An introduction to Tate's Thesis

James-Michael Leahy

Master of Arts

Department of Mathematics and Statistics

McGill University

Montreal, Quebec

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ABSTRACT

In the early 20th century, Erich Hecke attempted to find a further generalization of the Dirichlet L-series and the Dedekind zeta function. In 1920, he [14] introduced the notion of a Grössencharakter, an ideal class character of a number field, and established the analytic continuation and functional equation of its associated L-series, the Hecke L-series. In 1950, John Tate [27], following the suggestion of his advisor, Emil Artin, recast Hecke's work. Tate provided a more elegant proof of the functional equation of the Hecke L-series by using Fourier analysis on the adeles and employing a reformulation of the Grössencharakter in terms of a character on the ideles. Tate's work now is generally understood as the GL(1) case of automorphic forms [2]. The thesis provides a thorough analysis of the approach taken by Tate in his own thesis. Background information is furnished by theory concerning topological groups, Pontryagin duality, the restricted-direct topology, and the adeles and ideles.

ABRÉGÉ

Au début du 20ème siècle, Erich Hecke a essayé de trouver une nouvelle généralisation de la série L de Dirichlet et de la fonction zêta de Dedekind. En 1920, il [14] a introduit la notion de Grössencharakter, un caractère des classes d'idéaux d'un corps de nombres, et établi le prolongement analytique et l'équation fonctionnelle de sa série L associée, la série L de Hecke. En 1950, John Tate [27], suivant la suggestion de son directeur de thèse, Emil Artin, a remanié le travail de Hecke. Tate a fourni une preuve plus élégante de l'équation fonctionnelle de la série L de Hecke en employant l'analyse de Fourier sur les adèles et en utilisant une reformulation du Grössencharakter en termes de caractère sur les idèles. Le travail de Tate est maintenant généralement compris comme le cas de GL(1) de la théorie des formes automorphes [2]. Cette thèse donne un apercu de l'approche adoptée par Tate dans sa propre thèse. Ceci inclut une introduction à la théorie des groupes topologiques, la dualité de Pontryagin, la topologie des produits restreints, et les adèles et les idèles.

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Introduction

Most students of mathematics have some familiarity with the Riemann zeta function $\zeta(s)$, which is defined by the absolutely convergent series

$$\zeta(s) = \sum_{n=1} \frac{1}{n^s}$$

for complex numbers s such that $\Re(s) > 1$. In letting $s = 1$, the series that results, $\zeta(1)$, is the Harmonic series, which diverges. Despite being named the Riemann zeta function, Leonhard Euler was the first to study the function $\zeta(s)$ for $s \in \mathbb{R}$. He established the Euler product expansion,

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

which is valid in the domain $\Re s > 1$. In the third century B.C.E., Euclid understood that there are infinitely many primes. Since the Riemann zeta function is divergent at $s = 1$, then by the Euler product expansion we obtain

$$\prod_p \frac{1}{1 - p^{-1}} > \infty,$$

which provides an alternate proof of the infinitude of primes. By analyzing the Laurent expansion of $\pi \cot \pi z$, it can be determined that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The Euler product expansion then yields

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2} = 0.607927102\cdots.$$

In a non-rigorous fashion, the probability that a natural number, chosen at random, is divisible by a fixed prime p is $1/p$. More rigorously, one should choose random natural number from the first N numbers and then let N approach infinity. The probability that two natural numbers, chosen at random, are divisible by p is $1/p^2$. Then $1 - (1/p^2)$ represents the probability that two natural numbers chosen at random are either both relatively prime to p or, at most, one is not relatively prime to p. For distinct primes p and q, the probability that a natural number chosen at random is divisible by both p and q is $1/pq$. Hence, the divisibility events are independent and the product

$$\prod_p \left(1 - \frac{1}{p^2}\right)$$

represents the probability that two natural numbers, chosen at random, are relatively prime. It can be determined, by applying the Euler product of the Riemann zeta function, that this probability is $\frac{6}{\pi^2}$.

In 1859, Georg Friedrich Bernhard Riemann [25] showed that the function $\zeta(s)$ can be analytically continued to the complex plane to a function that is holomorphic for $s \neq 1$. The residue at the simple pole $s = 1$ is 1. He also proved the functional equation

$$\xi(s) = \xi(1 - s), \tag{1}$$

where $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and where

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

It can be shown that $\Gamma(s)$ is absolutely convergent for $\Re(s) > 1$ and can be meromorphically continued to the whole s plane with simple poles at negative integers. Further, $\Gamma(s) \neq 0$ for $s > 0$. Dividing both sides of equation 1 by $\pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})$, we obtain

$$\zeta(1 - s) = (2\pi)^{-s} 2\cos(\frac{\pi}{2}s)\Gamma(s)\zeta(s).$$

This functional equation shows that in the domain $\Re s < 0$, the function $\zeta(s)$ has simple zeroes at $-2, -4, -6, \cdots$. In addition to proving the functional equation, Riemann showed that $\zeta(s)$ has infinitely many zeros on the critical strip $0 \leq \Re s \leq 1$ and hypothesized that all these zeroes lie on the line $\Re s = 1/2$. This is known today as the *Riemann hypothesis*, which has not yet been proved. It is also interesting to note that when evaluating the Riemann zeta function at positive even integer points and at negative integer points, one obtains

$$\zeta(2n) = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},$$

and

$$\zeta(-n) = -\frac{B_{n+1}}{n+1},$$

respectively, where B_m for $m \in \mathbb{Z}$ are the Bernoulli numbers. The *prime number function* $\pi(x)$ is defined for $x \in \mathbb{R}$ to be the number of primes p such that $p \leq x$. Riemann provided an explicit formula for $\pi(x)$ that depended on the location of the zeros of $\zeta(s)$. See [15], Chapter 27, for the formula. Riemann also proved the *prime number theorem*, which states that

$$\lim_{x\to\infty} \frac{\left(\frac{x}{\log x}\right)}{\pi(x)} = 1.$$

This can be proved by analyzing the Riemann zeta function and two other functions:

$$\phi(s) = \sum_p \frac{\log p}{p^s} \quad \text{and} \quad \theta(x) = \sum_{p\leq x} \log p.$$

As a consequence of Tate's thesis, we now are able to realize that the factor of $\pi^{-s/2}\Gamma(\frac{s}{2})$, appearing in the functional equation of $\zeta(s)$, "originates from" the infinite prime of $\mathbb{Q}$. Let

$$Z_\infty(s) = \int_{\mathbb{R}-\{0\}} e^{-\pi x^2}|x|^s \frac{dx}{|x|}.$$

Making the change of variable $u = \pi x^2$, we obtain

$$Z_\infty(s) = \int\limits_{\mathbb{R}-\{0\}} e^{-\pi x^2}|x|^s\frac{dx}{|x|} = 2\int_0^\infty e^{-\pi x^2}x^{s-1}dx = \pi^{-s/2}\int_0^\infty e^{-u}u^{s/2-1}du = \pi^{-s/2}\Gamma(s/2).$$

By the definition of $\xi(s)$ and the Euler product of $\zeta(s)$, we obtain

$$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = Z_\infty(s)\prod_p \frac{1}{1-p^{-s}}.$$

Now, let $Z_p(s) = \sum_{n=0}^{\infty} 1/p^{ns}$. The function $Z_p(s)$ is a geometric series and converges to $1/(1-p^{-s})$. As such, we obtain by the Riemann zeta functional equation that

$$\xi(s) = Z_\infty(s) \prod_p Z_p(s),$$

and hence

$$Z_\infty(s) \prod_p Z_p(s) = Z_\infty(1-s) \prod_p Z_p(1-s).$$

Later it will be shown that $Z_p(s)$ can be realized as the integral

$$Z_p(s) = \int\limits_{\mathbb{Q}_p - \{0\}} |x|_p^s \frac{dx_p}{|x|_p} = \int\limits_{\mathbb{Q}_p - \{0\}} |x|_p^{s-1} dx_p,$$

where $\mathbb{Q}_p$ is the p-adic numbers and where dx_p is the p-adic Haar measure.

Let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a homomorphism. Such a map is called a *Dirichlet character*. Since $a^{\phi(m)} = 1$, where ϕ is the Euler totient function, then $\chi(a)^{\phi(m)} = 1$, which implies that the image of χ must lie in S^1 and, moreover, in the $\phi(m)$th roots of unity. If the group $(\mathbb{Z}/m\mathbb{Z})^\times$ is cyclic, then any character χ is determined uniquely by its value at the generator. Note that $(\mathbb{Z}/m\mathbb{Z})^\times$ is cyclic if $m = p^i$ or $m = 2p^i$, where p is some odd rational prime. It is possible to lift a Dirichlet character to $\mathbb{Z}$ by redefining χ:

$$\chi(a) := \begin{cases} 0 & \text{if } \gcd(a,m) > 1 \\ \chi(a \bmod m) & \text{otherwise} \end{cases}$$

for all $a \in \mathbb{Z}^*$ so that $\chi(a) = \chi(b)$ whenever $a \equiv b \bmod m$. Later it will be shown that χ actually can be lifted to the projective limit $\hat{\mathbb{Z}}^\times = \prod_p \mathbb{Z}_p^\times$, where $\mathbb{Z}_p^\times$ is the ring of p-adic units. One defines the *Dirichlet L-function*, associated to a Dirichlet character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ for $s \in \mathbb{C}$ with $\Re s > 1$, to be

$$L(s,\chi) := \sum_{(n,m)=1} \frac{\chi(n)}{n^s}.$$

Since $|\chi(n)| = 1$, then the sum converges absolutely for $\Re s > 1$. The Dirichlet L-function also has an Euler product and it is given by

$$L(s,\chi) = \sum_{(n,m)=1} \frac{\chi(n)}{n^s} = \prod_{p \nmid m} \frac{1}{1-\chi(p)p^{-s}}.$$

A Dirichlet character χ is called *odd* if $\chi(-1) = 1$ and *even* if $\chi(-1) = 1$. If χ is a Dirichlet character modulo m and $m|m'$, then χ can be lifted to a Dirichlet character modulo m' by pulling back using the projection. A Dirichlet character χ is called *primitive* if it cannot be lifted from Dirichlet character character of smaller modulus. Let

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Let χ be a primitive character modulo m. Let

$$\Lambda(s,\chi) = (\pi/m)^{-(s+a)/2}\Gamma(\frac{s+a}{2})L(s,\chi).$$

The Dirichlet L-function satisfies the following functional equation:

$$\Lambda(1-s,\overline{\chi}) = \frac{i^a k^{1/2}}{\tau(\chi)}\Lambda(s,\chi),$$

where

$$\tau(\chi) = \sum_{n=1}^{m} \chi(n)e^{2\pi i n/m}.$$

This is a special case of the functional equation of the Hecke L-function. The sum $\tau(\chi)$ is a called a *Gauss sum*. We will see below that John Tate, in his thesis, introduced the concept of a generalized Gauss sum, which subsumes the definition of τ. These sums will appear in the functional equation of the Hecke L-function. The *Generalized Riemann Hypothesis* states that the for every Drichlet character χ, the zeroes of $L(\chi, s) = 0$ in the critical strip lie on the line $\Re(s) = 1/2$. Dirichlet's prime number theorem [8] states that for any two positive coprime integers a and d, there are infinitely many primes of the form $a + nd$, where $n \geq 0$. That is, there exist infinitely many primes that are congruent to a modulo d. This theorem was proved using the Dirichlet L-series. Let $\pi(x, a, d)$ denote the number of prime numbers in this progression, which are less than or equal to x. If the Generalized Riemann Hypothesis is assumed true, then

$$\pi(x,a,d) = \frac{1}{\phi(d)} \int_{2}^{x} \frac{1}{\log t} dt + O(x^{1/2+\epsilon})$$

as $x \to \infty$, where $O(x^{1/2+\epsilon})$ is a function $f(x)$ such that $|f(x)| \leq M|x^{1/2+\epsilon}|$ for large enough x. This is even stronger than the prime number theorem. As can be seen, studying L-functions over the field $K = \mathbb{Q}$ reveals strong results about the distribution of rational prime numbers.

Much of algebraic number theory is the result of the study of the diophantine equation $x^n + y^n = z^n$. This diophantine equation led scholars to investigate c*yclotomic extensions* and, more generally, to the study of unique factorization of elements in number fields. Indeed, note that

$$x^p + y^p = z^p \Longrightarrow \prod_{n=0}^{p-1} (x + ye^{2\pi in/p}) = z^p$$

leads us to a multiplicative problem in the number field $\mathbb{Q}(e^{2\pi i/p})$ and, more specifically, in the number ring $\mathbb{Z}\left[e^{2\pi i/p}\right]$. It can be shown that if cyclotomic extensions were unique factorization domains, then there would be no non-trivial integer solutions to $x^p + y^p = z^p$ for $p > 2$, almost proving *Fermat's Last Theorem*. However, this is not true; it first fails when $\mathbb{Q}(e^{2\pi i/23})$. Since the ring of integers of a number field is a Dedekind domain, then fractional ideals uniquely factor into prime ideals. Ernst Kummer was the first to investigate the ideals of a number field [10]. If the ring of integers of a number field is a principal ideal domain (i.e. every ideal is generated by a single element), then the ring of integers is a unique factorization domain. So, in a sense, it is precisely the failure of the ring of integers of a number field to be a principal ideal domain that prevents the ring of integers from being a unique factorization domain. The fractional ideals of a number field form a group under multiplication. The ideal class group of the number field is the quotient group of all fractional ideals by principal ideals, and thus is an object that measures how badly unique factorization fails. Studying the prime ideals of a number ring can help us to understand the ideal class group , and hence the failure of unique factorization. As the Riemann zeta function encodes information about the distribution of rational primes, Dedekind most likely generated the Dedekind zeta function [7] for a number field K hoping to gain a better understanding of the primes in a number ring, and thus make progress on solving Fermat's Last Theorem.

Let K be a number field with the ring of integers $\mathfrak{o}_K$. Define for $\Re s > 1$ the Dedekind zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq (0)} \frac{1}{N\mathfrak{a}^s},$$

where the sum runs over all nonzero ideals $\mathfrak{a}$ of $\mathfrak{o}_K$ and where $N\mathfrak{a}$ is the order of the group $\mathfrak{o}_K/\mathfrak{a}\mathfrak{o}_K$. We will show later that ζ_K has the Euler product formula given by

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq (0)} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}},$$

where $\mathfrak{p}$ runs over all nonzero prime ideals $\mathfrak{p}$ of $\mathfrak{o}_K$. The function $\zeta_K(s)$ converges absolutely for $\Re s > 1$ and satisfies a functional equation and can be meromorphically continued to the whole complex plane with a simple pole at $s = 1$. Suppose K has r_1 real embeddings and r_2 non-conjugate complex embeddings. Then the residue at $s = 1$ is given by

$$\mathrm{Res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|d_K|}},$$

where h_k is the class number of K and where R_K is the regulator of K. Immediately we can see that the abundance of information encoded within Dedekind's zeta function. Again, there is an extended Riemann hypothesis which states that the zeroes that lie in the critical strip lie on the line $\Re s = 1/2$. Much more can be said about the value of $\zeta(s)$ at integer points, and this connection is studied in algebraic K-theory. Before we move on, let us given an example of how the Dirichlet L-functions appear as irreducible pieces of the Dedekind zeta function.

Example 0.0.1. An element $a \in \mathbb{Z}$, such that he congruence $x^2 \equiv a \bmod p$ has a solution, is called a *quadratic residue* modulo p. If p is an odd prime, then the *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined to be 1 if a is a quadratic residue, -1 if a is a quadratic nonresidue modulo p, and

0 if $p|a$. Note that $\left(\frac{a}{p}\right)$ is multiplicative. Define χ_5 on $(\mathbb{Z}/5\mathbb{Z})^\times$ using the Legendre symbol: $\chi_5(a) = \left(\frac{a}{5}\right)$. It can be shown that $\left(\frac{a}{p}\right) = a^{p-1/2} \bmod p$. And so,

$$\chi_5(a) = \begin{cases} 1 & \text{if } a^2 \equiv 1 \bmod 5 \\ -1 & \text{if } a^2 \equiv -1 \bmod 5. \end{cases}$$

After doing some work, one can prove the following facts for distinct rational primes p and q:

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \text{ and } \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4} \text{ and } \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}. \tag{2}$$

The middle equation from the list above is called the *law of quadratic reciprocity*; it was conjectured by Euler and proved by Gauss [13] in 1801. One can generalize the Legendre symbol to the *Jacobi symbol* $\left(\frac{a}{m}\right)$, where m is not necessarily a prime. The symbol is multiplicative and satisfies the same relations (2) as the Legendre symbol. These symbols have deep connections with quadratic and cyclotomic number fields. However, we simply will note that for the number field $K = \mathbb{Q}(\sqrt{5})$ and the ring of integers $\mathfrak{o}_K = \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$ that one obtains the following result about the splitting of rational prime ideals, (p), in $\mathfrak{o}_K$:

$$(p) \text{ splits in } \mathfrak{o}_K \Longleftrightarrow \left(\frac{a}{5}\right) = 1$$

$$(p) \text{ remains prime in } \mathfrak{o}_K \Longleftrightarrow \left(\frac{a}{5}\right) = -1$$

$$(p) \text{ ramifies in } \mathfrak{o}_K \Longleftrightarrow \left(\frac{a}{5}\right) = 0.$$

Applying the Euler product of $\zeta_K(s)$, we obtain

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}} = \prod_{p} \prod_{\mathfrak{p}|p} \frac{1}{1 - N\mathfrak{p}^{-s}},$$

where p is a rational prime and the notation $\mathfrak{p}|p$ means that $\mathfrak{p}$ lies over p. If $\chi_5(p) = 1$, then p splits. As such, there are two primes lying above p and both of their norms are equal to p. If $\chi_5(p) = -1$, then one prime lies above p and its norm is equal to p^2. The discriminant of a quadratic number field $\mathbb{Q}(\sqrt{d})$ is d if $d \equiv 1 \bmod 4$ and otherwise it is $4d$. Furthermore, it can be shown that a rational prime is ramified if and only if it divides the discriminant. Since $5 \equiv 1 \bmod 4$, then 5 is the only ramified prime and $\sqrt{5}$ is the prime lying above. The norm of $\sqrt{5}$ is 5. As such, we obtain

$$\begin{aligned}
\zeta_K(s) &= \frac{1}{1-5^{-s}} \cdot \prod_{\chi_5(p)=1} \frac{1}{(1-p^{-s})^2} \cdot \prod_{\chi_5(p)=-1} \frac{1}{1-p^{-2s}} \\
&= \frac{1}{1-5^{-s}} \cdot \prod_{\chi_5(p)=1} \frac{1}{(1-p^{-s})^2} \cdot \prod_{\chi_5(p)=-1} \frac{1}{(1+p^{-s})(1-p^{-s})} \\
&= \prod_{p} \frac{1}{1-p^{-s}} \prod_{p} \frac{1}{1-\chi_5(p)p^{-s}} \\
&= \zeta(s)L(s,\chi_5).
\end{aligned}$$

Analogous to the way that Dirichlet generalized the Riemann zeta-function, Hecke wanted to generalize the Dedekind zeta function to an L-function of a character on a number field. Hecke did so by creating a very specific multiplicative function on the ideals of $\mathfrak{o}_K$, called a Grössencharakter. Constructing a character on the ideals is difficult because an ideal need not have a principal generator. In addition, even if the ideal is principal, there may be infinitely many generators from which to choose. Is the correct analogue of $(\mathbb{Z}/m\mathbb{Z})^\times$ in K given by $(\mathfrak{o}_K/\mathfrak{m}\mathfrak{o}_K)^\times$ for an ideal $\mathfrak{m}$ in $\mathfrak{o}_K$? Hecke eventually figured out the correct construction; he defined the Hecke L-function, associated to a Grössencharakter χ, to be

$$L(s,\chi) := \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}.$$

Just as with the previous L-functions, this function has a Euler product and it is given by

$$L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}}.$$

The correct analogue of $(\mathbb{Z}/m\mathbb{Z})^\times$ is the Ray class group of module $\mathfrak{m}$. Hecke [14] proved that the function $L(s,\chi)$ has meromorphic continuation to the whole s-plane and satisfies a functional equation. However, he was not able to explicitly describe the factors that arise in the functional equation. In 1940, Chevalley [3] introduced the notion of the idele-class group. In 1950, John Tate [27] used the idele-class group to define a refined notion of a Grössencharakter; his definition eliminated many of the difficulties associated with constructing a Grössencharakter. As the Grössencharakter subsumes the definition of a Dirichlet character in the case $K = \mathbb{Q}$, the idele-class character does, too. In addition to creating this new type of character, Tate completely revolutionized the approach to proving the functional equation. He did so by using a "local to global" type of method.

