\mathfrak{u}}_\lambda$ is a generator of λ^h. Then

$$\log_{\mathfrak{p}} \nu_K(\sigma_\lambda)^h = \log_{\mathfrak{p}} v_K\left((\tilde{\mathfrak{u}}_\lambda)\right) = \log_{\mathfrak{p}}\left(\tilde{\mathfrak{u}}_\lambda\right),$$

which implies that

$$\log_{\mathfrak{p}} v_K(\sigma_\lambda) = h^{-1} \log_{\mathfrak{p}}(\tilde{\mathfrak{u}}_\lambda) = \log_{\mathfrak{p}} \mathfrak{u}_\lambda,$$

where $\mathfrak{u}_\lambda = \tilde{\mathfrak{u}}_\lambda \otimes h^{-1} \in \mathcal{O}_K[1/\ell]^\times \otimes L$. Using this dictionary, and the previously defined basis, we find that

$$\left(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\psi_K \log_{\mathfrak{p}} \nu_K)\right)(\sigma_\lambda) = \begin{pmatrix} \psi_K(\sigma_\lambda)\log_{\mathfrak{p}} \mathfrak{u}_\lambda & 0 \\ 0 & \psi_K(\sigma_{\lambda'})\log_{\mathfrak{p}} \mathfrak{u}'_\lambda \end{pmatrix}.$$

Since we are considering the deformations consisting of constant determinants, we need to apply the canonical projection $p : W_g \to \tilde{W}_g$ given by $A \mapsto A - \frac{1}{2}Tr(A)I$. Hence the ℓ-th Fourier coeffiicient of the generalized eigenform is

$$\frac{1}{2}Tr\begin{pmatrix} \psi_K(\sigma_\lambda)\log_{\mathfrak{p}}(\mathfrak{u}_\lambda/\mathfrak{u}'_\lambda) & 0 \\ 0 & \psi_K(\sigma_{\lambda'})\log_{\mathfrak{p}}(\mathfrak{u}'_\lambda/\mathfrak{u}_\lambda) \end{pmatrix} = \frac{1}{2}\log_{\mathfrak{p}}(\mathfrak{u}_\lambda/\mathfrak{u}'_\lambda)\left(\psi_K(\sigma_\lambda) - \psi_K(\sigma'_\lambda)\right).$$

This is exactly the eigenform described by part 1 of Conjecture 4.3.1. Denote this normalized generalized eigenform by g_K. By the same construction, we also have g_M coming from the other imaginary quadratic field M.

By construction $H = KM$ is a biquadratic field. If ℓ splits completely in H, then the Frobenius map $\sigma_\lambda = 1$ for all primes λ in H above ℓ. This implies both $a_\ell(g_K)$ and $a_\ell(g_M)$ are zero, because $\psi_K(\sigma_\lambda) = \psi_K(\sigma_{\lambda'})$. Hence, the Fourier coefficients of g_K are supported on the primes ℓ that are split in K, but are inert in M. For such a prime ℓ, $\psi_K(\sigma_\lambda) = -\psi_K(\sigma_{\lambda'})$, which shows that $a_\ell(g_K)$ is non-zero.

Theorem 5.2.1. *In the scenario described above, the dimension of $S_1^{(p)}(N,\chi)[\![g_\alpha]\!]_0$, the space of normalized generalized eigenforms, is two.*

Proof. We showed that the Fourier coefficients of g_K are supported on primes ℓ that are split in K and inert in M. Similarly, the Fourier coefficients of g_M are supported on primes ℓ that are split in M and inert in K. Since these sets of primes are distinct, g_K and g_M are linearly independent. □

Let $g_{K,1}$ and $g_{K,2}$ denote the two normalized generalized overconvergent modular form constructed from Conjecture 4.3.1. We already know that $g_{K,1} = 2g_K$. An interesting problem would be to describe g_M as a linearly combination of $g_{K,1}$ and $g_{K,2}$.

Suppose $\omega_{K,1}, \omega_{K,2} \in W_g$ are a choice of pre-images of $g_{K,1}$ and $g_{K,2}$ under the isomorphism described in Conjecture 4.2.3. Recall that the construction of the general conjecture purely works with the field H and the representation ρ_g. To obtain $\omega_{K,1}, \omega_{K,2} \in W_g$ through the construction described in Section 4.3, we made a choice of basis $\{e_{K,1}, e_{K,2}\}$ of V_g and similarly for the field M.