The following thesis attempts to provide a working understanding of the underlying prerequisites for studying Tate's work. It is intended for use by first or second year graduate students, or even serious undergraduates, in their studies of Tate's thesis. The background knowledge and prerequisite information necessary for learning and studying Tate's thesis is dense and complicated; surmounting the information is a challenging task. It is a task, however, that is achievable and will prove useful, particularly for a graduate student who has a course in analysis, algebra or algebraic number theory. But beyond comprehending the background knowledge, lays the more difficult job of understanding how all of the details fit together and work in harmony within Tate's thesis.

The fields of research within mathematics are plentiful and it is useful to categorize them to allow students to navigate the mathematical web. However, to become an adept practicing mathematician, a student must understand how the various fields and subjects of mathematics connect to one another. Tate's thesis is a paramount piece of mathematical work to demonstrate the interconnectivity of mathematical fields, as it draws upon ideas from both algebra and analysis in order to achieve a very specific goal, namely a proof of the meromorphic continuation and functional equation of the Hecke L-function.

As specific as his goal was, Tate's work both inspired and led to the study of automorphic forms and representations and, more generally, to the Langland's Program, itself one of the most overarching theories in mathematics and number theory. As such, Tate's thesis represents an excellent starting point for a beginning graduate student in number theory. To facilitate an investigation into Tate's work, the following thesis will provide a thorough investigation into the necessary background knowledge, starting with an introduction to topological groups, the Haar measure, Pontryagin Duality, and the Fourier inversion formula in Chapter 1. In the section devoted to topological groups we will define and develop the theory needed for understanding these groups throughout the remainder of the thesis. A short introduction to the p-adic numbers and profinite groups is included in this section as well. In the section on Haar measures, a quick review of basic measure theory, a partial proof of the existence of the Haar measure for locally compact groups, and a proof of the uniqueness of the Haar measure in the abelian case is provided. Useful properties of the Haar measure also are discussed. In the final section of Chapter 1, an introduction will be provided to the Pontryagin dual of a topological group and its topology, the compact-open topology. Included also will be a statement of the Fourier inversion theorem for a locally compact group, although a proof will not be provided. Lastly, Pontryagin duality and its functorial properties will be discussed.

Provided in chapter two is a brief overview of global and local fields. The local fields section contains a short summary of how the existence and uniqueness of the Haar measure, explicitly the module of automorphism, can be used to classify locally compact fields. It is recommended highly that the reader consult other sources if he/she requires a more thorough introduction to global and local fields (locally compact non-discrete fields). Nonetheless, an effort is made to include at least statements of the theorems necessary for understanding the material found in Chapters 3 and 4.

The first part of Chapter 3 supplies an introduction to the restricted direct product and its topology; results about the quasi-characters, characters, the dual group, and the Haar measure of the restricted product also are proved. Furthermore, the integration and Fourier transform of functions defined on a restricted direct product is discussed in detail. The section on the restricted direct product is heavily reliant upon the material found in Chapter 1. In the second section of Chapter 3, there is found an introduction to the adeles and ideles of a number field K, denoted $\mathbb{A}_K$ and $\mathbb{I}_K$, respectively. The results concerning adeles and ideles that are required for understanding Tate's thesis are discussed in this section; the absolute value on the ideles is introduced and a proof of Artin's product formula also is provided. Furthermore, the idele-class group and the norm one idele-class group are introduced also.

The final chapter addresses the main subject - Tate's thesis. The chapter begins with a proof of the factorization of quasi-characters of local fields (i.e. elements of $\mathrm{Hom}(F^\times, \mathbb{C}^\times)$, where F is a local field). In the second section, a construction of non-trivial additive characters for Archimedean and non-Archimedean local fields of characteristic zero is provided. This section also includes a proof of the Pontryagin self-duality of local fields. The section that follows discusses the Haar measure on the multiplicative group $F^\times$, where F is a local field. The fourth section provides a proof that the Fourier transform is an automorphism of the vector space of Schwartz-Bruhat functions of a local field. The following section, section five, contains a proof of the meromorphic continuation and functional equation of the local zeta function. Exactly how the local epsilon factor is affected by a change of measure, additive character, and quasi-character is contained in a proposition found in the sixth section. In addition, local root numbers are discussed for non-Archimedean and Archimedean local fields. In the seventh section, the Schwartz-Bruhat functions on the adeles of a number field are introduced. The standard adelic character of a number field K is constructed from the standard additive local field characters previously discussed in section two. Further on in section seven is a proof of the self-duality of the adeles and a description of the dual group of a number field K, especially the case $K = \mathbb{Q}$, is provided in section seven. Also contained in this section is a proof that the Fourier transform is an automorphism of the adeles and is, moreover, an isometry of $L^2(\mathbb{A}_K)$. The rest of section seven is dedicated to a proof of the Poisson summation formula and its extension, the Riemann-Roch theorem. The seventh section is heavily dependent on the restricted direct product section found in Chapter 3. Introduced in the eighth section are the idele-class quasi-characters of number fields. In section eight, we will show how the idele-class character factors into a product of both a unitary character on the norm-one idele-class group and a character on $\mathbb{R}_+^\times$, the set of positive nonzero reals. An explanation of how the definition of an idele-class quasi-characters subsumes the definition of a Dirichlet characters for $K = \mathbb{Q}$ is provided. Further in this chapter, in the ninth section, is the establishment of the meromorphic continuation and functional equation of the global zeta. The main content of the tenth section is the proof of the meromorphic continuation and functional equation of the Hecke L-function attached to an idele-class quasi character. Also stated and proven is the functional equation for the Dedekind zeta function. In the final section, we compute the volume of the norm-one idele-class group and thereby provide the residues of the Hecke and Dedekind zeta function at $s = 1$.

CHAPTER 1
Topological Groups, the Haar Measure, and Pontryagin Duality

In this chapter, we primarily follow chapters 1, 2 and 3 of Ramakrishnan and Valenza's *Fourier Analysis on Number Fields* [24]. Also used was chapter 11 of Folland's book, *Real Analysis: Modern Techniques and Their Applications* for the section on the Haar measure and chapter 4 of Folland's [11] book, *A course in abstract harmonic analysis*, for the section on Pontryagin duality and the Fourier inversion formula. Many short proofs will be given. The reader will be referred to the source text when proofs are omitted for brevity.

1.1 Topological Groups and Fields

1.1.1 Definitions and Examples

Definition 1.1.1. A *topological group* is a group G with a topology such that the maps $(g, h) \mapsto gh$ from $G \times G$ (with the product topology) to G and $g \mapsto g^{-1}$ from G to G are continuous. The identity of G is denoted e.

For any set $S \subseteq G$, let $S^{-1} = \{x \in G : x^{-1} \in S\}$. If S is open in G, then S^{-1} is open since inversion is continuous. In what follows, a neighborhood $U \subseteq X$ of $x \in X$ is a subset of X of which x lies in the interior. Most importantly, U need not be open. The following proposition is an equivalent definition of a topological group.

Proposition 1.1.2. *A group G is a topological group if and only if for all $g, h \in G$ and any neighborhood W of gh^{-1}, there exists open neighborhoods U of g and V of h, such that $UV^{-1} \subseteq W$.*

Proof. ($\Rightarrow$) Assume that G is a topological group as defined above. Let W be a neighborhood of gh^{-1}. Without loss of generality, we make take W to be open, since, by definition of a neighborhood, there exists an open set W' containing gh^{-1} and contained in W. Let M be the multiplication map from $G \times G$ to G defined by $(g, h) \mapsto gh$. Since $M^{-1}(W)$ is open and $\{U \times V : U, V \text{ open in } G\}$ constitutes a basis of $G \times G$, then there exists open sets U and V'

of G, containing g and h^{-1}, respectively, such that $U \times V' \subseteq M^{-1}(W)$. Note that V'^{-1} is an open set containing h since inversion is continuous. Define $V = V'^{-1}$ so that $V^{-1} = V'$ is an open neighborhood of h^{-1}. Therefore, there exists an open neighborhood U of g and V of h such that $UV^{-1} \subseteq W$.

($\Leftarrow$) Let W be an open set in G. Let $g^{-1} \in W^{-1}$. Since $g = e * g \in W$, then there exists open neighborhoods U and V of e and g^{-1}, respectively, such that $UV^{-1} \subset W$. Since U is a neighborhood of the identity, then $V^{-1} \subseteq UV^{-1} \subseteq W$. So, V is an open neighborhood containing g^{-1} that is contained in W^{-1}. This shows that inversion is continuous. Again, let W be an open set in G. Let $(g, g') \in M^{-1}(W)$. Then, $gg' \in W$; hence, there exists open neighborhoods U of g and V^{-1} of g'^{-1} such that $UV \subset M^{-1}(W)$. Note that if V^{-1} is an open neighborhood of g'^{-1}, then V is an open neighborhood of g' by what was shown above. In other words, $U \times V \subset M^{-1}(W)$. Therefore, $M^{-1}(W)$ is open in $G \times G$, since every point is an interior point. □

If we impose the discrete topology on the group G, which we will often do, then G is obviously a topological group. It is also clear then that a topology is *translation invariant*. That is, if we consider left or right translation by a fixed element, which is a homeomorphism from G to G, then $\forall g \in G$ and $U \subseteq G$ the following are equivalent:

(i) U is open.

(ii) gU is open.

(iii) Ug is open.

Definition 1.1.3. Let X be a topological space and let S be a subset of Homeo(X), the set of all homeomorphisms from X to itself. Then X is said to be a *homogeneous* space under S if $\forall x, y \in X$, there exists $f \in S$ such that $f(x) = y$. If the subset S is the full set of homeomorphisms, then we simply say that X is a *homogeneous space.*

Proposition 1.1.4. *Every topological group is translation invariant and homogeneous under itself ($S = G$). Furthermore, a local neighborhood base at the identity determines a local base at all $g \in G$.*

Proof. For all $g \in G$ let $L_g : G \to G \times G$ be the continuous map defined by $h \mapsto (g, h)$. We know that $M : G \times G \to G$ defined by $(g, h) \mapsto gh$ is continuous, since G is a topological group. Therefore, $M \circ L_g$ is a continuous map from G to G and is defined by $h \mapsto gh$. Furthermore, $M \circ L_{g^{-1}}$ is a continuous map and is defined by $h \mapsto g^{-1}h$. Since $M \circ L_g \circ M \circ L_{g^{-1}}$ is the identity, then $M \circ L_g$ is a homeomorphism. Consequently, left multiplication by g is a homeomorphism. More specifically, left translation by a fixed element induces an injection of G into $\mathrm{Homeo}(G) \cap \mathrm{Aut}(G)$. For all $g, h \in G$, left translation by gh^{-1} sends h to g. This shows that G is homogeneous under itself.

Let $V(g)$ be a neighborhood filter of g in G, a collection of all neighborhoods of g, and $\{E_i\}_{i \in I}$ be a local base of $e \in G$. Let $V \in V(g)$. Then $g^{-1}V$ is a neighborhood of e, and so there exists $E^* \in \{E_i\}_{i \in I}$ such that $E^* \subset g^{-1}V \Rightarrow gE^* \subset V$. Therefore, $\{gE_i\}_{i \in I}$ is a local neighborhood base of g. Thus, a local neighborhood base at the identity $e \in G$ determines a local base at all $g \in G$ and, consequently, G. This is one of the most important properties of topological groups. □

Some examples of topological groups will follow. It is advantageous to use the topological structure to uncover facts about the algebraic structure, and vice versa. Many of the traditionally difficult theorems of algebraic number theory are proven easily using the adelic and idelic (topological) approach to algebraic number theory. These advantages enabled Tate to apply abelian harmonic analysis to establish the functional equation of the Hecke L-Functions. The following examples should serve as a stimulus to continue reading and, more importantly, to begin to think about topological groups. We provide the examples before the abstract theory so that the reader may have examples in mind before launching into the general theory.

Examples 1.1.5.

(i) Any group G endowed with the discrete topology is a topological group.

(ii) Any group G endowed with the trivial topology is a topological group.

(iii) Every subgroup of a topological group, endowed with the subspace topology, is a topological group.

(iv) With the Euclidean topology, $(\mathbb{R}, +)$ is a topological group. Consider first the addition map $(a, b) \mapsto a + b$. For all $\epsilon > 0$ and $a, b, c, d \in \mathbb{R}$, $|(a + b) - (c + d)| < \epsilon$ whenever $|a - c| < \epsilon/2$ and $|b - c| \leq \epsilon/2$. Second, consider the inversion map $a \mapsto -a$. For all $\epsilon > 0$

and $a, b \in \mathbb{R}$, we see that $|-a-(-b)| < \epsilon$ whenever $|b-a| \leq \epsilon$. From this example, we can see exactly when a metrizable group is a topological group. The result is stated in the following proposition.

(v) With the Euclidean topology, $(\mathbb{R}^{\times}, \cdot)$ is a topological group. Note that $\mathbb{R}^{\times} = \mathbb{R} - \{0\}$. Consider the multiplicative function $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $(x, y) \mapsto xy$. Pick $(a, b) \in \mathbb{R} \times \mathbb{R}$. Then

$$|xy - ab| = |(x-a)y + (y-b)a| \leq |x-a||y| + |y-b||a|.$$

For a fixed $\epsilon > 0$, consider the neighborhood ball $N_{\delta}(a, b) \subset \mathbb{R}^2$ of (a, b), where $\delta = \min(1, \epsilon/(2(1+|b|)), \epsilon/(2(1+|a|)))$. Then

$$|xy - ab| \leq |x-a|(|y-b| + |b|) + |y-b||a| < \frac{\epsilon}{2(1+|b|)}(1+|b|) + \frac{\epsilon}{2(1+|a|)} < \epsilon$$

whenever $(x, y) \in N_{\delta}(a, b)$. Since (a, b) and $\epsilon > 0$ were arbitrarily chosen, then the multiplicative function is continuous. For all $\epsilon > 0$ and $a, b \in \mathbb{R}^{\times}$ we have $|a^{-1} - b^{-1}| = |(b-a)(ab)^{-1}| < \epsilon$ whenever $|b-a| < \epsilon/|ab|$ ($a \neq 0$ and $b \neq 0$). Therefore, inversion is continuous.

(vi) The groups $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, with the subspace topology induced by the Euclidean topology on $\mathbb{R}$ are topological groups. Also, the group $(\mathbb{Q}^{*}, \cdot)$ is a topological group with the subspace topology induced by the Euclidean topology on $\mathbb{R}^{\times}$.

(vii) The groups $(\mathbb{C}, +)$ and $(\mathbb{C}^{\times}, \cdot)$ with the complex norm topology are topological groups.

(viii) The groups $\mathbb{R}^m = \prod_{i=1}^{m} \mathbb{R}$ with $m \in \mathbb{N}$ are topological groups with vector addition and the product topology, or equivalently, the Euclidean topology.

As a generalization of the above examples, the following theorem provides necessary and sufficient conditions for a group G with a topology induced by a metric to be a topological group.

Proposition 1.1.6. *Let G be a group and assume the topology on G is induced from a metric, d. Then G is a topological group if and only if the following two conditions hold:*

(i) For all $\epsilon > 0$ and $g_1, g_2 \in G$ there exists $\delta > 0$ such that $d(g_1 g_2, h_1 h_2) < \epsilon$ whenever $d(g_1, h_1) < \delta$ and $d(g_1, h_2) < \delta$.

(ii) For all $\epsilon > 0$ and $g \in G$ there exists $\delta > 0$ such that $d(g^{-1}, h^{-1}) < \epsilon$ whenever $d(g, h) < \delta$.

Proof. The proof is trivial. □

We also have the following general theorem about the direct product of two topological groups.

Proposition 1.1.7. *Let G_1 and G_2 be topological groups. The direct product $G_1 \times G_2$ endowed with the product topology and componentwise group operation is a topological group.*

Proof. Let W be a neighborhood of

$$(g_1, g_2) \cdot (h_1, h_2)^{-1} = (g_1, g_2) \cdot (h_1^{-1}, h_2^{-1}) = (g_1 h_1^{-1}, g_2, h_2^{-1}).$$

Since

$$\{W_1 \times W_2 : W_1 \text{ open in } G_1 \text{ and } W_2 \text{ open in } G_2\}$$

constitutes a basis of $G_1 \times G_2$, then there exists open neighborhoods $W_1 \subseteq G_1$ and $W_2 \subseteq G_2$ of $g_1 h_1^{-1}$ and $g_2 h_2^{-1}$, respectively, and $W_1 \times W_2 \subseteq W$. Since G_1 and G_2 are topological groups, then there exists open sets U_1, V_1, U_2, V_2 of g_1, h_1, g_2, h_2, respectively, such that $U_1 V_1^{-1} \subseteq W_1$ and $U_2 V_2^{-1} \subseteq W_2$. In other words, $U_1 \times U_2$ is a neighborhood of (g_1, g_2) and $V_1 \times V_2$ is a neighborhood of (h_1, h_2) such that $(U_1 \times U_2)(V_1 \times V_2)^{-1} \subseteq (W_1 \times W_2) \subseteq W$. Therefore, $G_1 \times G_2$ is a topological group. □

Probably the most important examples of topological groups are Lie groups. A Lie group is a group that is also a finite-dimensional differentiable manifold, in which the group operations are smooth maps. Since smooth implies continuous, Lie groups are examples of topological groups. However, not all topological groups are Lie groups. For example, $\mathbb{Q}$ endowed with the subspace topology inherited from the Euclidean topology on $\mathbb{R}$, is a non-discrete countable topological group that is not a Lie group.

Examples 1.1.8.

(i) Recall that $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ are the multiplicative group of invertible $n \times n$ matrices with entries in $\mathbb{R}$ and $\mathbb{C}$, respectively. So, we have that $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{M}_{(n,n)}(\mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C}) \cong \mathrm{GL}(2, \mathbb{R}) \subset \mathrm{M}_{(2n,2n)}(\mathbb{R})$, where $\mathrm{M}_{(n,n)}(\mathbb{R})$ is the set of $n \times n$ matrices.

It readily is apparent that $\mathrm{M}_{(n,n)}(\mathbb{R})$ and $\mathrm{M}_{(2n,2n)}(\mathbb{R})$ are differential manifolds via their identification with $\mathbb{R}^{n^2}$ and $\mathbb{R}^{4n^2}$, respectively. As such, both $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ can be given the Euclidean topology from their identification with vectors in $\mathbb{R}^{n^2}$ and $\mathbb{R}^{4n^2}$. Consider $\det : \mathrm{M}_{(n,n)}(\mathbb{R}) \to \mathbb{R}$, the determinant of an $n \times n$ matrix, defined by

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Since the determinant of an $n \times n$ matrix is a polynomial in the matrix coefficients, then it is a continuous function , and hence $\det^{-1}(0)$ is closed. Using the fact that $\mathrm{GL}(n,\mathbb{R}) = \mathrm{M}_{(n,n)} - \det^{-1}(0)$ and that $\mathrm{GL}(n,\mathbb{C}) = \mathrm{M}_{(2n,2n)} - \det^{-1}(0)$, we can see that both general linear groups are open. Since open subsets of differentiable manifolds are themselves differential manifolds (restriction of charts to open subset), then $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are real differential manifolds. Multiplication of matrices is a polynomial in matrix coefficients , and hence is a smooth map from $\mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ (same for $\mathbb{C}$). By Cramer's rule, we can see that every entry of the matrix A^{-1} is a polynomial in matrix coefficients. Therefore, matrix inversion is continuous. Consequently, $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are real Lie groups and, thus, topological groups.

(ii) All subgroups of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ are topological groups. However, by Cartan's Theorem, only closed subgroups of Lie groups are guaranteed to be Lie groups. This does not preclude open subgroups of Lie groups from being Lie groups. It can be shown that both $\mathrm{O}(n,\mathbb{R})$ and $\mathrm{O}(n,\mathbb{C})$, the subgroups of orthogonal matrices ($AA^T = I$), are real Lie groups. In addition, $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$, the special linear groups ($\det A = 1$), are real Lie groups. Also, $\mathrm{SO}(n,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{C})$, the subgroups of special orthogonal matrices, are real Lie groups. Lastly, $\mathrm{U}(n,\mathbb{C})$, the complex unitary matrices ($AA^* = I$), is a real lie group. This list is not exhaustive.

(iii) There are complex Lie groups and, of course, $\mathrm{GL}(n,\mathbb{C})$ is the prototypical example. A complex Lie group is a complex manifold whose group operations are analytic. $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$ are complex Lie groups. However, $\mathrm{U}(n,\mathbb{C})$ is not a complex lie group.

1.1.2 **Some Theory of Topological Groups**

Now that we have provided a sufficient number of examples of topological groups, we develop some general theory of topological groups. A subset of S of G is *symmetric* if $S^{-1} = S$.

Proposition 1.1.9. *Let G be a topological group. Then the following assertions hold:*

(i) Every neighborhood U of the identity contains a neighborhood V of the identity such that $VV \subseteq U$.

(ii) Every neighborhood U of the identity contains a symmetric neighborhood V of the identity.

(iii) If H is a subgroup of G, so is its closure.

(iv) Every open subgroup of G is also closed.

(v) If K_1 and K_2 are compact subsets of G, so is K_1K_2.

Proof.

(i) If we can prove this statement for U is open, then we are done because $\text{int}(U) \subseteq U$. There exists open subsets V_1 and V_2 of U such that $(e, e) \in V_1 \times V_2$ and $V_1V_2 \subset U$. Let $V = V_1 \cap V_2$. Therefore, V is neighborhood of the identity such that $VV \subset U$.

(ii) It is clear that $U \cap U^{-1}$ is a symmetric neighborhood of e that is contained in U.

(iii) Consider $g, h \in \overline{H}$. So, $g = \lim g_\alpha$ and $h = \lim h_\alpha$ where g_α and h_α are nets in H. Since G is a topological group, then gh is the limit of the convergent net $h_\alpha g_\alpha$. Therefore, $gh \in \overline{H}$.

(iv) Since H is a subgroup, then g is a disjoint union of cosets of H–$G = \coprod gHG$. We know that in the disjoint union, one of the $g's$ is the identity. And so,

$$H = G - \coprod_{g \neq e} gH.$$

If H is open, then so are the gH in the union, and hence so is the union. Thus, H is closed because it is the complement of an open set.

(v) Under the continuous multiplicative map, K_1K_2 is the image of $K_1 \times K_2$. Since K_1 and K_2 are compact, then $K_1 \times K_2$ is compact in the product topology. Therefore, K_1K_2 is compact because it is the image of a compact set under a continuous mapping

$\square$

Corollary 1.1.10. *From 1 and 2 above, every neighborhood U of the identify contains a symmetric neighborhood V such that $VV \subseteq U$.*

Let f be a function on a group G. We define left and right translates of f by $L_h f(g) = f(h^{-1}g)$ and $R_h f(g) = f(gh)$, respectively. If f is a continuous function from G to $\mathbb{R}$ or $\mathbb{C}$, then we say that f is *left uniformly continuous* if, for all $\epsilon > 0$, there exists a neighborhood V of the identity such that

$$||L_h f - f||_u < \epsilon \;\; \forall h \in V$$

where $||\ ||_u$ is the uniform, or supremum, norm. And *right uniform continuity* is defined similarly. Let $C_c(G)$ be the space of continuous functions on G with compact support.

Proposition 1.1.11. *Let G be a topological group. Every function $f \in C_c(G)$ is both left and right uniformly continuous.*

Proof. Let $K = \text{supp}(f)$ and pick $\epsilon > 0$. Since f is continuous, then for all $g \in K$, there exists an open neighborhood, U_g, of the identity such that for all $h \in U_g$

$$|f(gh) - f(g)| < \epsilon.$$

Alternatively, $|f(g') - f(g)| < \epsilon$ if $g^{-1}g' \in U_g$. By the corollary above we know that there exists a symmetric neighborhood, V_g, of the identity such that $V_g V_g \subseteq U_g$. Consider the cover $\{gV_g\}_{g\in K}$ of K, which we can reduce to a finite subcover, $\{g_i V_{g_i}\}_{i=1,2,\ldots n}$, by compactness. Let $V = \cap_{i=1}^n V_{g_i}$. Let $h \in V$ and $g \in K$. If $g \in K$, then there exists an $i \in \{1, \ldots, n\}$ such that $g \in g_i V_{g_i}$. From the triangle inequality we get

$$|f(gh) - f(g)| \leq |f(gh) - f(g_i)| + |f(g_i) - f(g)|.$$

Both terms on the right are bounded by ϵ because $g_i^{-1}g \in V_{g_i} \subseteq V_{g_i}V_{g_i} \subseteq U_{g_i}$ and $g_i^{-1}gh \in V_{g_i}V_{g_i} \subseteq U_{g_i}$. Thus, f is right uniformly continuous in K. If g is not in K, then we need to bound $|f(gh)|$. If $f(gh) \neq 0$, then $gh \in \text{supp}(f)$, and hence $gh \in g_j V_j$ for some $j \in \{1, \ldots, n\}$. Therefore, $|f(gh) - f(g_j)| < \epsilon$. Also, $g_j^{-1}g = g_j^{-1}(gh)h^{-1} \in V_{g_j}V_{g_j} \subseteq U_{g_j}$, so $|f(g_j)| < \epsilon$. Finally, $|f(gh) - f(g)| \leq 2\epsilon$. This completes the proof. $\square$

Proposition 1.1.12. *Let G be a topological group. Then the following assertions are equivalent:*

(i) G is T_1.

(ii) G is Hausdorff.

(iii) The identity e is closed in G.

(iv) Every point of G is closed in G.

Proof. (i)⇒(ii) If G is T_1, then for all $g, h \in G$, $g \neq h$, there exists an open neighborhood U of e such that $gh^{-1} \notin U$. From Corollary 1.1.10, there exists V, a symmetric open neighborhood of the identity, such that $VV \subseteq U$. Consider the open neighborhoods of g and h, Vg and Vh, respectively. Suppose there exists $g' \in Vg \cap Vh$. Then $g' = v_1 g$ and $g' = v_2 h$, which implies

$$v_1 g = v_2 h \Rightarrow gh^{-1} = v_1^{-1} v_2 \in V^{-1}V = VV \subseteq U.$$

Contradiction. Therefore, G is Hausdorff.

(ii)⇒(iii) Suppose not the above, then $G - \{e\}$ would not be open. So, there exists some point $g \in G - \{e\}$ such that every open neighborhood U of g intersects e. Contradiction.

(iii)⇒(iv) Let $x \in G$. Left multiplication by x^{-1} is a homeomorphism from x to e. So, if e is closed, then x is closed.

(iv)⇒(i) Let $g, h \in G$. Since $\{g\}$ is closed, then $G - \{g\}$ is open, so there exists an open neighborhood $U \subset G - \{g\}$ of h. We similarly can find an open neighborhood $V \subset G - \{h\}$ of g.

□

Let H be a subgroup of G, a topological group. Let G/H, the set of left cosets, have the quotient topology. The quotient topology is the finest topology such that $\rho : g \mapsto gH$ is continuous. In other words, U is open in G/H if and only if $\rho^{-1}(U)$ is open in G. Under coset multiplication, G/H is a group if and only if H is a normal subgroup of G. In this case, G/H is a topological group with respect to the quotient topology.

Proposition 1.1.13. *Let G be a topological group and let H be a subgroup of G. Then the following assertions hold:*

(i) The quotient space G/H is homogeneous under G.

(ii) The canonical projection $\rho : G \to G/H$ is an open map.

(iii) The quotient space G/H is T_1 if and only if H is closed.

(iv) The quotient space G/H is discrete if and only if H is open. Moreover, if G is compact, then H is open if and only if G/H is finite.

(v) If H is normal in G, then G/H is a topological group with respect to coset multiplication and the quotient topology.

(vi) Let H be the closure of $\{e\}$ in G. Then H is normal in G, and the quotient group G/H is Hausdorff with respect to the quotient topology.

Proof.

(i) It is clear that any element $g \in G$ acts on G/H by left translation. Following this, we need to show that G, acting by left translation, is a subset of Homeo(G/H). Since the inverse map has a similar form, then it suffices to show that the left translation map is open in the quotient topology. Let $\overline{U}$ be an open subset of G/H. By the definition of the quotient topology, $U = \rho^{-1}(\overline{U})$ is open in G. Also, $\rho^{-1}(g\overline{U}) = g\rho^{-1}(\overline{U}) = gU$ is open in G. Therefore, $g\overline{U}$ is open in G/H, and thus left translation by G is an open map. Then, for any $xH, yH \in G/H$, $x \neq y$, left translation by yx^{-1} sends xH to yH. This shows that G/H is homogeneous under H.

(ii) Let V be open in G. We need to show that $\rho(V)$ is open in G/H; i.e. that $\rho^{-1}(\rho(V))$ is open in G. We know that $\rho^{-1}(\rho(V)) = VH$. Let $x \in VH$. So, $x = vh$ for some $v \in V$ and $h \in H$. Since V is open, there exists U, an open neighborhood of v in V. Then Uh is an open subset of VH that contains x. Thus, VH is open, and consequently, ρ.

(iii) As we saw in the above proposition, G/H is T_1 if and only if every point of G/H is closed. We know that $\rho^{-1}(H) = H$, so each coset of G/H is closed if and only if each coset is a closed subset of G. From the homogeneity of G is G/H, we know that this is true if and only if H is a closed subset of G.

(iv) From part (ii) we know H is an open subset of G if and only if H is an open coset of G/H. From homogeneity, H is an open coset of G/H if and only if every point of G/H is open (G/H discrete). Assume that G is compact. Since ρ is continuous, then G/H is compact. By what was just shown, H is open if and only if G/H is discrete, and G/H

is discrete if and only if G/H is finite since G is compact. Note that G/H finite implies discreteness only by convention.

(v) If H is normal, then G/H is a group. By part (ii) we have that ρ is an open map. We also have

$$\begin{tikzcd} G \arrow[r, "T_g"] \arrow[d, "\rho"'] & G \arrow[d, "\rho"'] \\ G/H \arrow[r, "T_{\rho(g)}"] & G/H \end{tikzcd}$$

and

$$\begin{tikzcd} G \arrow[r, "inv(g \mapsto g^{-1})"] \arrow[d, "\rho"'] & G \arrow[d, "\rho"'] \\ G/H \arrow[r, "inv"] & G/H. \end{tikzcd}$$

Therefore, G/H is a topological group.

(vi) Since $\{e\}$ is a subgroup, then so is its closure, denoted H, by Proposition 1.1.9. It is also the smallest closed subgroup containing $\{e\}$. Consider gHg^{-1}. Note that $e \in gHg^{-1}$ and that gHg^{-1} is closed. And gHg^{-1} has the same number of elements of H or less. Therefore, $gHg^{-1} = H$, and H is normal. By part (iii) we know that G/H is T_1, and by part (v) we know that G/H is a topological group with respect to coset multiplication and the quotient topology. Therefore, G/H is Hausdorff. Thus, we see that every topological group G projects onto a Hausdorff topological group.

□

Proposition 1.1.14. *Let G° be the connected component of the identity of some topological group G. That is, G° is the maximal connected subset of the identity, e, in G. Then G° is a normal subgroup of G. Moreover, the quotient space G/G° is totally disconnected (every point of G is its own connected component).*

Proof. Let $x \in G^\circ$. Then $x^{-1}G^\circ$ is connected, since G is homogeneous. Also, $e \in x^{-1}G^\circ$. Since G° is the maximal connected component of e, then $x^{-1}G^\circ \subseteq G^\circ$. This shows that G° is closed under inversion. Let $y \in G^\circ$. We see that xG° is connected and contains the identity since $x^{-1} \in G^\circ$. Consequently, $xy \in G^\circ$. Similarly, $yG^\circ y^{-1} \subseteq G^\circ$, proving that G^0 is a normal subgroup of G. By the previous proposition, G/G° is a topological group with respect to the quotient operation and quotient topology. From the homogeneity of G, the elements of G/G° are exactly the connected components of G. Suppose F is a connected component of G/G° that contains more than one element. Consider $\rho^{-1}(F)$ where ρ is the canonical projection $\rho : G \to G/G^\circ$, which is continuous by definition of the quotient topology. Let $h \in \rho^{-1}(F)$. Let H be the connected component of h. $\square$

Proposition 1.1.15. *Let G be a Hausdorff topological group. Then:*

(i) The product of a closed subset F and a compact subset K is closed.

(ii) If H is a compact subgroup of G, then $\rho : G \to G/H$ is a closed map.

Proof.

(i) Let $z \in \overline{FK}$. So, z is the limit of a convergent net $\{f_j k_j\}_{j\in I} \subset FK$, where $\{f_j\}_{j\in I} \in F$ and $\{k_j\}_{j\in I} \in K$. Since K is compact, there exists a convergent subnet $\{\kappa_j\} \in K$ that converges to a point $k \in K$. Note that since $\{f_j k_j\}$ converges, then we can replace $\{f_j k_j\}$ with $\{f_j \kappa_j\}$. Consider U an open neighborhood of e in G. As shown above, there exists an open neighborhood V of e such that $VV \subseteq U$. The nets $\{z^{-1} f_j \kappa_j\}$ and $\{\kappa_j^{-1} k\}$ both converge to e and thus lie in V. Since $VV \subseteq U$, then the product of the nets,$\{z^{-1} f_j k\}$, eventually lie in U. Consequently, $\lim f_j = zk^{-1}$ and $z = zk^{-1}k \in FK$.

(ii) The first part of the proof mimics part (ii) of the previous proposition. Let C be closed in G. Then we must show that $\rho(C)$ is closed in G/H. However, under the quotient topology, this reduces to showing that $\rho^{-1}(\rho(C)) = CH$ is closed in G. By part (i), CH is closed in G since H is compact and C is closed.

$\square$

Remark 1.1.16. Compactness is necessary in the above theorem. Let $G = \mathbb{R}^2$ and H be the y axis. Then $G/H \cong \mathbb{R}$ and, under this identification, $\rho(x, y) = x$. If we let C be the graph of $y = 1/x$, obviously closed, then the projection onto the x-axis is $\mathbb{R}^\times$, which is open, but not closed.

1.1.3 Locally Compact Groups and Fields

Before discussing locally compact groups, we will quickly define the notion of topological ring and field, since locally compact fields are of the utmost importance in Tate's thesis.

Definition 1.1.17. A ring R with operations " $+$ " and " $\cdot$ " such that $(R, +)$ is a topological group and such that $M : R \times R \to \mathbb{R}$ defined by $(r, s) \mapsto r \cdot s$ is continuous is called a *topological ring.* A field F with operations " $+$ " and " $\cdot$ " such that $(F, +)$ and $(F^\times, \cdot)$ are topological groups is called a *topological field.*

Definition 1.1.18. A topological space is *locally compact* if every point of the space admits a compact neighborhood. A topological group G that is both locally compact and Hausdorff is called a *locally compact group.* A topological field F that is both locally compact and Hausdorff is called a *locally compact field.*

Examples 1.1.19.

(i) The topological fields $\mathbb{R}$, $\mathbb{C}$ with the Euclidean topology are locally compact fields. Furthermore, they are both non-discrete. A non-discrete locally compact field is called a *local field.*

(ii) The topological field $\mathbb{Q}$ is a discrete locally compact field.

(iii) The topological group $\mathbb{Z}^d$ is a discrete locally compact group.

(iv) The topological group S^1 is a locally compact group with the subspace topology induced by the Euclidean topology.

Proposition 1.1.20. *Any locally compact subset of a Hausdorff space is the set theoretic difference of two closed sets or, equivalently, is the intersection of an open and closed set . Consequently, any locally compact dense subset of a Hausdorff space is open.*

Proof. Let S be a compact subset of a Hausdorff space X. We can find an open neighborhood U in S of $s \in S$ such that $Cl_s U$ is compact in S. Since U is open in S, then there exists V, open in X, such that $U = V \cap S$. Then $Cl_X(V \cap S) \cap S = Cl_X U \cap S = Cl_s U$ is compact. So, $Cl_x(S \cap V) \cap S$ is closed in X and contains $S \cap V$, and thus $Cl_X(S \cap V)$. Therefore, $Cl_X(S \cap V) \subset S \Rightarrow Cl_X(S) \cap V \subset S$. Hence, $Cl_X S \cap V$ is a neighborhood of s in $Cl_X S$, which is contained in S. Therefore, S is open in $Cl_X(S)$. Any open set in $Cl_X(S)$ has the form $B \cap CL_X(S)$ where B is open in X. Therefore, $S = S \cap Cl_X(S) = B \cap Cl_X(S)$, where B is open in X. Also, $S = (B^c)^c \cap Cl_X(S) = Cl_X(S) - B^c$. If S is dense and locally compact, then, as shown, $S = O \cap C$, where O and C are, respectively, open and closed in X. Since $S = A - T$ where A, T are closed in X, then pick $x \in A^c$, which is open. Let U be an open neighborhood of x in A^c. This, however, contradicts the density of S. Therefore, $A^c = \emptyset$ and $A = X$, which implies $S = X - T$. Consequently, S is open. $\square$

Proposition 1.1.21. *Let G be a Hausdorff topological group. Then a subgroup H of G is locally compact (in the subspace topology) if and only if H is closed. In particular, every discrete subgroup of G is closed.*

Proof. Let K be compact neighborhood of e in H. Since H is also Hausdorff, then K is closed in H. Being closed in H implies the existence of C, a closed neighborhood of e in G, such that $K = C \cap H$. Also, $C \cap H$ is compact in G and hence closed. There exists a neighborhood V of the identity such that $VV \subseteq C$ by Proposition refTopGroup. We know that $\overline{H}$ is a subgroup of G by Proposition 1.1.9. So, if $x \in \overline{H}$, then every neighborhood of x^{-1} intersects H non-trivially. Hence, there exists $y \in Vx^{-1} \cap H$. If we can show that $yx \in C \cap H$, then both y and yx lie in H and, consequently, $x \in H$. Since $C \cap H$ is closed, then we only need to show that every neighborhood of yx meets H. If W is a neighborhood of yx, then clearly $y^{-1}W$ contains x. So, $y^{-1}W \cap xV$ is a neighborhood of x. Since $x \in \overline{H}$, then there exists $z \in y^{-1}W \cap xV \cap H$. Now, the product yz lies in W and H and $y \in Vx^{-1}$ and $z \in xV$. Therefore, $yz \in VV \subseteq C$ and so $yz \in W \cap (C \cap H)$. Finally, every neighborhood of yx meets H, which implies y and yx lie in H, which then implies $x \in H$. This proves $\overline{H} \subseteq H$, consequently, $\overline{H} = H$. Let H be a closed subgroup of G. Let $x \in H$. Let K be a compact neighborhood of x in G. Then $K \cap H$ is a compact neighborhood of x in H. This completes the proof. $\square$

Proposition 1.1.22. *Let $\{G_i\}_i \in I$ be a set of locally compact groups such that G_i is compact for all but finitely many $i \in I$. Then*

$$\prod_{i \in I} G_i$$

is locally compact.

Proof. Let $S = \{i \in I : G_i \text{ not compact}\}$. By hypothesis, this set is finite. By Tychonoff's theorem, the possibly infinite product $\prod_{nu \notin S} G_i$ is compact. See Chapter 5 of Munkres [22] for a proof of Tychonoff's theorem. Furthermore, since G_i, $i \in S$ is locally compact, then the finite product $\prod_{i \in S} G_i$ is locally compact. Indeed, let $(g_i)_{i \in S}$ be a point in $\prod_{i \in S} G_i$. Since

G_i, $i \in S$ is locally compact, then for all $i \in S$, there exists a locally compact neighborhood $K_i \subset G_i$ of g_i. Let $K = \prod_{i \in S} K_i$. Then K is compact neighborhood of $(x_i)_{i \in S}$ in the direct product $\prod_{i \in S} G_i$ since a product of finitely many compact sets is compact. As such, the full product $\prod_{i \in I} G_i$ is locally compact. □

Proposition 1.1.23. *If G is locally compact group and H is a closed subgroup, then G/H is a locally compact group.*

Proof. If K is a compact neighborhood of 1 in G, then $\rho_H(xK)$ is a compact neighborhood of $\rho(x)$ in G/H since ρ_H is continuous. Part (iii) of Proposition 1.1.13 and Proposition 1.1.13 show that G/H is Hausdorff. This completes the proof. □

1.1.4 **P-adic Numbers and Topology**

We will now examine and investigate the p-adic numbers, non-discrete topological fields, which are important examples to have in mind when working with the completions of global fields at primes. Many of the integrals in Tate's Thesis involve integration over locally compact non-Archimedean fields, of which the p-adic numbers are an example. The completion of a number field at a finite place is a finite extension of the p-adic numbers. Such a completion is also a non-Archimedean field. For this reason, it is important to develop an intuition and understanding of the p-adic numbers and their topology. Although a quick summary of the p-adic numbers will be given below, a more detailed exposition can be found in Govêa's book *An introduction to the p-adic numbers.* We will follow Vladimirov and Zelenov's [28] text, *p-Adic Analysis and Mathematical Physics.*

For p, a rational prime, and $x \in \mathbb{Q}$, define $|\cdot|_p$ by

$$|0|_p = 0 \quad |x|_p = p^{-\nu_p},$$

where $\nu_p = \nu_p(x)$ is defined from the representation

$$x = p^{\nu_p}\frac{m}{n}$$

with $(m, p) = (n, p) = 1$.