Let $A \in \mathrm{End}(V_g)$ be the endomorphism of V_g sending $e_{M,1}$ to $e_{K,1}$ and $e_{M,2}$ to $e_{K,2}$. Then A induces an endomorphism of W_g. Suppose

$$\omega_{M,1} = a_1\omega_{K,1} + a_2\omega_{K_2} + a_3\omega(1)_K \text{ for some } a_1, a_2, a_3 \in L,$$

where $\omega(1)_K$ denotes the choice of $\omega(1)$ with respect to the field K. Since $\omega(1)_K$ and $\omega(1)_M$ span the same one dimensional L-vector space, a_1 and a_2 are not both zero. By applying the isomorphism given by Conjecture 4.2.3, we find that

$$g_M = \frac{1}{2}g_{M,1} = \frac{1}{2}a_1 g_{K,1} + \frac{1}{2}a_2 g_{K,2}.$$

Chapter 6
Conclusion

Let g be an ordinary newform of weight one that is irregular at a prime p. Suppose g_α is its unique p-stabilization. In this thesis, we studied the relative tangent space of the eigencurve at g_α. More specifically, by Proposition 4.2.1, the elements of the tangent space can be naturally identified with the space of normalized generalized eigenforms $S_1^{(p)}(N,\chi)\,[\![g_\alpha]\!]_0$. Following the general $\mathcal{R} = \mathbb{T}$ philosophy, we hope to produce an isomorphism between the tangent space of all ordinary deformations with constant determinants, with the relative tangent space of the eigencurve at g_α. This led Darmon, Lauder and Rotger to conjecture that we should be able to describe the Fourier coefficients of the normalized generalized eigenforms in terms of logarithms of algebraic numbers, via this conjectural isomorphism. The explicit statement of this conjecture was given in Conjecture 4.2.3. In the case where g was a theta series attached to a Hecke character on an imaginary quadratic field, we made a reduction of the conjecture, described in Conjecture 4.3.1.

In the special scenario the thesis considered, we obtained two Hecke characters ψ_K and ψ_M on imaginary quadratic fields K and M that cut out a D_4-extension (the dihedral group of order 8). Additionally, the weight one theta series attached to the two characters are equal by construction. This implies that the two Hida families of theta series attached to the character as described in Section 3.9, intersect at weight one. By explicitly computing the tangent space attached to the Hida families, we proved Conjecture 4.3.1. That is, the relative tangent space of the eigencurve at g_α is of dimension two, and the Fourier coefficients of the normalized generalized eigenforms can be described explicitly in terms of logarithms of global units, as conjectured.

Let E be an elliptic curve over $\mathbb{Q}$. By modularity, there exists a normalized newform of weight 2 satisfying $L(f,s) = L(E,s)$. Suppose

$$\rho : G_{\mathbb{Q}} \to \mathrm{Aut}(V_\rho) \cong GL_n(L)$$

is an Artin representation with coefficients in come finite extension $L \subseteq \mathbb{C}$ of $\mathbb{Q}$. Suppose ρ factors through a finite extension $\mathrm{Gal}(H/\mathbb{Q})$. The Birch and Swinnerton-Dyer conjecture predicts that the analytic rank of the Hasse-Weil-Artin L-function $L(E, \rho, s)$ is the same as the algebraic rank defined to be

$$\dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(V_\rho, E(H) \otimes L).$$

Let g and h be weight one newforms with associated Artin representations ρ_g and ρ_h. Assume that ρ is an irreducible constituent of $\rho_{gh} = \rho_g \otimes \rho_h$ and assume that ρ_{gh} is self-dual. In [10], the authors proposed the elliptic Stark conjecture, which gives a formula describing a p-adic iterated integral attached to (f, g, h), which is a p-adic avatar of the special value of $L(E, \rho_{gh}, 1)$, in terms of formal group logarithms of global points on $E(H)$.

In the statement of the elliptic Stark conjecture, there is an underlying hypothesis that the space of overconvergent generalized eigenforms $S_1^{(p)}(N, \chi)[\![g_\alpha]\!]$ consists of only classical forms. This hypothesis is expected to not hold when g is irregular at p. However, if Conjecture 4.2.3 holds true, we can explicitly describe the Fourier coefficients of normalized generalized eigenforms, which forms a natural linear complement $S_1^{(p)}(N, \chi)[\![g_\alpha]\!]_0$ of the space of classical forms inside $S_1^{(p)}(N, \chi)[\![g_\alpha]\!]$. This will allow us to define and study an alternative version of the elliptic Stark conjecture. Furthermore, in [10], the authors are able to prove the elliptic Stark conjecture in the case where g and h are theta series associated to a common imaginary quadratic field K in which p splits. This result gives us great hope that the conjecture can also be proven in the irregular case if we assume g is as described in Section 5.1 and h is a theta series associated to K.

Appendix A- Dirichlet's Unit Theorem

Theorem. *(Dirichlet's S-Unit Theorem). Let K be a number field. Suppose S is a finite set of places of K, that includes all the infinite ones. Let K^S be the multiplicative group given by*

$$K^S = \left\{ \alpha \in K^{\times} : |\alpha|_{\mathfrak{p}} = 1 \, \textit{for all } \mathfrak{p} \notin S \right\}.$$

Then K^S is the direct sum of a cyclic group with a free group of rank $|S| - 1$.
More specifically, the map

$$\begin{aligned} \lambda : K^S &\rightarrow \prod_{\mathfrak{p} \in S} \mathbb{R} \\ a &\mapsto (\log |a|_{\mathfrak{p}})_{\mathfrak{p} \in S} \end{aligned}$$

has kernel $\mu(K)$ (the roots of unities of K) and its image is a complete lattice

$$\left\{ (a_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in S} \mathbb{R} : \sum_{\mathfrak{p} \in S} a_{\mathfrak{p}} = 0 \right\}$$

of dimension $|S| - 1$.

Proof. See Chapter VI, Proposition 1.1 of [31] or Chapter II, Section 18 of [3]. ☐

In this thesis, we will need to know more than the rank of K^S as a group, but also its structure as a Galois module. Let $G = \mathrm{Gal}(K/\mathbb{Q})$. We will consider the special case where T is a set of place of $\mathbb{Q}$ containing the infinite places, and S is the set of places of K lying above those in T. Let Y_S be free abelian group generated by S. We have a natural action of G on Y_S by permuting the places $\lambda \in S$ above $\mathfrak{p} \in T$. By fixing a place $\lambda \in S$ above each $\mathfrak{p} \in T$, we can additionally say that

$$Y_S \cong \oplus_{\mathfrak{p} \in T} \mathrm{Ind}_{G_\lambda}^{G} \mathbb{Z},$$

where G_λ is the decomposition group of λ, and acts trivially on $\mathbb{Z}$. Let

$$X_S = \left\{ \sum_{\lambda \in S} a_\lambda \lambda \in Y_S : \sum_{\lambda \in S} a_\lambda = 0 \right\}.$$

Then we have a natural exact sequence of G-modules

$$0 \to X \to Y \xrightarrow{e} \mathbb{Z} \to 0,$$

where $e : \sum a_\lambda \lambda \to \sum \lambda$ is the augmentation map. With this exact sequence, we can then show that by viewing $\mathbb{C}X_S$ and $\mathbb{C}Y_S$ as $\mathbb{C}[G]$-modules, their associated characters satisfy

$$\chi_{Y_S} = \chi_{X_S} + 1_G$$

where 1_G is the trivial character. See Section 3.2 of [12] for the proof and more details. In this thesis, we are most interested in the cases where K is an imaginary quadratic field, and

1. S is the set of infinite places
2. S is the set of infinite places and the primes above some rational prime $p \in \mathbb{Q}$ that splits completely

In the first case, $K^S = \mathcal{O}_K^\times$, which is just the normal Dirichlet unit theorem. Furthermore, as G-modules, we have the isomorphism

$$\mathcal{O}_K^\times \otimes \mathbb{C} \cong \mathrm{Ind}_{\langle c \rangle}^G \mathbb{C} - 1_{\mathbb{C}},$$

where c is a complex conjugation, and $1_{\mathbb{C}}$ is the trivial representation. In the latter case, $K^S = \mathcal{O}_K[\frac{1}{p}]^\times$. We have an isomorphism of G-modules

$$\mathcal{O}_K[1/p]^\times \otimes \mathbb{C} \cong \mathrm{Ind}_{\langle c \rangle}^G \mathbb{C} \oplus \mathrm{Ind}_{G_\lambda}^G \mathbb{C} - 1_{\mathbb{C}},$$

where λ is a prime in K above p and G_λ is its decomposition group.

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