Proposition 1.1.24. $|\cdot|_p$ *satisfies the following characteristics:*

(i) $|x|_p \geq 0, \;\; |x|_p = 0 \Leftrightarrow x = 0.$

(ii) $|xy|_p = |x|_p|y|_p.$

(iii) $|x+y|_p \leq \max(|x|_p, |y|_p) \leq |x|_p + |y|_p$ *and* $|x+y|_p = \max(|x|_p, |y|_p)$ *when* $|x|_p \neq |y|_p$.

Proof. Let us prove (c) because (a) and (b) are clear. Let $x = p^{\gamma}\frac{m}{m}$ and $y = p^{\gamma'}\frac{m'}{n'}$ be the given where m, n, m', n' are not divisible by p. Thus, $|x|_p = p^{-\gamma}$ and $|y|_p = p^{-\gamma'}$. Without loss of generality let $\gamma = \min(\gamma, \gamma')$. Then

$$x + y = p^{\gamma}\frac{m}{n} + p^{\gamma'}\frac{m'}{n'} = p^{\gamma}\frac{mn' + nm'p^{\gamma'-\gamma}}{nn'}.$$

The integer nn' is not divisible by p, but it is possible that the numerator, $mn' + nm'p^{\gamma'-\gamma}$, is divisible by p. Consequently, $\gamma(x+y) \geq \gamma = \min(\gamma, \gamma')$. This implies that

$$|x+y|_p = p^{-\gamma(x+y)} \leq p^{-\min(\gamma,\gamma')} = \max(p^{-\gamma}, p^{-\gamma'}) = \max(|x|_p, |y|_p).$$

Suppose that $\gamma' > \gamma$, then $mn' + nm'p^{\gamma'-\gamma}$ is not divisible by p. Therefore, $\gamma(x+y) = \gamma$ and $|x+y|_p = \max(|x|_p, |y|_p)$. For $p = 2$ and $|x|_2 = |y|_2$, we see that nn' is odd and that $mn' + m'n$ must be even because an odd number plus an odd number is even. So, $\gamma(x+y) \geq \gamma + 1$, which implies that $|x+y|_2 \leq 1/2|x|_2$. □

From these three properties, we see that $|\cdot|_p$ is a norm and takes the countable set of values $p^{\gamma}, \; \gamma \in \mathbb{Z}$. We call $|x|_p$ the *p-adic norm.*

Definition 1.1.25. A field F with a norm $|\cdot|_F$ is said to be Archimedean if, for any nonzero $x \in F$, there exists an $n \in \mathbb{N}$ such that

$$|x + \cdots + x|_F > 1,$$

where x is being summed n times.

Definition 1.1.26. The field $\mathbb{Q}_p$ of *p-adic numbers* is defined as the completion of $\mathbb{Q}$ with respect to the p-adic norm. The unit disc, $\{\alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1\}$, denoted $\mathbb{Z}_p$ is called the *p-adic integers.* Note that $\mathbb{Q}_p$ is a non-Archimedean field by part (iii) of the previous proposition.

With the p-adic topology induced by the p-adic norm, $(\mathbb{Q}_p, +)$ is an abelian topological group. Indeed, for $a, b, c, d \in \mathbb{Q}_p$, we have $|a + b - (c + d)|_p \leq \max\{|a - c|_p, |b - d|_p|\} < \epsilon$ whenever $|a - c|_p < \epsilon$ and $|b - d|_p < \epsilon$. Also, we have $|-a - (-b)|_p < \epsilon$ whenever $|b - a|_p < \epsilon$. Furthermore, the group $\mathbb{Q}_p^*$ forms a topological group under multiplication. As such, $\mathbb{Q}_p$ is a topological field.

Endowing $\mathbb{Z}_p$ with the subspace topology, we see that $\mathbb{Z}_p$ is a topological ring. Every ideal of $\mathbb{Z}_p$ is of the form $p^m\mathbb{Z}_p^\times$. The ideal $\{x : |x|_p < 1\}$ in $\mathbb{Z}_p$ is the principal ideal generated by p. Furthermore, (p) is the unique maximal ideal of $\mathbb{Z}_p$. A principal ideal domain having exactly one nonzero prime ideal is called a *discrete valuation ring.* The set of units of $\mathbb{Z}_p$, denoted $\mathbb{Z}_p^\times$, is precisely the set $\{x : |x|_p = 1\} = \mathbb{Z}_p - (p)$. Also, one can show that $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$.

Theorem 1.1.27. *Every p-adic number $\alpha \in \mathbb{Q}_p$ has a unique p-adic expansion*

$$\alpha = \alpha_{-r}p^{-r} + \alpha_{1-r}p^{1-r} + \cdots + \alpha_{-1}p^{-1} + \cdots \alpha_0 + \alpha_1 p + \cdots$$

with $\alpha_n \in \mathbb{Z}$ and $0 \leq \alpha_n \leq p - 1$. Also, $\alpha \in \mathbb{Z}_p$ if and only if $\alpha_{-r} = 0$ whenever $r > 0$.

Proof. See Proposition 2.23 in Chapter 2 for a more general result and a reference for the proof. □

The residue field $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ consists of p elements. We see this because, by the above theorem, every element of $\alpha \in \mathbb{Z}_p$ has the form $\alpha = a_0 + p(a_1 + a_2 p + a_3 p^2 + \cdots)$ with $0 \leq a_0 \leq p - 1$.

Definition 1.1.28. Let $\gamma \in \mathbb{Z}$. The *closed disc centered at α of radius p^γ* is

$$\underset{\gamma}{B}(\alpha) = \{\beta \in \mathbb{Q}_p : |\beta - \alpha|_p \leq p^\gamma\}.$$

Note that the closed disc centered at α of radius p^γ is also an an *open disc centered at α of radius $p^{\gamma+1}$*. Finally, the *circle centered at α of radius p^γ* is

$$\underset{\gamma}{S}(\alpha) = \{\beta \in \mathbb{Q}_p : |\beta - \alpha|_p = p^\gamma\}.$$

The following proposition is a direct consequence of the p-adic absolute value being non-Archimedean.

Proposition 1.1.29. *If $\beta \in B_\gamma(\alpha)$, then*

$$\underset{\gamma}{B}(\beta) = \underset{\gamma}{B}(\alpha).$$

Therefore, every element of $B_{o,\gamma(\alpha)}$ is a center.

Proof. Let $\beta' \in B_\gamma(\alpha)$. Then

$$|\beta - \beta'|_p = |(\beta - \alpha) + (\alpha - \beta')|_p \leq \max\{|\beta - \alpha|_p, |\alpha - \beta'|_p\} \leq p^\gamma.$$

This shows that $B_\gamma(\alpha) \subseteq B_\gamma(\beta)$. Since $\alpha \in B_\gamma(\beta)$, then the same argument shows that $B_\gamma(\beta) \subseteq B_\gamma(\alpha)$. Therefore, $B_\gamma(\alpha) = B_\gamma(\beta)$. □

The following facts are useful and can be proven easily:

(i) $S_\gamma(\alpha) = B_\gamma(\alpha) - B_{\gamma-1}(\alpha)$.

(ii) $B_\gamma(\alpha) = \cup_{\gamma' \leq \gamma} S_{\gamma'}(\alpha)$.

(iii) $\cap_{\gamma \in \mathbb{Z}} B_\gamma(\alpha) = \{\alpha\}$.

(iv) $\cup_{\gamma \in \mathbb{Z}} B_\gamma(\alpha) = \cup_\gamma S_\gamma(\alpha) = \mathbb{Q}_p$.

Because of (i), $B_\gamma(\alpha)$ is both open and closed. The following are corollaries of the above proposition.

Corollary 1.1.30.

(i) The discs $\overline{B_\gamma(\alpha)}$ and $S_\gamma(\alpha)$ are both open and closed sets in $\mathbb{Q}_p$.

(ii) Every point of the disc $\overline{B_\gamma(\alpha)}$ is its center.

(iii) Any two discs in $\mathbb{Q}_p$ are either disjoint or one is contained in another.

(iv) Every open set in $\mathbb{Q}_p$ is a union of, at most, a countable set of disjoint discs.

Proposition 1.1.31. *If a set $M \subseteq \mathbb{Q}_p$ contains two points α and β and $\alpha \neq \beta$, then it can be represented as a union of disjoint closed and open sets M_1 and M_2, both in M, such that $\alpha \in M_1$ and $\beta \in M_2$.*

Proof. We prove this proposition by considering three separate cases.

(i) $\alpha = 0$, $|\beta|_p = p^\gamma$. For M_1 and M_2, we can take sets $M_1 = M \cap \mathrm{B}_\gamma$ and $M_2 = M \cap (\mathbb{Q}_p - \mathrm{B}_\gamma)$.

(ii) $|\alpha|_p = p^\gamma$, $|\beta|_p = p^{\gamma'}$, $\gamma' > \gamma$. Then we may take $M_1 = M \cap B_\gamma$ and $M_2 = M \cap (\mathbb{Q}_p - \mathrm{B}_\gamma)$.

(iii) $|\alpha|_p = p^\gamma = |\beta|_p$. Using the representation of p-adic numbers, we may express α and β as follows:

$$\alpha = p^{-\gamma}(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \cdots), \quad \beta = p^{-\gamma}(\beta_0 + \beta_1 p + \beta_2 p^2 + \cdots),$$

where $\alpha_0 = \beta_0, \alpha_1 = \beta_1, \cdots, \alpha_{k-1} = \beta_{k-1}, \alpha_k \neq \beta_k$. Thus, $|\alpha - \beta|_p = p^{\gamma-k}$. Set $M_1 = M \cap \mathrm{B}_{\gamma-k-1}(\alpha)$, and $M_2 = M \cap (\mathbb{Q}_p - \mathrm{B}_{\gamma-k-1}(\alpha))$.

□

Remark 1.1.32. It follows from the above proposition that any set that contains more than one point is disconnected, making the p-adic numbers, $\mathbb{Q}_p$, totally disconnected.

Proposition 1.1.33. *A set $K \subseteq \mathbb{Q}_p$ is compact in $\mathbb{Q}_p$ if and only if it is closed and bounded in $\mathbb{Q}_p$.*

Proof. Let K be a compact set in $\mathbb{Q}_p$. Let $y \in \mathbb{Q}_p - K$. Since $\mathbb{Q}_p$ is Hausdorff, then for all $k \in K$ there exist open neighborhoods U_k of k and V_k of y such that $U_k \cap V_k = \emptyset$. The collection $\{U_k\}_{k\in K}$ is an open covering of K. Since K is compact, then there exist elements $k_1, k_2, \ldots k_n$ such that $K \subseteq \cup_{i=1}^n U_{k_i}$. Also, $\cap_{i=1}^n V_{k_i}$ is disjoint from $\cup_{i=1}^n U_{k_i}$. If $z \in \cup_{i=1}^n U_{k_i}$, then $z \in U_{k_i}$ for some i, implying that $z \notin V_{k_i}$, and further that $z \notin \cap_{i=1}^n V_{k_i}$. So $V = \cap_{i=1}^n V_{k_i}$ is an open neighborhood of y disjoint from K. Therefore, K is closed. Consider the collection of open sets $\{B_{p^n}\}_{n\in\mathbb{N}}$, whose union is $\mathbb{Q}_p$. Since K is compact, then some finite subcollection covers K. Thus, $K \subset B_{p^m}$ for some $m \in \mathbb{N}$. This shows that K is bounded. So far we have proven the necessity of the above condition. Since $\mathbb{Q}_p$ is a complete metric space, then it is sufficient to show that every infinite set $M \subseteq K$ contains at least one limit point. See Munkres [22], Theorem 28.2, for a proof of this fact. Let $x \in M$. Then $|x|_p = p^{-\gamma(x)} \leq C$ because K is bounded, which implies that $\gamma(x)$ is bounded from below. If $\gamma(x)$ is not bounded from above on M, then there exists a sequence $\{x_k\} \subseteq M$ such that $\gamma(x_k) \to \infty$ as $k \to \infty$. In other words, $0 \in \mathbb{Q}_p$ and is a limit point of K. On the other hand, if $\gamma(x)$ is bounded from above on M, then there exists a number γ_0 such that M contains an infinite set of points of the form

$$p^{\gamma_0}(x_0 + x_1 p + \cdots), \ \ 0 \leq x_j \leq p-1, \ \ x_0 \neq 0.$$

This follows from the assumption that M is infinite and that there are only a finite number of γ that are possible since $\gamma(x)$ is bounded above and below. Since x_0 takes only $p-1$ values, then there exists an integer $a_0, \ 1 \leq a_0 \leq p-1$, such that M contains an infinite set of points of the form $p^{\gamma_0}(a_0 + x_1 p + \cdots)$, and so on. Hence, we obtain a sequence $\{a_j\}_{j\in\mathbb{N}}, \ 0 \leq a_j \leq p-1, \ a_0 \neq 0$. The limit point is the p-adic number represented by $p_0^{\gamma}(a_0 + a_1 p + a_2 p^2 + \cdots)$. Since K is closed, then this point is in K. This completes the proof. □

Corollary 1.1.34.

(i) Every disc $B_\gamma(\alpha)$ and circle $S_\gamma(\alpha)$ is compact.

(ii) Every compact set in $\mathbb{Q}_p$ can be covered by a finite number of disjoint discs of fixed radius.

(iii) The Heine-Borel Lemma is valid for $\mathbb{Q}_p$.

Definition 1.1.35. Two absolute values $|\cdot|$ and $|\cdot|'$ on a field F are *equivalent* if there is a positive constant t such that $|a|' = |a|^t$ for all $a \in F$. A *place* of F is an equivalence class of absolute values.

Recall that when completing $\mathbb{Q}$ with respect to the usual absolute value, we obtain $\mathbb{R}$.

Theorem 1.1.36. Otrowski's Theorem *Every nontrivial place of $\mathbb{Q}$ is represented by either the usual absolute value or a p-adic one for some rational prime p.*

Proof. See Ramakrishnan and Valenza [24], Chapter 4, Section 4. □

A similar statement holds for number fields. We will say more about this in Chapter 2.

1.1.5 Profinite Topology

Let I be a nonempty set. We say that I is *preordered* with respect to the relation $\leq$ if the given relation is reflexive and transitive. We say that a preordered set is a *directed set* if every finite subset of I has an upper bound in I. As an example, the integers are preordered and directed with respect to division. Note that a finite collection of integers is bounded by its least common multiple. The antisymmetric property (i.e., $i \leq j$ and $j \leq i$ implies $i = j$) does not hold for the integers because $1|-1$ and $-1|1$, but $1 \neq 1$. Thus, the integers under divisibility are not partially ordered. Now, fix I, a preordered set of indices, and let $\{G_i\}_{i \in I}$ be a family of groups with the discrete topology. Furthermore, for each pair of indices $i, j \in I$ with $i \leq j$ let $\phi_{ij} : G_j \to G_i$ be given subject to the following conditions:

(i) $\phi_{ii} = 1_{G_i}$ for all $i \in I$

(ii) $\phi_i j \circ \phi_{jk} = \phi_{ik}$ for all $i, j, k \in I$, $i \leq j \leq k$.

The above system (G_i, ϕ_{ij}) is called a *projective or inverse system.*

Definition 1.1.37. Let (G_i, ϕ_{ij}) be a projective system of groups. Then we define the *projective or inverse limit* of the system, denoted $\lim\limits_{\leftarrow} G_i$, by

$$\lim_{\leftarrow} G_i = \{(g_i) \in \prod_{i \in I} G_i : i \leq j \Rightarrow \phi_{ij}(g_j) = g_i\}.$$

Since the identity element of the direct product lies the projective limit, then the projective limit of groups is non-empty and a group with respect to the componentwise group operation. Since $\lim\limits_{\leftarrow} G_i \subset \prod_{i \in I} G_i$, then for each group G_i there exists a projection map (hence the name projective limit) $p_i : \lim\limits_{\leftarrow} G_i \to G_j$. One can remember the "inverse" part because the association $i \mapsto G_i$ is a contravariant functor. The projective limit satisfies the following universal property:

Let H be a nonempty set and let $(\psi_i : H \to G_i)_{i \in I}$ be a system of maps such that for each pair of the indices $i, j \in I$ with $i \leq j$, the following diagram commutes:

$$\begin{tikzcd} & H \arrow[dl, "\psi_j"'] \arrow[dr, "\psi_i"] & \\ G_j \arrow[rr, "\phi_{ij}"] & & G_i. \end{tikzcd}$$

Then there exists a unique map $\Psi : H \longrightarrow \lim\limits_{\leftarrow} G_i$ such that for each $i \in I$ the diagram

$$\begin{tikzcd} H \arrow[r, "\Psi"] \arrow[dr, "\psi_i"'] & \lim\limits_{\leftarrow} G_i \arrow[d, "p_i"] \\ & G_i \end{tikzcd}$$

also commutes. The map ψ is defined by $h \mapsto (\psi_i(h))_{i\in I}$. We could have constructed the projective limit in the category of sets, replacing homomorphisms with set mappings and replacing groups with sets (forgetting the group structure). Similarly, one can take the projective limit of a projective system of topological spaces, replacing homomorphisms with continuous mappings. However, one does not know that the projective limit is non-empty in the category of sets and topological spaces. However, the projective limit is non-empty in the category of topological groups , and hence is a topological group. Consider a projective system of finite groups endowed with the direct topology. Their projective limit will acquire the subspace topology induced by the product topology on the full direct product.

Definition 1.1.38. A topological group that is isomorphic to the projective limit of a projective system of finite groups is a called a *profinite* group.

Proposition 1.1.39. *Let G b e profinite group, given as a projective limit of the projective system (G_i, ϕ_{ij}). Then the following assertions hold.*

(i) G is Hausdorff with respect to the profinite topology.

(ii) G is a closed subset of the direct product $\prod G_i$.

(iii) G is compact.

Proof.

(i) Every element of a discrete topological group is closed. By Proposition 1.1.12, a discrete group is Hausdorff. The direct product of Hausdorff spaces is Hausdorff, and subspaces of Hausdorff spaces are Hausdorff. Since the projective limit is a subset of the direct product of discrete groups, then the projective limit is Hausdorff.

(ii) In order to show that the projective limit is closed, we will show that its complement in the direct product is open. We see that

$$G^c = \bigcup_i \bigcup_{j\geq i} \{(g_k) \in \prod G_k : \phi_{ij}(g_j) \neq g_i\}.$$

For a fixed i and j in I, $\{(g_k) \in \prod G_k : \phi_{ij}(g_j) \neq g_i\}$ is open since any homomorphism of finite groups is continuous with respect to the discrete topology. Therefore, G^c is open, proving that G is closed.

(iii) The direct product of compact groups is compact by Tychonoff's theorem, and a closed subset of a compact space is compact. A discrete finite group is itself compact. Hence, the direct product of discrete finite groups is compact by Tychonoff's. Applying 2., we get that G is compact since it is a closed subset of the compact direct product.

□

Theorem 1.1.40. *Let G be a topological group. Then G is profinite if and only if G is compact and totally disconnected.*

Proof. See Theorem 1-14 in Ramakrishnan and Valenza [24]. □

Theorem 1.1.41. *Let G be a profinite group and let H be a subgroup of G. Then H is open if and only if G/H is finite. Moreover, the following are equivalent to one another:*

(i) H is closed.

(ii) H is profinite.

(iii) H is the intersection of a family of open subgroups.

If (i), (ii), or (iii) are satisfied, then G/H is compact and totally disconnected.

Proof. See Theorem 1-18 in Ramakrishnan and Valenza [24]. □

Examples 1.1.42.

(i) For a rational prime p, set $G_m = \mathbb{Z}/p^m\mathbb{Z},\ m \geq 1$. There exists the canonical projection

$$\phi_{mn} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$$

whenever $m \leq n$. The system $(\mathbb{Z}/p^m\mathbb{Z}, \phi_{mn})$ is a projective system of topological groups, where we endow the groups with the discrete topology. We then form the projective limit to obtain the ring

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^m\mathbb{Z} = \{(g_i) \in \prod_{m \geq 1} \mathbb{Z}/p^m\mathbb{Z} : m \leq n \Rightarrow \phi_{mn}(g_j) = g_i\}.$$

This ring is called the p-adic integers and is isomorphic to the p-adic integers defined above. This follows from Proposition 1.1.27. In our discussion of p-adic numbers we discovered that $\mathbb{Q}_p$, with the p-adic topology, is totally disconnected. As such, $\mathbb{Z}_p \subseteq \mathbb{Q}_P$ is totally disconnected. This fact agrees with the above theorem about profinite groups.

(ii) Let $G_n = \mathbb{Z}/n\mathbb{Z},\ n \geq 1$. There exists the canonical projection

$$\phi_{mn} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$$

whenever $m \leq n$. We endow the groups with the discrete topology The system $(\mathbb{Z}/m\mathbb{Z}, \phi_{mn})$ is a projective system of topological groups. The projective limit

$$\hat{\mathbb{Z}} = \lim_{\leftarrow}\ \mathbb{Z}/m\mathbb{Z}$$

is a topological ring. It can also be shown that $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$. This fact follows directly from that fact that a commutative profinite group is the direct product of its Sylow subgroups. See Corollary 1.24 in Ramakrishnan and Valenza [24].

(iii) Similarly, we can define the profinite groups

$$\hat{\mathbb{Z}}^\times = \lim_{\leftarrow}\ (\mathbb{Z}/m\mathbb{Z})^\times \quad \text{and} \quad \hat{\mathbb{Z}}_p^\times = lim_{\leftarrow}\ (\mathbb{Z}/p^m\mathbb{Z})^\times .$$

Again, one finds that $\hat{\mathbb{Z}}^\times \cong \prod_p \hat{\mathbb{Z}}_p^\times$. This isomorphism in combination with the decomposition of the idele-class group of $\mathbb{Q}$ will enable us to show that Dirichlet characters are subsumed in the definition of idele-class characters.

1.2 Haar Measure

Recall that the Borel σ-algebra for a topological space X is the smallest σ-algebra containing all open sets. A *positive measure* μ on a measure space $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to \mathbb{R}_+ \cup \{\infty\}$ that is countably additive. Note that $\mathbb{R}_+$ is the set of nonnegative reals. That is,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

for $\{A_n\}$, a collection of disjoint sets in $\mathcal{M}$. Let μ be a Borel measure on a X, a locally compact Hausdorff space, and let E be a Borel subset of X. We say that μ is outer regular on E if

$$\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$$

and that μ is inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}.$$

A measure μ is *regular* if every Borel set in X is both outer and inner regular.

Definition 1.2.1. A *Radon measure* on a X, a locally compact Hausdorff space, is a Borel measure that is finite on compact sets, outer regular on all Borel sets, and inner regular on all open sets.

Let G be a locally compact topological group and μ be a Borel measure on G. A measure μ is said to be *left (respectively, right) translation invariant* if, for all Borel measurable sets E in G,

$$\mu(s \cdot E) = \mu(E) \quad (\text{respectively}, \mu(E \cdot s) = \mu(E))$$

for all $s \in G$.

Definition 1.2.2. A *left Haar measure* (respectively, *right Haar measure*) on a locally compact group G is a nonzero Radon measure μ, which is left translation invariant (respectively, right translation invariant). A *bi-invariant Haar measure* on a locally compact group G is a nonzero Radon measure that is both right and left invariant.

Examples 1.2.3.

(i) The Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^n$ is a bi-invariant Haar measure with respect to addition.

(ii) If G is discrete, then the counting measure is a bi-invariant Haar measure.

(iii) Let λ be the Lebesgue measure on $\mathbb{C}$. Let $f : [0, 2\pi) \to S^1$ be defined by $f(\theta) = e^{i\theta}$. Then $\lambda \circ f^{-1}$ is a left and a right Haar measure on S^1.

The following proposition will highlight a few important properties of the Haar measure. The third property is particularly useful. Also, recall that since a Haar measure is a Radon measure, then it is finite on compact sets, which is another property that will be used often.

Proposition 1.2.4. *Let G be a locally compact group with nonzero Radon measure μ. Then the following assertions are true:*

(i) The measure μ is a left Haar measure on G if and only if the measure $\tilde{\mu}$, defined by $\tilde{\mu}(E) = \mu(E^{-1})$, is a right Haar measure on G.

(ii) The measure μ is a left Haar measure on G if and only if

$$\int_G L_s f d\mu = \int_G f d\mu$$

for all $f \in \mathcal{C}_c^+$ and for $s \in G$.

(iii) If μ is a left Haar measure on G, then μ is positive on all nonempty open subsets of G. Furthermore,

$$\int_G f d\mu > 0 \quad \forall\, f \in \mathcal{C}_c^+.$$

(iv) If μ is a left Haar measure on G, then $\mu(G)$ is finite if and only if G is compact.

Proof. (i) Since inversion is a homeomorphism, then E^{-1} is Borel if and only if E is Borel. Then we have that $\tilde{\mu}(Es) = \tilde{\mu}(E)$ for all $s \in G$ and for all Borel sets E if and only if $\mu(s^{-1}E^{-1}) = \mu(E^{-1})$ for all $s \in G$ and all Borel sets E.

(ii) If μ is a Haar measure on G, then for all characteristic functions $\mathbf{1}_U$, U compact and $U \subseteq G$, we obtain

$$\int_G L_s \mathbf{1}_U d\mu = \mu(sU) = \mu(U) = \int_G \mathbf{1}_U d\mu.$$

The same is true for all finite linear combinations of compactly supported characteristic functions on G. Since simple functions of compact support are dense in $\mathcal{C}_c^+$, then passing to the limit and using the linearity property of the integral, we obtain that the above relation is true for all $f \in \mathcal{C}_c^+$. Conversely, suppose that the integral equality holds. Consider the linear functional defined by $\Lambda(f) = \int_G f d\mu$ on $\mathcal{C}_c(G)$. From this positive linear functional, we can, via the Riesz representation theorem, recover the Radon measure of any open set $U \subseteq G$ as follows:

$$\mu(U) = \sup\{\int_G f d\mu : f \in \mathcal{C}_c(G),\ ||f||_u \leq 1,\ \text{and } \mathrm{supp}(f) \subseteq U\}.$$

Since $||f||_u = ||L_s f||_u$ and since $\text{supp}(L_s f) \subseteq sU$ if and only if $\text{supp}(f) \subseteq U$, then

$$\begin{aligned}\mu(U) &= \sup\{\int_G L_s f d\mu : L_s f \in \mathcal{C}_c(G),\ ||L_s f||_u \leq 1,\ \text{and } \text{supp}(L_s f) \subseteq sU\} \\ &= \sup\{\int_G g d\mu : g \in \mathcal{C}_c(G),\ ||g||_u \leq 1,\ \text{and } \text{supp}(g) \subseteq sU\} \\ &= \mu(sU).\end{aligned}$$

The result then extends to all Borel sets by outer regularity of the radon measure μ.

(iii) Since $\mu \neq 0$, then $\mu(G) > 0$. As such, there exists a $K \subseteq G$ such that $\mu(K) > 0$ by inner regularity. Let U be an open set in G. Since K is compact, then there exists $s_1, s_2, \ldots, s_n$ in G such that $K \subseteq \cup_{i=1}^{n} s_i U$. **The idea that a compact set is covered by finitely many translates of an open set is a key idea in the proof of existence of the Haar measure on a locally compact group.** Also, μ is left invariant, so $\mu(s_i U) = \mu(U)$, which implies that $\mu(U) > 0$. If $f \in \mathcal{C}_c^+$, then there exists a compact set K' such that $f > 0$. Furthermore, there exits a set $U' \subseteq K$ with $\mu(U) > 0$ such that $f > R$ for some constant $R > 0$. Then

$$\int_G f d\mu \geq R\mu(U) > 0.$$

(iv) If G is compact, then, since μ is a Radon measure, we have that $\mu(G) < \infty$. Conversely, assume that $\mu(G) < \infty$. Suppose, by contradiction, that G is not compact. Let K be a compact set whose interior contains e. If there were a finite number of translates of K that cover G, then G would be compact. Therefore, there exists an infinite sequence $\{s_j\}$ in G such that

$$s_n \notin \bigcup_{j<n} s_j K.$$

From Proposition 1.1.10, we have that there exists a symmetric neighborhood U of e such that $UU \subset K$. Suppose that there exists a $u, v \in U$ such that $s_i u = s_j v$ for $i < j$. Then

$$s_j = s_i u v^{-1} \in s_i U U^{-1} = s_i U U \subseteq s_i K$$

for $i < j$, which is contradiction. Therefore, the translates s_jU are disjoint and $\mu(s_iU) = \mu(U) > 0$ by part (iii). Then

$$\mu(G) \geq \mu(\bigcup_{j=1}^{\infty} s_jK) \geq \sum_{j=1}^{\infty} \mu(s_jUU) \geq \sum_{j=1}^{\infty} \mu(s_jU) = \sum_{i=1}^{\infty} \mu(U) > \infty,$$

which is a direct contradiction.

$\square$

Theorem 1.2.5. *Every locally compact group G admits a left (or right) Haar measure. Furthermore, this measure is unique up to multiplication by a positive real constant.*

Remark 1.2.6. The uniqueness of the Haar measure is just as important as the existence. Indeed, let ϕ be a continuous automorphism of G. Then $\mu \circ \phi$ is also a Haar measure on G. As such, there is a unique positive real constant, call it $\text{mod}_G(\phi)$, such that $\mu \circ \phi = \text{mod}_G(\phi) \cdot \mu$. In fact, $\text{mod}_G(\cdot)$ is a homomorphism from $\text{Aut}(G)$ to $\mathbb{R}_+^\times$, where the domain is a group under composition. This construction can be used to classify all locally compact fields. Every element of a locally compact field defines an automorphism of the additive group of the field. Let $l \in K$. The automorphism associated to l is the map $T_l : a \mapsto la$. In addition to the map $\text{mod}_k(\cdot) : k \to \mathbb{R}_+^\times$ being a homomorphism, it can also be shown that the map is continuous. Identifying $n \cdot 1_k$ with n, one can study how mod_k acts on $\mathbb{N}$. One can completely uncover classify all types of locally compact fields using this construction. We will say more about this in Chapter 2.

We will state below the Riesz Representation Theorem, as taken from Rudin [26], Chapter 2; it is the essential ingredient in the proof of the existence of a Haar measure for locally compact groups.

Theorem 1.2.7. *Let X be a locally compact Hausdorff space and let Λ be a positive linear functional on $\mathcal{C}_c(X)$. Then there exists a σ-algebra $\mathcal{M}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathcal{M}$, which represents Λ in the sense that:*

(i) $\Lambda f = \int_X f d\mu \quad \forall\, f \in \mathcal{C}_c(X)$,

and which satisfies the following properties:

(ii) $\mu(K) < \infty$ *for all compact sets* $K \subset X$.

(iii) μ *is outer regular on* $E \in \mathcal{M}$*.*

(iv) μ *is inner regular on all open sets and all* $E \in \mathcal{M}$ *such that* $\mu(E) < \infty$

(v) *If* $E \in \mathcal{M}$*,* $A \subset E$ *and* $u(E) = 0$*, then* $A \in \mathcal{M}$*. We say a measure is* complete *if it satisfies this property.*

A set E in a topological space is called *σ-compact* if it is a countable union of compact sets. A set E in a measure space is said to have *σ-finite measure* if E is a countable union of sets E_i with $\mu(E_i) < \infty$. A measure μ is σ-finite if X is of σ-finite measure with respect to μ.

Corollary 1.2.8. *Let* μ *and* $\mathcal{M}$ *be as above.*

(i) μ *is a Radon measure.*

(ii) *Every* σ*-compact set has* σ*-finite measure.*

(iii) *If* $E \in \mathcal{M}$ *and* E *has* σ*-finite measure, then* E *is inner regular.*

(iv) *If* X *is* σ*-compact, then* μ *is regular.*

(v) *If* μ *is* σ*-finite, then* μ *is regular.*

Proof.

(i) This follows directly from the definition of a Radon measure.

(ii) Since compact sets are of finite measure, then every σ-compact set E is of σ-finite measure.

(iii) If $\mu(E) < \infty$, then we are done. Suppose $\mu(E) = \infty$. Since E has σ-finite measure, then E is an increasing union of sets E_i with $\mu(E_i) < \infty$ and $\mu(E_i) \to \infty$. That is, for all $N \in \mathbb{N}$ there exists an $M > 0$ such that $\mu(E_j) > N$ for all $i > M$. Since the E_i are inner regular, then there exists compact sets $K \subset E_i$ with $\mu(K) > N$. Therefore, μ is inner regular on sets of σ-finite measure.

(iv) If X is σ-compact, then X has σ-finite measure. Furthermore, all $E \in \mathcal{M}$ have σ-finite measure. Therefore, by part (ii), μ is inner regular on $\mathcal{M}$ and hence a regular measure.

(v) If μ is σ-finite, then X has σ-finite measure and so any set in $\mathcal{M}$ has finite measure. Thus, μ is regular.

□

Proof. (Theorem 1.2.5) Let $\mathcal{C}_c^+(G) = \{f \in \mathcal{C}_c(G) : f(s) \geq 0 \; \forall s \in G \text{ and } ||f||_u > 0\}$. Note that $f \in \mathcal{C}_c^+(G)$ must be real valued. Let $f, \phi \in \mathcal{C}_c^+$. Let $U = \{s \in G : \phi(s) > ||\phi||_u/2\}$. Clearly

U is an open set since ϕ is continuous. **Since the support of f is compact, then a finite number of translates of U will cover the support of f.** That is $\text{supp}(f) \subseteq \cup_{i=1}^{n} s_i U$, where $s_i \in G$ for $i = 1, \ldots, n$. Consider $g = \frac{2}{||\phi||_u} \sum_{i=1}^{n} L_{s_i}\phi$. Let $u \in U$. Then

$$\begin{aligned} g(s_j u) = \frac{2}{||\phi||_u} \sum_{i=1}^{n} L_{s_i}\phi(s_j u) &= \frac{2}{||\phi||_u} \sum_{i=1}^{n} \phi(s_i^{-1} s_j u) \\ &> \frac{2}{||\phi||_u} \left(||\phi||_u/2 + \sum_{i=1}^{j-1} \phi(s_i^{-1} s_j u) + \sum_{i=j+1}^{n} \phi(s_i^{-1} s_j u) \right) \\ &= 1 + \sum_{i=1}^{j-1} \phi(s_i^{-1} s_j u) + \sum_{i=j+1}^{n} \phi(s_i^{-1} s_j u) \geq 1, \end{aligned}$$

for all $u \in U$ and s_i, $i = 1, \ldots, n$. Therefore,

$$f \leq ||f||_u \leq g||f||_u = \frac{2|f||_u}{||\phi||_u} \sum_{i=1}^{n} L_{s_i}\phi,$$

for some $s_1, s_2, \cdots, s_n \in G$. As such, we can define $(f : \phi)$, the *Haar covering number* of f with respect to ϕ, to be

$$(f : \phi) = \inf\{\sum_{i=1}^{n} c_j : 0 < c_1, \ldots, c_n \text{ and } f \leq \sum_{j=1}^{n} c_j L_{s_j}\phi \text{ for some } s_1, \ldots, s_n \in G\}.$$

Note that $(f : \phi) > 0$ since $||f||_u > 0$.

The following assertions easily are verified:

(i) $(f : \phi) = (L_s f : \phi)$ for all $s \in G$

(ii) $(f_1 + f_2 : \phi) \leq (f_1 : \phi) + (f_2 : \phi)$

(iii) $(cf : \phi) = c(f : \phi)$ for any $c > 0$

(iv) $(f_1 : \phi) \leq (f_2 : \phi)$ whenever $f_1 \leq f_2$

(v) $(f : \phi) \geq ||f||_u/||\phi||_u$

(vi) $(f_1 : \phi) \leq (f_1 : f_0)(f_0 : \phi)$

The Haar covering yields an invariant approximate functional–in the sense that it is only subadditive–defined as follows: fix $f_0 \in \mathcal{C}_c^+$ and define

$$I_\phi(f) = \frac{(f : \phi)}{(f_0 : \phi)} \quad (f, \phi \in \mathcal{C}_c^+).$$

By property (vi), we get that

$$(f_0 : f)^{-1} \leq I_\phi(f) \leq (f : f_0).$$

To finish the proof, one shows that I_ϕ is an approximate functional and that it becomes more linear as the support of ϕ shrinks. More specifically, as an application of Urysohn's lemma for locally compact Hausdorff spaces, one can prove that given $f_1, f_2 \in \mathcal{C}_c^+$, for every $\epsilon > 0$ there is a neighborhood V of the identity e of G such that

$$I_\phi(f_1) + I_\phi(f_2) \leq I_\phi(f_1 + f_2) + \epsilon,$$

whenever the support of ϕ lies in V. A bona fide invariant linear functional I is obtained as a limit in the space

$$X = \prod_{f \in \mathcal{C}_c^+} [(f_0 : f)^{-1}, (f : f_0)].$$

Indeed, every point of the compact set X is a nonzero positive real valued function I_ϕ defined on$\mathcal{C}_c^+$. See Folland [12], Chapter 11, Lemma 1.7 for the complete proof.

We will provide only a proof for uniqueness in the case that G is an **abelian** locally compact group. See Folland [12] for a full proof of uniqueness. Let dx, dy be two left Haar measures on G. Let $f \in \mathcal{C}_c(G)$ and let $g \in \mathcal{C}_c^+(G)$. Then

$$\int_G f(x)dx \int_G g(y)dy = \int_G \int_G f(x)g(y)\mu(dx)dy = \int_G \int_G f(x+y)g(y)dxdy,$$

by Fubini's theorem and property (ii) of the above Proposition 1.2.4. Since both functions are of compact support, then we can exchange the order of the integral and apply part (ii) to obtain

$$\begin{aligned} \int_G fd\mu \int_G gd\nu &= \int_G \int_G f(x+y)g(y)dydx = \int_G \int_G f(y)g(y-x)dydx \\ &= \int_G \int_G f(y)g(-x)dxdy = \int_G fd\nu \int_G g \circ (-1)d\mu \end{aligned}$$

Define c by $c = \int_G gd\nu / \int_G g \circ (-1)d\mu$. Since $g \in \mathbb{C}_c^+$, then $c \in \mathbb{R}_+^\times$. Furthermore, $c \int_G fd\mu = \int_G fd\nu$ for all $f \in \mathcal{C}_c(G)$. Appealing to the Riesz Representation theorem, we obtain $\nu = c\mu$. $\square$

It follows that if G is an abelian locally compact group, then G admits a bi-invariant Haar measure.

Proposition 1.2.9. *Let G be a locally compact group. For $f \in \mathcal{C}_c(G)$ the functions from G to $\mathbb{C}$, defined by*

$$h \mapsto \int_G L_h f d\mu$$

and

$$h \mapsto \int_G R_h f d\mu,$$

are continuous.

Proof. Let K=supp(f). Then hK=supp$(L_h f)$ and

$$\begin{aligned}
\left| \int_G L_h f d\mu - \int_G L_e f d\mu \right| &= \left| \int_G (L_h f - f) d\mu \right| = \left| \int_{K \cup hK} (L_h f - f) d\mu \right| \\
&\leq \int_{K \cup hK} ||L_h f - f||_u d\mu = \mu(K \cup hK) ||L_h f - f||_u \\
&= M ||L_h f - f||_u
\end{aligned}$$

since K compact implies $\mu(K \cup hK) = M < \infty$. Since $f \in \mathcal{C}_c(G)$, then by Proposition 1.1.11 we know that for all $\epsilon > 0$ there exists a neighborhood of the identity $V \subset G$, such that $M||L_h f - f||_u < M\epsilon$ whenever $h \in V$ Therefore, the function is continuous. The proof for $h \mapsto \int_G R_h$ is similar. $\square$

1.3 Pontryagin Duality and the Fourier Inversion Theorem

In this section we will discuss the topological group of continuous homomorphisms from a group G into S^1. We call such a homomorphism a continuous character of G. The group of continuous characters form a group under multiplication and this group is called the Pontryagin dual of G and is denoted $\hat{G}$. The content of Pontryagin Duality is that G and $\hat{G}$ are mutually dual. This construction is analogous to the dual vector space of a finite-dimensional vector space. A vector space and its dual are not naturally isomorphic, but their endomorphism algebras are anti-isomorphic via the transpose. In a similar way, we will see that G is not always naturally isomorphic to $\hat{G}$, however, the group algebra $C(G)$ is isomorphic to $C(\hat{G})$, if one correctly specifies the group algebras, via the Fourier transform. The canonical isomorphism of G to $\hat{\hat{G}}$ given by evaluation of a character of G at an element is the same as the canonical isomorphism of a finite dimensional vector space to its double dual.

Let us now restrict our attention to an abelian topological group G. We will write the group operation multiplicatively. Define $\hat{G}$, the *multiplicative group of continuous complex characters of* G, to be the set of all continuous homomorphisms of G into the circle group, $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, of the complex numbers. That is, $\hat{G} = \mathrm{Hom}_{\mathrm{cont}}(G, S^1)$. We also call $\hat{G}$ the *Pontryagin dual* of G. Let us compute the dual groups of a few well-known abelian groups.

Proposition 1.3.1.
(i) $\hat{\mathbb{R}} \cong \mathbb{R}$, *with the pairing* $< x, \xi >= e^{2\pi i \xi x}$
(ii) $\hat{S}^1 \cong \mathbb{Z}$, *with the pairing* $< \alpha, n >= \alpha^n$.
(iii) $\hat{\mathbb{Z}} \cong S^1$, *with the pairing* $< n, \alpha >= \alpha^n$
(iv) $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$, *with the pairing* $< m, k >= e^{2\pi i \frac{mk}{n}}$
Note that when speaking about the Pontryagin dual of a field, such as $\mathbb{R}$, *we mean the dual group of the additive group of the field* $(< \mathbb{R}, + >)$.

Proof.
(i) If $\phi \in \hat{\mathbb{R}}$, then $\phi(0) = 1$. Since ϕ is continuous, then there exists an $a \in \mathbb{R}$ such that

$$A = \int_0^a \phi(t)dt \neq 0.$$

Since ϕ is a group homomorphism, we have

$$\phi(x)A = \int_0^a \phi(t + x)dt = \int_x^{a+x} \phi(t)dt.$$

By the fundamental theorem of calculus, ϕ is differentiable and

$$\phi'(x) = A^{-1}(\phi(a + x) - \phi(x)) = A^{-1}(\phi(a)\phi(x) - \phi(x)) = A^{-1}(\phi(a) - 1)\phi(x).$$

Letting $c = A^{-1}(\phi(a) - 1)$, we obtain $\phi(t) = e^{ct}$. However, since $|\phi(x)| = 1$, then we know $c = 2\pi i\xi$ for some $\xi \in \mathbb{R}$.

(ii) First, we identify S^1 with $\mathbb{R}/\mathbb{Z}$ via the topological group isomorphism $e^{-2\pi i(\cdot)} : \mathbb{R}/\mathbb{Z} \mapsto S^1$. To compute the dual group of S^1, we can simply compute the dual group of $\mathbb{R}/\mathbb{Z}$. The main idea is that the dual group of $\mathbb{R}/\mathbb{Z}$ is, in fact, isomorphic to the subgroup of characters of $\mathbb{R}$ that are trivial on $\mathbb{Z}$. It is certainly clear that the subgroup of characters of this type induce a character on $\mathbb{R}/\mathbb{Z}$, but what is not immediately clear is that all characters on $\mathbb{R}/\mathbb{Z}$ are of this form. For now, we will assume that there is a one-to-one correspondence, but we will say more about that is later in this section. In part (i), we found that all continuous characters on $\mathbb{R}$ can be written in the form $\chi(x) = e^{2\pi i\xi x}$ for a some fixed $\xi \in \mathbb{R}$. The characters of the form $\chi(x) = e^{2\pi inx}$ for some fixed $n \in \mathbb{Z}$ are precisely the characters of $\mathbb{R}$ that are trivial on $\mathbb{Z}$. As such, $\hat{S^1} = \mathbb{Z}$.

(iii) It is unfortunate, but the dual group of $\mathbb{Z}$, denoted $\hat{\mathbb{Z}}$, is the same notation used for the projective limit $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. For $\phi \in \hat{\mathbb{Z}}$, let $\alpha_\phi = \phi(1) \in S^1$. Since $\mathbb{Z}$ is a cyclic group generated by 1, then $\phi(n) = \phi(1)^n = \alpha_\phi^n$. Every character of $\mathbb{Z}$ is completely determined by its value at 1. As such, to generate a character of $\mathbb{Z}$, we simply pick an element $\zeta \in S^1$ and define $\phi(1) = \zeta$. Therefore, $\hat{\mathbb{Z}} \cong S^1$.

(iv) As explained in part (ii), the dual group of $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the subgroup of character of $\mathbb{Z}$ that are trivial on the ideal $n\mathbb{Z}$. From part (iii), we see that the characters ϕ of $\mathbb{Z}$ such that $\phi(1)$ is an nth root of unity are the only characters of $\mathbb{Z}$ that are trivial on $n\mathbb{Z}$. The set of kth roots of unity are isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Therefore, $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$. In fact, for all finite abelian groups G, one has $\hat{G} \cong G$. We direct the reader to Keith Conrad's expository paper on the *Characters of finite abelian groups* [4] for a basic proof of this fact.

$\square$

Secondly, the dual group of the locally compact field $\mathbb{R}$ is isomorphic to itself. We will see in chapter 4 that locally compact non-discrete fields (local fields) are self-dual. For example, the dual group (with respect to addition) of the locally compact field $\mathbb{Q}_p$ is $\mathbb{Q}_p$. In fact, $\mathbb{R}$ and $\mathbb{Q}_p$, in some sense, make up the "majority" of local fields. We will say more about this is in chapter 2. We will prove below that the dual group of a locally compact group is a locally compact group. Thirdly, the dual group of the compact group S^1 is the discrete group $\mathbb{Z}$, and the dual group of the discrete group $\mathbb{Z}$ is the compact group S^1. This is true in general and will also be proved below.

Proposition 1.3.2. *The Pontryagin dual of* $G_1 \times G_2$ *is isomorphic to* $\hat{G}_1 \times \hat{G}_2$.

Proof. This is obvious. □

Let G be a group and X a subset of G. For $n \in \mathbb{N}$ define $X^{(n)} \subseteq G$ as follows:

$$X^{(n)} = \{\prod_{j=1}^{n} x_j : x_j \in X,\ j = 1, \ldots, n\}.$$

This is certainly different than the n-fold Cartesian product of X with itself.

We will endow $\hat{G}$ with *compact-open topology.* In this topology, sets of the form

$$W(K, V) = \{\chi \in \hat{G} : \chi(K) \subseteq V\}, \tag{1.1}$$

where K is a compact subset of G and V is a neighborhood of the identity in S^1, constitute a neighborhood base of the trivial character in $\hat{G}$. Since $\hat{G}$ is homogeneous, then a basis at the identity defines basis for the topology of $\hat{G}$. Since S^1 is metrizable, then the compact-open topology coincides with the topology of compact convergence. That is, χ_n converges to χ in $\hat{G}$ if and only if for each compact set K in G $\chi_n|_K$ converges uniformly to χ_K. If G is compact, then the compact open topology coincides the topology of uniform convergence. If G is finite, then the compact-open topology coincides with the topology of pointwise convergence. Lastly, if G is a separable locally compact abelian group, then $\hat{G}$ is metrizable.

Recall that the universal covering space of S^1 is $\mathbb{R}$ with the continuous surjective map

$$\phi : \mathbb{R} \to S^1$$
$$x \mapsto e^{2\pi i x}.$$

The kernel of ϕ is $\mathbb{Z}$. For $\epsilon \in \mathbb{R}$ such that $0 < \epsilon \leq 1$ define $N(\epsilon) \subseteq S^1$ by

$$N(\epsilon) = \phi\left((-\epsilon/3, \epsilon/3)\right).$$

The following technical lemma will be important in establishing key properties about the compact-open topology on $\hat{G}$.

Lemma 1.3.3. *Let m be a positive integer and suppose that $x \in \mathbb{C}$ such that $x, x^2, \ldots, x^m$ lie in $N(1)$. Then $x \in N(1/m)$. Consequently, if U is a subset of G containing the identity and $\chi : G \to S^1$ is a group homomorphism (not necessarily continuous) such that $\chi(U^{(m)}) \subseteq N(1)$, then $\chi(U) \subseteq N(1/m)$.*

Proof. We prove this by induction. For $m = 1$ the statement is obvious. Let r be a positive integer and suppose that $x^r \in N(1)$. Then $x^r = e^{2\pi i x}$ for some $x \in (-1/3, 1/3)$, which implies that there exists a $y \in N(1/r)$ such that $x^r = y^r$. Hence, the quotient x/y is an rth root of unity. Since $\phi(q/r) = e^{2\pi i q/r}$ is an rth root of unity for all $q \in \mathbb{Z}$ such that $0 \leq q < r$, then $x \in N(1/r)\phi(q/r)$. Let us investigate sets of the form $N(1/r)\phi(q/r)$. We claim that for all $r > 0$,

$$N\left(\frac{1}{r}\right) \cap N\left(\frac{1}{r+1}\right)\phi\left(\frac{q}{r+1}\right) \neq \emptyset \Longrightarrow q = 0.$$

By definition, we have that

$$N\left(\frac{1}{r+1}\right)\phi\left(\frac{q}{r+1}\right) = \{e^{2\pi i t/3} : t \in \left(\frac{3q-1}{r+1}, \frac{3q+1}{r+1}\right)\}$$

and

$$N\left(\frac{1}{r}\right) = \{e^{2\pi i/3} : t \in \left(-\frac{1}{r}, \frac{1}{r}\right)\}.$$

The above sets have no intersection unless

$$\frac{1}{r} > \frac{3q-1}{r+1} \Longleftrightarrow 1 + \frac{1}{r} > 3q - 1 \Longleftrightarrow 2r + 1 > 3qr,$$

which cannot not hold unless $q = 0$.

Suppose $x \in N(1/r)$ and $x^{r+1} \in N(1)$. Applying the opening argument again, we obtain $x \in N(1/(r+1))\phi(q/(r+1))$ where $0 \leq q < r+1$. Then $x \in N(1/r) \cap N(1/(r+1))\phi(q/(r+1))$, which implies $q = 0$, and hence $x \in N(1/(r+1))$. Consequently, it follows by induction that if $x, x^2, \ldots, x^m$ lie in N(1), then $x \in N(1/m)$.

Let $g \in U \subseteq G$ and $e \in U$. Then $g, g^2, \ldots, g^m \in U^{(m)}$ by definition. Therefore, if $\chi(U^{(m)}) \subseteq N(1)$, then $\chi(g), \chi(g)^2, \ldots, \chi(g)^m$ lie in $N(1)$. Applying the first part of the theorem, we obtain $\chi(g) \in N(1/m)$ and thus $\chi(U) \subseteq N(1/m)$. $\square$

In the remaining chapters, the topological fields and groups that we will consider are locally compact. Indeed, the completion of global field at place is a locally compact field. In the following proposition we will prove that the dual group of a locally compact group is locally compact. Furthermore, we will show that the dual group of a compact group is discrete and that the dual group of a discrete group is compact. These three facts will be essential in proving that a local field is isomorphic to its dual.

Proposition 1.3.4.

(i) A group homomorphism $\chi : G \to S^1$ is continuous, and hence a character of G, if and only if $\chi^{-1}(N(1))$ is a neighborhood of the identity in G.

(ii) The family $\{W(K, N(1))\}_K$, indexed over all compact subsets of G, is a neighborhood base of the trivial character for the compact-open topology of $\hat{G}$.

(iii) If G is discrete, then $\hat{G}$ is compact.

(iv) If G is compact, then $\hat{G}$ is discrete.

(v) If G is locally compact, then $\hat{G}$ is locally compact.

Proof.

(i) Suppose $\chi^{-1}(N(1)) = U$, where $U \subseteq G$ a neighborhood of the identity in G. By Proposition 1.1.9, there exists an open neighborhood V of the identity in G such that $V^{(m)} \subseteq U$. Then we have $\phi(V^{(m)}) \subset \phi(U) \subseteq N(1)$. From the previous lemma we have that $V \subseteq N(1/m)$. Therefore, χ is continuous. If χ is continuous, then $\chi^{-1}(N(1)) = U$, where U is an open neighborhood of the identity.

(ii) Since $\{N(1/m)\}_m$, $m \in \mathbb{Z}_+^*$, constitutes a neighborhood basis of the identity in S^1, $W(K, N(1/m))$ constitutes a neighborhood basis of the trivial character in the compact-open topology. Therefore, we need to show that there exists a compact set K_1 of G for every compact set K of G and every positive integer m such that

$$W(K_1, N(1)) \subseteq W(K, N(1/m)).$$

Consider $K_1 = K^{(m)}$, which is compact because it is the continuous image of the compact set K^m under the multiplication map. Let $\chi \in W(K_1, N(1))$. Then $\chi(x), \chi(x)^2, \ldots \chi(x)^m \in N(1)$ for all $x \in K$. From the technical lemma, we have that $\chi(x) \in N(1/m)$, and hence $\chi \in W(K, N(1/m))$.

(iii) If G is discrete, then $\hat{G} = \mathrm{Hom}_{\mathrm{cont}}(G, S^1) = \mathrm{Hom}_(G, S^1)$. In the case that G is discrete, compact sets are finite. Thus a neighborhood basis of the trivial character is

$$W(g, V) = \{\chi \in \hat{G} : \chi(g) \subset V\},$$

where V is an open neighborhood of the identity in S^1 and where $g \in G$. This basis induces the topology of pointwise convergence. The space of all maps from G to S^1, $(S^1)^G$, with the product topology (topology of pointwise convergence) is compact by Tychonoff's theorem. Let $\{\chi_i\}_{i \in I}$ be a sequence of characters that converge pointwise to some $f \in (S^1)^G$. Then $\chi_i(s+t) \to f(s+t)$ and $\chi_i(s)\chi_i(t) \to f(s)f(t)$ as $i \to \infty$. Since $\chi_i(s+t) = \chi_i(s)\chi(t)$ for all i, then $f(s+t) = f(s)f(t)$, which implies that f is a homomorphism from G to S^1. Therefore, $\hat{G} = \mathrm{Hom}(G, S^1)$ is a closed subset of the compact set $(S^1)^G$ and is hence, compact.

(iv) For any character χ, $\chi(G)$ is a compact subgroup of S^1. Since any subgroup of S^1 cannot be totally contained in $N(\epsilon)$, $0 < \epsilon \leq 1$, then $W(G, N(1))$ must consist of only the trivial character. Therefore, the trivial character is an open set and thus by homogeneity of the topological group $\hat{G}$, every element of $\hat{G}$ is open, making $\hat{G}$ discrete.

(v) By part (i) it suffices to check that for any fixed compact neighborhood K of the identity in G,

$$W(K, \overline{N(1/4)})$$

is a compact neighborhood of the identity in $\hat{G}$. Let G_0 be isomorphic as a group to G, but with the discrete topology. As such, only the finite sets of G_0 are compact, and hence the compact-open topology on $\hat{G}_0$ coincides with the topology of pointwise convergence. Otherwise said, $\hat{G}_0 \cong \mathrm{Hom}(G, S^1)$ with the topology of pointwise convergence. By part (iii), $\hat{G}_0$ is compact. Analogous to W, define W_0 by

$$W_0 = \{\chi \in \hat{G}_0 : \chi(K) \subseteq \overline{N(1/4)}\}.$$

Since $\hat{G}_0$ has the topology of pointwise convergence, then W_0 is closed and hence compact in $\hat{G}_0$. Let $\chi \in W_0$. Then $K \subseteq \chi^{-1}(N(1/4)) \subseteq \chi^{-1}(N(1))$, where K was chosen as a compact neighborhood of the identity. By part (i) χ is continuous and thus $\chi \in W$. However, we also know that $W \subseteq W_0$ since $\hat{G}_0$ ignores continuity. Therefore, $W = W_0$. If we can show that τ_0, the topology induced on W by $\hat{G}_0$, is finer than τ, the topology induced on W by $\hat{G}$, then the compactness of W in τ_0 will imply the compactness of W in τ. Since it is also clear that τ is finer than τ_0, it will actually follow that the two topologies are the same. Indeed, the compact-open topology is finer than the topology of pointwise convergence. Let K_1 be a compact subset of G and let m be a positive integer. Define

$$W(\chi) = (\chi W(K_1, N(1/m))) \cap W$$

for $\chi \in W$. $W(\chi)$ is a neighborhood base of χ in τ. If we can show that $W(\chi)$ is an open neighborhood of χ with respect to τ_0, then τ has a neighborhood base at χ contained in τ_0. Since K is a neighborhood of the identity, then there exists an open neighborhood of the identity, V, in G, such that $V^{(2m)} \subseteq K$ (see first chapter). Note that since K_1 is a compact set of G, then the cover $\subseteq \cup_{g\in G} gV$ of G reduces to a finite cover, implying the existence of a finite set F of G such that $K_1 \subseteq F\dot{V}$. Consider the subsets W_0 of W defined as follows:

$$W_0(\chi) = (\chi W_0(F, N(1/(2m)))) \cap W,$$

where $W_0(F, N(1/(2m))$ is the set of characters in $\hat{G}_0$ that map into $N(1/(2m))$. Clearly $W_0(F, N(1/(2m))$ is open in $\hat{G}_0$. We need to show that $W_0(\chi) \subseteq W(\chi)$. If so, then $W_0(\chi)$ is a τ_0-neighborhood of χ contained in $W(\chi)$, hence proving that $W(\chi)$ is open in τ_0.
Let $\mu \in W_0(\chi))$. Then $\mu = \chi\mu_0 \in W$ for $\mu_0 \in G_0$ such that $\mu_0(F) \subseteq N(1/(2m))$. Since $\chi \in W$, then $\chi^{-1} = \overline{\chi} \in W$ and hence $\mu_0 \in W^{(2)}$. Consequently,

$$\mu_0(K) = \omega_1(K)\omega_2(K) \subseteq \overline{N(1/4)N(1/4)} \subseteq N(1/2) \subset N(1).$$

By part (i) we know that μ_0 is continuous. Also, since $V^{(2m)} \subseteq K$ and $\mu(K) \subseteq N(1)$, then by the previous lemma $\mu_0(V) \subseteq N(1/(2m))$. Then

$$\mu_0(K_1) \subseteq \mu_0(F) \cdot \mu_0(V) \subseteq N(1/(2m))N(1/(2m)) = N(1/m).$$

Then $\mu_0 \in W(K_1, N(1/m))$, which implies that $\mu \in W(\chi)$. Therefore, $W_0(\chi) \subseteq W(\chi)$, and hence τ_0 is finer than τ. Consequently, W is compact in the compact-open topology, indicating that $\hat{G}$ is locally compact.

$\square$

Let G be a locally compact group and let dy be the Haar measure on G. We say a function $f : G \to \mathbb{C}$ is *absolutely integrable* if

$$||f||_1 := \int |f(y)|dy < \infty.$$

With respect to function addition, the space of absolutely integrable functions forms a complex vector space. In fact, $|| \cdot ||_1$ is a semi-norm of this vector space. To make $|| \cdot ||_1$ a bona fide norm, we identify functions $f, g : G \to \mathbb{C}$ if $||f - g||_1 = 0$ and denote the vector space by $L_1(G)$. Let $f \in L_1(G)$. Then we define $\hat{f} : \hat{G} \to \mathbb{C}$, the *Fourier transform* of f, to be

$$\hat{f}(\chi) = \int_G f(y)\overline{\chi}(y)dy \text{ for } \chi \in \hat{G}.$$

Note that

$$\left|\hat{f}(\chi)\right| = \left|\int_G f(y)\overline{\chi}(y)dy\right| \leq \int_G |f(y)|\,|\overline{\chi}(y)|\,dy = \int_G |f(y)|\,dy = ||f||_1 < \infty,$$

so the Fourier transform makes sense for $f \in L_1(G)$.

Examples 1.3.5.

(i) For $G = \mathbb{R}$, we know that $\hat{\mathbb{R}} \cong \mathbb{R}$, and hence we can identify each $t \in \mathbb{R}$ with the character

$$x \mapsto e^{2\pi ixt}.$$

Let dx be the Lebesgue measure on $\mathbb{R}$. Let $f \in L^1(\mathbb{R})$. In this case, the Fourier transform reduces to

$$\hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi ixt}dx.$$

Although we are used to thinking of the Fourier transform as a function on $\mathbb{R}$, it is actually a function defined on $\hat{\mathbb{R}}$. The $'t'$ is really representing the character $\chi : \mathbb{R} \to S^1$ given by $\chi(x) = e^{-2\pi ixt}$.

(ii) For $G = \mathbb{Z}/n\mathbb{Z}$, we know that $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$, and hence we can identify each $m \in \mathbb{Z}/n\mathbb{Z}$ with the character

$$k \mapsto e^{2\pi i\frac{mk}{n}}.$$

Since G is finite, then the counting measure is the Haar measure on G. Let $f \in L^1(\mathbb{Z}/n\mathbb{Z})$. In this case, the Fourier transform reduces to

$$\hat{f}(m) = \sum_{k=0}^{n-1} f(k)e^{-2\pi i \frac{mk}{n}}.$$

(iii) For $G = S^1$, we know that $\hat{S^1} \cong \mathbb{Z}$, and hence we can identify each $n \in \mathbb{Z}$ with the character

$$\theta \mapsto e^{in\theta}.$$

Recall that with $g : [0, 2\pi) \to S^1$ defined by $g(\theta) = e^{i\theta}$, the measure $d\theta = dx \circ g^1$ is the Haar measure on S^1. Let $f \in L^1(S^1)$. In this case, the Fourier transform reduces to

$$\hat{f}(n) = \int_0^{2\pi} f(\theta)e^{-in\theta}d\theta.$$

(iv) For $G = \mathbb{Z}$, we know that $\hat{\mathbb{Z}} \cong S^1$ and we can identify each $e^{i\theta} \in S^1$ with the character

$$n \mapsto e^{in\theta}.$$

Since G is finite, then the counting measure is the Haar measure on G. Let $f \in L^1(\mathbb{Z})$. In this case, the Fourier transform reduces to

$$\hat{f}(\theta) = \sum_{n=-\infty}^{\infty} f(n)e^{-in\theta}.$$

Let $\mathfrak{B}(G)$ denote the set of functions $f \in L^1(G)$ such that f is continuous and $\hat{f} \in L^1(\hat{G})$.

Theorem 1.3.6 (The Fourier Inversion Theorem)**.** *There exists a Haar measure $d\chi$ on $\hat{G}$ such that for all $f \in \mathfrak{B}(G)$,*

$$f(y) = \int_{\hat{G}} \hat{f}(\chi)\chi(y)d\chi.$$

Note the drop of the conjugation on χ. That is, $\hat{\hat{f}}(y) = f(-y)$. In addition, the Fourier transform $f \mapsto \hat{f}$ identifies $\mathfrak{B}(G)$ with $\mathfrak{B}(\hat{G})$.

Proof. See Folland [11], Chapter 4, Theorem 4.32. □

When defining the Fourier transform on a locally compact group, we must fix a Haar measure. Since the Haar measure is unique up to a positive constant, then any Fourier transform, no matter what measure is fixed when defining it, will only differ from another defined Fourier transform by a constant. Suppose we fix a measure dx on a locally compact group G when defining the Fourier transform. Then the Fourier inversion theorem guarantees the existence of a measure $d\chi$ on $\hat{G}$ such that the Fourier inversion theorem holds for all $f \in \mathfrak{B}(G)$. However, if we fix a measure, say $c \cdot dx$ on G, then the dual measure to this measure is precisely $d\chi/c$.

In addition to the choice of measure, in the above examples, we saw that it was convenient and more consistent with the 'less abstract' theory to fix the 'form' of character when defining the Fourier transform. More precisely, we fixed an isomorphism of the dual group to another group. For example, in the case $G = \mathbb{R}$, we fixed an isomorphism $\mathbb{R} \to \hat{\mathbb{R}}$ given by $t \mapsto e^{-2\pi ixt}$ when defining the Fourier transform. We also chose the standard Lebesgue measure on $\mathbb{R}$. In this case, the dual measure to the Lebesgue measure on $\hat{G} = \hat{\mathbb{R}} = \mathbb{R}$ is precisely the Lebesgue measure. When a locally compact group is isomorphic to its dual and a Haar measure is the dual of itself in the sense of the Fourier inversion theorem, we call the measure self-dual. However, if we chose $t \mapsto e^{-ixt}$ as our isomorphism , and hence $x \mapsto e^{-ixt}$ as our 'form' of character, then the dual measure to the Lebesgue measure dx on $\mathbb{R}$ is $dx/2\pi$. To check that a given measure is a dual measure to another measure, we only need check one function, since the Haar measure is unique up to a constant. In chapter 4, we will construct 'standard' characters on local fields, show that local fields are self-dual, and moreover, explicitly show that certain measures on local fields are self-dual when defining the fourier transform with respect to the 'standard characters'. We will now highlight a few examples of the Fourier inversion theorem.

Examples 1.3.7.

(i) Let $G = S^1$. The counting measure on $\mathbb{Z}$ is dual to the normalized Lebesgue measure $d\theta/2\pi$ on S^1. That is, for $f \in \mathfrak{B}(S^1)$ and

$$\hat{f}(n) = \int_0^{2\pi} f(\theta)e^{-in\theta}\frac{d\theta}{2\pi},$$

we have

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta}.$$

(ii) Let $G = \mathbb{Z}/n\mathbb{Z}$. The dual of the counting measure is the counting measures divided by n. That is, for $f \in \mathfrak{B}(\mathbb{Z}/n\mathbb{Z})$ and

$$\hat{f}(m) = \sum_{k=0}^{n-1} f(k)e^{-2\pi i\frac{km}{n}},$$

we have

$$f(k) = \frac{1}{n}\sum_{m=0}^{n-1} \hat{f}(m)e^{-2\pi\frac{km}{n}}.$$

With the help of the Fourier inversion theorem and an extension of the Fourier transform to an isometry on $L^2(G)$, one can prove Pontryagin duality. We will not prove Pontryagin duality, but will direct the reader to a proof. Pontryagin duality and its derivatives will be used multiple times in Tate's thesis and especially in the proof of self-duality of local fields.

Theorem 1.3.8 (Pontryagin Duality)**.** *The map that associates to $g \in G$ the character $\chi \mapsto \chi(g)$ of $\hat{G}$ is an isomorphism of topological groups G and $\hat{G}$. Hence G and $\hat{G}$ are mutually dual.*

Proof. See Chapter 3, Section 4, of Ramakrishnan and Valenza [24]. □

Let H be a closed subgroup of G. We define $H^\perp$ to be the subgroup of characters of G that restrict to the identity on H:

$$H^\perp = \{\chi \in \hat{G} : \chi|_H = 1\}.$$

Proposition 1.3.9. *The subgroup $H^\perp$ of $\hat{G}$ is closed. Furthermore, $H = (H^\perp)^\perp$.*

Proof. Let U be a neighborhood of 1 in S^1 such that U contains no subgroup other than the trivial subgroup. Then $H^\perp = W(H,U)$, so $H^\perp$ is an open , and hence closed subgroup of $\hat{G}$. By Pontryagin duality, $H \subseteq (H^\perp)^\perp$. Indeed, if $h \in H$ and $\chi \in H^\perp$, then $\alpha(h)(\chi) = \chi(h) = 1$, where α is the isomorphism $\alpha : G \to \hat{\hat{G}}$. We would now like to show the reverse inclusion.

We show this as an application of the Gelfand-Raikov theorem. The theorem says: if G is any locally compact group, the irreducible unitary representations of G separate points on G. That is, if x and y are distinct points on G, there is an irreducible representation π such that $\pi(x) \neq \pi(y)$. See Chapter 3, Theorem 3.34, of Folland [11] for a proof of the Gelfand-Raikov theorem. Let $x_0 \notin H$. Since H is closed, then by Proposition 1.1.23, the quotient group G/H is locally compact. As such, we can apply the Gelfand-Raikov Theorem to G/H in order to assert the existence of a character (one-dimensional representation) χ of G/H such that $\chi(\rho_H(x_0)) \neq 1$, where ρ_H is the projection $G \to G/H$. Then $\chi \circ \rho_H \in H^\perp$, and $(\chi \circ \rho)(x_0) \neq 1$, so $x_0 \notin (H^\perp)^\perp$. Therefore, $(H^\perp)^\perp \subseteq H$, which completes the proof. □

Theorem 1.3.10. *Suppose H is a closed subgroup of G. Define $\Phi : \widehat{G/H} \to H^\perp$ and $\Psi : \hat{G}/H^\perp \to \hat{H}$ by*

$$\Phi(\chi) = \chi \circ \rho_H \quad \text{and} \quad \Psi(\eta H^\perp) = \eta|_H,$$

where $\rho_H : G \to G/H$ is the canonical projection. Then Φ and Ψ are isomorphisms of topological groups.

Proof. See Theorem 4.39 in Folland [11]. □

Corollary 1.3.11. *If H is a closed subgroup of G, then every character on H extends to a character on G.*

Proof. This follows from the fact that Ψ is surjective. Notice that this is a sort of Hahn-Banach theorem for locally compact abelian groups. □

We will now briefly discuss some functorial properties of Pontryagin duality. Let G_1 and G_2 be locally compact abelian groups and ϕ a continuous homomorphism of G_1 into G_2. Then

$$(\hat{\phi}\chi)(g) = \chi(\phi(g)) \quad \text{for } \chi \in \hat{G}_2,\ g \in G_1$$

defines a continuous homomorphism $\hat{\phi}$ of $\hat{G}_2$ into $\hat{G}_1$. Consequently, $\widehat{\cdot}$ is a contravariant functor of the category of locally compact abelian groups onto itself. Since G is locally compact, then so is every closed subgroup H of G and every quotient group G/H. Applying the contravariant functor $\widehat{\cdot}$ to the short exact sequence

$$1 \longrightarrow H \xrightarrow{\text{inc}} G \xrightarrow{\rho_H} G/H \longrightarrow 1,$$

we obtain

$$1 \longrightarrow \widehat{G/H} \cong H^{\perp} \longrightarrow \hat{G} \longrightarrow \hat{G}/H^{\perp} \cong \hat{H} \longrightarrow 1,$$

by Theorem 1.3.10, which is also a short exact sequence. Therefore, $\widehat{\cdot}$ is an exact functor.

CHAPTER 2
Global and Local Fields

2.1 Global Fields

Definition 2.1.1. A *global field* is one of the following:

(i) a field K that is a finite field extension of $\mathbb{Q}$;

(ii) a finitely generated function field in one variable over a finite field $k = \mathbb{F}_q$.

We call (i) a *number field* and (ii) a *function field*.

In this exposition, we will not not discuss function fields, please see Lorenzini [21], *An invitation to arithmetic geometry* for a text on function fields. A field K is said to be *separable* over a field F if the minimal polynomial of every element of K has no multiple roots. A finite extension K of a field F is said to be *simple* if there exists a single element α such that $K = F(\alpha)$. The primitive element theorem states that if K/F is a finite extension, then K is simple if and only if there exists finitely many subfields of K containing F. Moreover, if K/F is finite and separable, then K/F is simple. See Dummit and Foote [9], Abstract Algebra, Section 14.4. A polynomial $f(x) \in F[x]$ has multiple a root α if and only if α is a root of the $D_x f(x) \in F[x]$, where D_x is the algebraic derivative of the polynomial. In particular, a polynomial does not have multiple roots if it is relatively prime to its derivative. If F is a field of characteristic zero, then every irreducible polynomial is relatively prime to its derivative, hence, making every extension of F separable. Since $\mathbb{Q}$ has characteristic zero, then every number field K is separable , and hence there exists an element $\alpha \in \overline{\mathbb{Q}}$ such that K is generated by α, $K = \mathbb{Q}(\alpha)$. Then $K = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(m_\alpha(x))$, where $m_\alpha(x) = c_n x^n + \cdots c_1 x + c_0 \in \mathbb{Q}[x]$ is the minimal polynomial of α and where $n = [K : \mathbb{Q}]$. The elements $1, \alpha, \alpha^2, \ldots, \alpha^n$ constitute a basis of the vector space of K over $\mathbb{Q}$, and hence

$$K = \mathbb{Q}(\alpha) = \{c_0 + c_1\alpha + \cdots + c_{n-1}\alpha^{n-1} : c_0, c_1, \ldots, c_{n-1} \in \mathbb{Q}\}.$$

A field extension K/F is said to be Galois if $\mathrm{Card}(\mathrm{Aut}(K/F)) = [K : F]$; that is, if the number of the F-automorphisms of K (automorphisms of K that fix F) is equal to the dimension of K as a vector space over F. This is equivalent to: K/F is Galois if and only if the fixed field of $\mathrm{Aut}(K/F)$ if F. This is also equivalent to: K/F Galois if and only if K is the splitting field of some separable polynomial over F. In the case of K/F is a separable extension, in which case, $K = F(\alpha)$, then K is Galois if and only if it contains all roots of $m_\alpha(x)$.

Let K be a number field. An *algebraic integer* is an element of K which is a root of a monic polynomial with coefficients in $\mathbb{Z}$. It can be shown that the product and sum of two algebraic integers is an algebraic integer. We denote the ring of algebraic integers in a number field K by $\mathfrak{o}_K$. The ring of integers, $\mathfrak{o}_K$, is clearly an integral domain because it lies inside K. Since every integer $m \in \mathbb{Z}$ is an algebraic integer, then mx, where $x \in \mathfrak{o}_K$ is in $\mathfrak{o}_K$. Therefore, $\mathfrak{o}_K$ is a $\mathbb{Z}$-module. More abstractly, the ring of integers is defined to be the *integral closure* of $\mathbb{Z}$ in K.

The field of fractions of $\mathfrak{o}_K$ is precisely K. Indeed, since K has no zero divisors, then $\mathfrak{o}_K$ has no zero divisors. Thus, the field of fractions of $\mathfrak{o}_K$ is contained in K. Let $k \in K$ and $m_k(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$ be the minimal polynomial of k. We may take $a_r \in \mathbb{Z}$, or else we can multiply the coefficients by the denominator of $a_r \in \mathbb{Q}$. Then $a_r k$ satisfies the polynomial $x^r + a_r a_{r-1} x^{r-1} + \cdots a_r^{n-1} a_1 x + a_r^n a_0$, and is hence an algebraic integer. Since $a_n \in \mathbb{Z}$, then k is a quotient of algebraic integers. Already, from this fact, it is clear that $\mathfrak{o}_K$ is an important subring that one should consider when studying an algebraic number field K.

Since $\mathbb{Z}$ is a principal ideal domain and $K/\mathbb{Q}$ is separable, then every finitely generated $\mathfrak{o}_K$-module of K is a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}]$. Therefore, $\mathfrak{o}_K$ is free $\mathbb{Z}$-module of K and moreover, every finitely generated $\mathfrak{o}_K$-module admits a $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_n$. See Neukirch [23], Chapter I, Section 2, Proposition 2.10 for a proof of this fact. As a consequence, we may speak of an *integral basis of* $\mathfrak{o}_K$ *over* $\mathbb{Z}$. That is, there exists $\omega_1, \ldots, \omega_n \in \mathfrak{o}_K$ such that every $\beta \in \mathfrak{o}_K$ can be written uniquely as a linear combination $\beta = m_1\omega_1 + m_2\omega_2 + \cdots m_n\omega_2$, where

$n = [K : \mathbb{Q}]$ and $m_1, m_2, \cdots, m_n \in \mathbb{Z}$. Note well that an integral basis, or basis for an ideal $\mathfrak{a}$ of $\mathfrak{o}_K$, is a basis of K over $\mathbb{Q}$ because the length of the basis always agrees with $[K : \mathbb{Q}]$.

Definition 2.1.2. Let K be a number field. The *discriminant* of a basis $\alpha_1, \ldots, \alpha_n$ of K over $\mathbb{Q}$ is defined by

$$d(\alpha_1, \ldots, \alpha_n) = \det((\sigma_i \alpha_j))^2,$$

where σ_i, $i = 1 \ldots, n$ varies over the $\mathbb{Q}$-embeddings $K \to \overline{\mathbb{Q}}$.

If we denote the primitive element of K by α, then we know the set $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ constitutes a basis of K over $\mathbb{Q}$. Letting $\sigma_i \alpha = \alpha_i$, we see that the discriminant of this basis is the determinant of the Vandermonde matrix, which is given by

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{vmatrix} = \prod_{i<j} (\alpha_i - \alpha_j)^2.$$

Suppose that $\alpha_1, \ldots, \alpha_n$ are $\beta_1, \ldots, \beta_n$ are two different integral basis of a finitely generated $\mathfrak{o}_K$-submodule of K. Then the change of basis matrix, T, between these two basis as free $\mathbb{Z}$-modules necessarily has integer coefficients, and hence $\det(T) = \pm 1$. Using the fact that an embedding is additive, multiplicative, and fixes $\mathbb{Q}$, we obtain

$$d(\beta_1, \ldots, \beta_n) = \det(T)^2 d(\alpha_1, \ldots, \alpha_n) = d(\alpha_1, \ldots, \alpha_n).$$

Therefore, we define the *discriminant* of an algebraic number field K to be $d_K = d(\mathfrak{o}_K) = d(\alpha_1, \ldots, \alpha_n)$. Now, suppose one takes two different finitely generated $\mathfrak{o}_K$ submodules I and J of K, such that $I \subseteq J$. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be the integral basis of I and J, respectively. In this case, the change of a basis matrix between the two basis' does not necessarily have integral entries. The matrix may contain rational entries. This is because it only makes sense to consider a change of basis when treating both basis' as basis' of K over $\mathbb{Q}$. The determinant of the base change matrix is precisely $(J : I)$. Thus, we obtain the following result about the discriminant:

$$d(I) = (J : I)^2 d(J).$$

Definition 2.1.3. The *trace* and *norm* of an element $x \in K$ are defined to be the trace and determinant, respectively, of the endomorphism

$$T_x : K \to K, \quad T_x(\alpha) = x\alpha$$

of K as $\mathbb{Q}$-vector space. That is,

$$\mathrm{Tr}_{K/\mathbb{Q}}(x) = \mathrm{Tr}(T_x), \quad N_{K:\mathbb{Q}}(x) = \det(T_x).$$

Furthermore, the trace and norm have the following Galois theoretic interpretation:

$$N_{K/\mathbb{Q}}(x) = \prod_{\sigma} \sigma(x) \quad \text{and} \quad \mathrm{Tr}_{K/\mathbb{Q}}(x) = \sum_{\sigma} \sigma(x), \tag{2.1}$$

where σ varies over the different $\mathbb{Q}$-embeddings of K into the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. See Neukirch [23], Chapter 2, Proposition 2.6, for a proof of this fact. If

$$m_x(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0 \in \mathbb{Q}[t]$$

is the minimal polynomial of x over $\mathbb{Q}$, then $N_{K/\mathbb{Q}}(x) = (-1)^n a_0^{n/d}$ and $\mathrm{Tr}_{K/\mathbb{Q}}(x) = -\frac{n}{d} a_{d-1}$, where $n = [K : \mathbb{Q}]$. If $x \in \mathfrak{o}_K$, then both the trace and norm of x lie in $\mathbb{Z}$ because the minimal polynomial of x over $\mathbb{Q}$ is a monic polynomial with coefficients in $\mathbb{Z}$. Let $\alpha_1, \ldots, \alpha_n$ be a basis of K over $\mathbb{Q}$. Then one obtains the following relationship between the discriminant and trace:

$$d(\alpha_1, \ldots, \alpha_n) = \det(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j)).$$

Indeed,

$$\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j) = \sum_k (\sigma_k \alpha_i)(\sigma_k \alpha_j),$$

and hence the matrix $(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j))$ is the product of $(\sigma_k \alpha_i)^t$ and $(\sigma_k \alpha_j)$. Using the fact that the determinant is multiplicative and that the determinant of the transpose of a matrix is the same as the determinant of the matrix, we obtain the desired result.

Proposition 2.1.4. *Let $\alpha_1, \ldots, \alpha_n$ be a basis of K over $\mathbb{Q}$. Then*

$$(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(xy)$$

is a nondegenerate bilinear form on the $\mathbb{Q}$-vector space K. Hence,

$$d(\alpha_1, \ldots, \alpha_n) \neq 0.$$

Proof. Clearly, (x, y) is bilinear. Let θ be the primitive element of K over $\mathbb{Q}$. Let $[K : \mathbb{Q}] = n$. Fix $E := \{1, \theta, \ldots, \theta^{n-1}\}$ as a ordered basis for K over $\mathbb{Q}$. Let

$$M = (\mathrm{Tr}_{K/\mathbb{Q}}(\theta^i \theta^j))_{i,j=0,\ldots,n-1}.$$

As such,

$$(x, y) = [x]_E \, M \, [y]_E^t \, ,$$

where $[x]_E$ is the vector corresponding to the coefficients of x written with respect to the ordered basis E and where $[y]_E$ is defined similarly. Let $\theta_i = \sigma_i \theta$. Since $K/\mathbb{Q}$ is separable, then

$$\det(M) = d(1, \theta, \ldots, \theta^{n-1}) = \prod_{i<j} (\theta_i - \theta_j)^2 \neq 0$$

, and hence (x, y) is nondegenerate. The final statement follows from the relation

$$d(\alpha_1, \ldots, \alpha_n) = \det(\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j)).$$

$\square$

Theorem 2.1.5. *The ring $\mathfrak{o}_K$ is noetherian, integrally closed, and every prime ideal $\mathfrak{p}$ is a maximal ideal.*

Proof. Since every ideal of $\rtimes_K$ is a finitely generated $\mathbb{Z}$-module, then every ideal is a finitely generated $\mathfrak{o}_K$-module. This is one of the many equivalent conditions of being noetherian. Since $\mathfrak{o}_K$ is the integral closure of $\mathbb{Z}$ in K, then $\mathfrak{o}_K$ is integrally closed. The ideal $\mathfrak{p} \cap \mathbb{Z}$ must be prime ideal in $\mathbb{Z}$. Let $\mathfrak{p} \cap \mathbb{Z} = (p)$. Suppose $y \in \mathfrak{p}$ and $y \neq 0$. Then the constant term of the minimal monic polynomial for y over $\mathbb{Z}$ must be nonzero and divisible by p. Therefore, the integral domain $\mathfrak{o}_K/\mathfrak{p}$ is an algebraic extension of $\mathbb{Z}/p\mathbb{Z}$. As such, $\mathfrak{o}_K/\mathfrak{p}$ is a field, and hence $\mathfrak{p}$ is maximal. $\square$

Definition 2.1.6. A noetherian, integrally closed integral domain in which every nonzero prime ideal is maximal is called a *Dedekind* domain.

Therefore, the above theorem tells us that $\mathfrak{o}_K$ is a Dedekind domain. Another equivalent definition of a Dedekind domain is the following: A ring R is a Dedekind domain if for each maximal ideal M of a ring R, the localization ring $A_M = \{a/n : a \in R, m \in A - M\}$ is a discrete valuation ring, and each nonzero element of A is contained in only finitely many prime ideals. See Appendix B, Theorem B-5, in Ramakrishnan and Valenza [24]. Recall that a discrete valuation ring is a principal ideal domain having exactly one nonzero prime ideal. Recall that the p-adic integers, denoted $\mathbb{Z}_p$, are a discrete valuation ring with (p) as the unique prime ideal. The localization ring $\mathbb{Z}_{(p)} = \{a/n : a \in \mathbb{Z}, m \in \mathbb{Z} - (p)\}$ is similar to $\mathbb{Z}_p$. This is because completion and localization are similar operations.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $\mathfrak{o}_K$. We say $\mathfrak{a}|\mathfrak{b}$ if $\mathfrak{b} \subseteq \mathfrak{a}$. The sum of ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined to be

$$\mathfrak{a} + \mathfrak{b} = \{a + b : a \in \mathfrak{a},\ b \in \mathfrak{b}\}.$$

This the smallest ideal containing $\mathfrak{a}$ and $\mathfrak{b}$. By our divisibility relation, we can interpret this as the greatest common divisor of $\mathfrak{a}$ and $\mathfrak{b}$. We define the product $\mathfrak{a}$ and $\mathfrak{b}$ by

$$\mathfrak{a}\mathfrak{b} = \{\sum_i a_i b_i : a_i \in \mathfrak{a},\ b_i \in \mathfrak{b}\}.$$

Theorem 2.1.7. *Let K be a number field. Every ideal $\mathfrak{a}$ of $\mathfrak{o}_K$ different from (0) or (1) admits a factorization*

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_r$$

into nonzero prime ideals $\mathfrak{p}_i$ of $\mathfrak{o}_K$, which is unique up to the order of its factors.

Proof. See Neukirch [23], Chapter 1, Theorem 3.3. □

Definition 2.1.8. A *fractional ideal* of K is a finitely generated $\mathfrak{o}_K$ submodule $\mathfrak{a} \neq 0$ of K.

Proposition 2.1.9. *The fractional ideals of K form an abelian group under multiplication of ideals. We denote this group by J_K. The identity element is $(1) = \mathfrak{o}_K$ and the inverse of a* *fractional ideal* $\mathfrak{a}$ *is given by*

$$\mathfrak{a}^{-1} = \{x \in K | x\mathfrak{a} \subseteq \mathfrak{o}_K\}.$$

Proof. See Neukirch, Chapter 3, Proposition 3.8. □

Combining the previous theorem and proposition, we obtain that every fractional ideal $\mathfrak{a}$ admits a unique representation, up to order, as a product

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_p},$$

where $\nu_p \in \mathbb{Z}$ and $\nu_p = 0$ for all but finitely many $\mathfrak{p}$. As such J_K is a free abelian group on the set of nonzero prime ideals $\mathfrak{p}$ of $\mathfrak{o}_K$. Denote by P_K the group of principal ideals of K. They form a subgroup of J_K and the quotient group

$$Cl_K = J_K/P_K,$$

is called the ideal class group. The units in $\mathfrak{o}_K$, denoted by $\mathfrak{o}_K^\times$, fits into the following exact sequence

$$1 \longrightarrow \mathfrak{o}_K^\times \to K^* \longrightarrow J_K \longrightarrow Cl_K \longrightarrow 1.$$

Denote by h_K the order of Cl_K. One typically uses Minkowski lattice theory to show that h_k is finite. We direct the reader to Neukirch [23], Chapter 6.

Definition 2.1.10. The *inverse different* of K is defined to be the set

$$\mathfrak{D}_K^{-1} = \{x \in K | \mathrm{Tr}_{K/\mathbb{Q}}(xy) \in \mathbb{Z},\ \forall y \in \mathfrak{o}_K\}.$$

One can show that the inverse different of K is a proper $\mathfrak{o}_K$ submodule of K containing $\mathfrak{o}_K$. As such, $\mathfrak{D}_K$ is a fractional ideal.

When referring to the norm of an ideal of a number field K, one typically distinguishes between the relative and absolute norm. The relative norm of a fractional ideal $\mathfrak{a}$ of K is denoted $N_{K/\mathbb{Q}}(\mathfrak{a})$, and is defined to be the image of $\mathfrak{a}$ under the norm map introduced above. The absolute norm of an ideal $\mathfrak{a}$ of $\mathfrak{o}_K$ is denoted $|N_{K/\mathbb{Q}}|(\mathfrak{a})$, and is defined to be $[\mathfrak{o}_K : \mathfrak{a}\mathfrak{o}_K]$. In the case the ideal $\mathfrak{a}$ is principal, say generated by a, then $|N_{K/\mathbb{Q}}|(\mathfrak{a}) = |N_{K/\mathbb{Q}}(a)|$. Indeed, let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be an integral basis of $\mathfrak{o}_K$. Then $a\alpha_1, a\alpha_2, \cdots, a\alpha_n$ is an integral basis of $\mathfrak{a} = a\mathfrak{o}_K$. Let A be the transition matrix, as basis' of K over $\mathbb{Q}$, from the basis $a\alpha_i$ to α_i. As stated above, $|\det(A)| = (\mathfrak{o}_k : (a))$. By definition, $\det(A) = N_{K/\mathbb{Q}}(a)$. Therefore, $|N_{K/\mathbb{Q}}|(\mathfrak{a}) = |N_{K/\mathbb{Q}}(a)|$. As an application of the Chinese remainder theorem, one can show that for $\alpha = \mathfrak{p}_1^{\nu_1} \cdots \mathfrak{p}_r^{\nu_r}$, $|N_{K/\mathbb{Q}}|(\alpha) = |N_{K/\mathbb{Q}}|(\mathfrak{p}_1)^{\nu_1} \cdots |N_{K/\mathbb{Q}}|(\mathfrak{p}_r)^{\nu_r}$. This implies the multiplicativity of the absolute norm. And hence, we can extend the absolute norm to fractional ideals of K. Although, not trivially, one can show that $N_{K/\mathbb{Q}}(\mathfrak{D}_K) = d_K$, where d_K is the ideal of $\mathbb{Z}$ generated by the discriminants of all bases of $K/\mathbb{Q}$ which are contained in $\mathfrak{o}_K$. See Neukirch, Chapter 3, Theorem 2.9 for a proof of this. We call d_K the relative discriminant. The discriminant in Proposition 2.1.4 will henceforth be denoted $|d_K|$ and called the absolute discriminant. See Chapter 3, Section 2, in Neukirch [23], for more information about the different and discriminant.

Since $\mathfrak{o}_K$ is a Dedekind domain, then the localization ring $\mathfrak{o}_\mathfrak{p} = \{a/b : a \in \mathfrak{o}_K,\ b \in \mathfrak{o}_K - \mathfrak{p}\}$ is a discrete valuation ring with the unique prime ideal $\mathfrak{p}\mathfrak{o}_\mathfrak{p}$. We call an element π a *uniformizing parameter* if π generates the unique prime ideal $\mathfrak{p}\mathfrak{o}_\mathfrak{p}$. Every element $x \in K$ can be expressed as $\pi^\nu \cdot \frac{a}{b}$, where $a, b \in \mathfrak{o} - \mathfrak{p}$; define $|x|_\mathfrak{p} = (N\mathfrak{p})^{-\nu}$. We can extend $|\cdot|_\mathfrak{p}$ to K, the field of fractions of $\mathfrak{o}_K$, in the obvious way. It follows that $|x|_\mathfrak{p} = 0$ if and only if $x = 0$ and that $|x|_\mathfrak{p}$ is multiplicative. Furthermore, it is not difficult to show that $|x+y|_\mathfrak{p} \leq \max\{|x|_\mathfrak{p}, |y|_\mathfrak{p}\}$. As such, we call $|\cdot|_\mathfrak{p}$ the $\mathfrak{p}$-adic absolute value. The $\mathfrak{p}$-adic absolute value is non-Archimedean. This absolute value subsumes the definition of the p-adic absolute value in the case $K = \mathbb{Q}$. Let $K_\mathfrak{p}$ be the completion of K with respect to the $\mathfrak{p}$-adic absolute value. That is, $K_\mathfrak{p}$ is the equivalence class of Cauchy sequences of K with respect to the $\mathfrak{p}$-adic norm. Also, note that K clearly embeds into the completion $K_\mathfrak{p}$.

Equivalence classes of absolute values of K are called *places* of K. Two absolute values, $|\cdot|_1$ and $|\cdot|_2$, on K are equivalent if $|\cdot|_1 = |\cdot|_2^t$ for some positive constant t. It can be shown that every non-Archimedean equivalence class of places of K is represented by a $\mathfrak{p}$-adic absolute value. Non-Archimedean places of K are referred to as *finite places.* Archimedean places of K are induced by $\mathbb{Q}$-embeddings of K into $\mathbb{C}$. The absolute value corresponding to an embedding is the composition of the embedding with the complex absolute value. Since $|z| = |\overline{z}|$, it is then immediately clear why complex conjugate embedding induce equivalent norms. The Archimedean places of K are called *infinite places.* The completion of K with respect to an absolute value induced by a complex embedding is precisely $\mathbb{C}$. The completion of K with respect to an absolute value induced by real embedding is precisely $\mathbb{R}$. As such, for the infinite places, we distinguish between real and complex places. The generalized form of Ostrowski's theorem, which we stated for $\mathbb{Q}$ in Chapter 1, states that every place of a number field K is either represented by a $\mathfrak{p}$-adic absolute value or an absolute value corresponding to a real or complex embedding. We will discuss the structure of the completions of K in the following section on local fields (i.e. non-discrete locally compact fields). We direct the reader to Keith Conrad's article, *Otrowski for Number Fields* [5], for a proof the Otrowski's theorem for number fields.

2.2 **Local Fields**

Theorem 2.2.1. *Let k be a locally compact non-discrete field (local field). Then*

(i) If $\operatorname{char}(k) = 0$, then k is $\mathbb{R}$ or $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for some rational prime p.

(ii) If $\operatorname{char}(k) = p > 0$, then k is ultrametric and isomorphic to a field of formal power series in one variable over a finite field.

Proof. We will provide a detailed sketch of the proof. In doing so, we will establish important facts about local fields, which play an important role in Tate's thesis. See Chapter 4, Sections 1,2,and 3, of Ramakrishnan and Valenza [24] for a full proof.

Since k is a locally compact abelian group under addition, then k admits an additive bi-invariant Haar measure, μ. Let X be a Borel set of k. For all $\alpha \in k^*$, we have that $\alpha \cdot X$ is a Borel set of k because multiplication is a one-to-one and continuous and hence, an open mapping. Since left multiplication by an element α is an automorphism of the additive group of k, then $\mu \circ \alpha$ is a bi-invariant Haar measure. The uniqueness of the Haar measure implies that there exists a positive real constant c_α such that $\mu \circ \alpha = c_\alpha \mu$. We write $c_\alpha = \operatorname{mod}_k(\alpha)$ and call $\operatorname{mod}_k(\alpha)$ the *module of automorphism* of $\alpha \in K^*$. We can extend this construction to $0 \in k$ by defining $\operatorname{mod}_k(0) = 0$. Let $\alpha, \beta \in k$. Then $\operatorname{mod}_k(\alpha \cdot \beta) = \operatorname{mod}_k(\alpha) \operatorname{mod}_k(\beta)$ by associativity of multiplication. One can show that $\operatorname{mod}_k : k \to \mathbb{R}_+$ is a continuous homomorphism and moreover, an open homomorphism of $k^\times$ onto a closed subgroup Γ of $\mathbb{R}_+^\times$. Furthermore, one can show that if k is non-discrete, then mod_k is unbounded and consequently, that k it not compact. In addition, one obtains the following result, which we state as a proposition:

Proposition 2.2.2. *The sets $B_m = \{\alpha \in k : \mathrm{mod}_k(\alpha) \leq m\}$ with $m \in \mathbb{Z}_+$ constitute a local base at zero for the topology of k.*

Proof. See Ramikrishnan and Valenza [24], Chapter 4, Proposition 4-7. $\square$

As a corollary, one obtains that $\lim_{n\to\infty} a^n$ if and only if $\mod_k(a) < 1$. This is the first step along the journey to classifying non-discrete locally compact fields. Next, one obtains that there exists a positive real constant $A \geq 1$ such that

$$\mathrm{mod}_k(\alpha \cdot \beta) \leq A \cdot \sup\{\mathrm{mod}_k(\alpha), \mathrm{mod}_k(\beta)\}, \tag{2.2}$$

and that if $A = 1$, then $\Gamma = \mathrm{mod}(k^*)$ is discrete in $\mathbb{R}_+$. If $A = 1$, then k is ultrametric. Furthermore, if mod_k is bounded on the prime ring $\{n \cdot 1_k : n \in \mathbb{N}\}$, then $\mathrm{mod}_k \leq 1$ on the prime ring and k is ultrametric. As it turns out, since mod_k is multiplicative function on $\mathbb{N} \cong \{n \cdot 1_k : n \in \mathbb{N}\}$ and satisfies (2.2), one can show that **either** $\mathrm{mod}_k(m) \leq 1$ **for all** m **(ultrametric), or** $\mathrm{mod}_k(m) = m^\lambda$ **for some positive constant** λ **and for all** m. Suppose $\mathrm{char}(k) = p$. Then $\mathrm{mod}_k(p) = 0$, which does not equal p^λ for any positive real. Therefore, if k has positive characteristic, then $\mathrm{mod}_k(m) \leq 1$ for all m.

A *topological vector space* is a topological group such that scalar multiplication is a continuous mapping. Let V be a topological vector space over a non-discrete locally compact field k, and let W be a finite-dimensional subspace of V of dimension n. Fix a basis $w_1, \ldots, w_n$ of W. Let ρ_j be the continuous projection map from the jth component of k^n to k. Also, let ψ_j be the map from k to W defined by $a \mapsto aw_j$. This map is continuous because W is a topological vector space, as inherited from V. Define $\phi_j = \psi_j \circ \rho_j$, which is continuous because a composition of continuous maps is continuous. Let $\phi = \sum_{j=1}^n \phi_j$. That is,

$$\begin{aligned} \phi : k^n &\longrightarrow W \\ (a_j) &\mapsto \sum a_j w_j. \end{aligned}$$

The map ϕ is continuous because a sum of continuous functions is continuous. Since both W and k^n are finite topological vector spaces of the same dimension, then ϕ is a continuous isomorphism of topological vector spaces.

Proposition 2.2.3. *Let $k, V, W,$ and ϕ be as above.*

(i) Let U be any open neighborhood of zero in V. Then $W \cap U \neq \{0\}$.

(ii) The mapping ϕ is homeomorphism. Consequently, W admits precisely one structure as a topological vector space k.

(iii) W is closed and locally compact.

(iv) If V is itself locally compact, then V is finite-dimensional over k and $\mathrm{mod}_V(a) = \mathrm{mod}_k(a)^{\dim V}$ for all $a \in V$.

Proof. See Ramakrishnan and Valenza [24], chapter 4, Proposition 4-13. □

Let k be ultrametric. Then $\mathrm{mod}_k(m) \leq 1$ for all m and $\{m \cdot \mathbf{1}_k : m \in \mathbb{N}\} \subseteq B_1$. Furthermore, since B_1 is compact, then there exists a limit point a of $\{m \cdot \mathbf{1}_k : m \in \mathbb{N}\}$ (limit point compactness). Thus, for every $\epsilon > 0$, there exists infinitely many m such that $\mathrm{mod}_k(m \cdot \mathbf{1}_k - a) \leq \epsilon$. Let m_1 and m_2 be two such integers. Then by the ultrametric inequality, $\mathrm{mod}((m' - m) \cdot \mathbf{1}_k) \leq \epsilon$. As such, there exists an $n \in \mathbb{N}$ such that $\mathrm{mod}_k(n) < 1$. Let p be the smallest such positive integer. It follows that p must be prime since p is minimal and mod_k is multiplicative. By the ultrametric property $\mathrm{mod}_k(np) < 1$ for all $n \in \mathbb{N}$. Let r be any positive integer less than p. As such, $\mathrm{mod}_k(r) \geq 1$. Then

$$\mathrm{mod}_k(j + mp) \leq \sup\{\mathrm{mod}_k(j), \mathrm{mod}_k(mp)\} = 1.$$

So, if $n \in \mathbb{N}$ is prime to p, then $\mathrm{mod}_k(n) = 1$. Therefore, p is the unique prime such that $\mathrm{mod}_k(p) < 1$. If $\mathrm{char}(k) > 0$, then $\mathrm{mod}_k(\mathrm{char}(k)) = 0$, which implies that $p = \mathrm{char}(k)$. If $\mathrm{char}(k) = 0$, then $\mathrm{mod}_k(p) \neq 0$, so there exists a positive real number t such that $\mathrm{mod}_k(p) = p^{-t}$. We may express every $n \in \mathbb{N}$ as mp^r with $(m, p) = 1$ and hence,

$$\mathrm{mod}_k(n) = \mathrm{mod}_k(mp^r) = \mathrm{mod}_k(m) \bmod (p)^r = p^{-tr} = (p^{-r})^t = |p^r|_p^t = |mp^r|_p^t = |n|_p^t,$$

where $|\cdot|_p$ is the p-adic norm on $\mathbb{Q}$ introduced in the first chapter.

Now, suppose $\mathrm{mod}_k(m) = m^\lambda$ for all m. Then $\mathrm{mod}_k(m) = m^\lambda = |m|^\lambda$ for all $m \in \mathbb{N}$. In general, in the case $\mathrm{char}(k) = 0$, we have

$$\mathrm{mod}_k(n) = |n|_\nu^t,$$

where ν is either a finite prime or the infinite prime. The isomorphism of algebras

$$\mathbb{Z} \to \mathbb{Z} \cdot \mathbf{1}_k$$
$$n \mapsto m \cdot \mathbf{1}_k$$

extends to a unique isomorphism $\mathbb{Q} \to \mathbb{Q} \cdot \mathbf{1}_k \subseteq k$. Since the sets B_t constitute a local base at 0 in k, then the topological structure of $\mathbb{Q} \cdot \mathbf{1}_k \cong \mathbb{Q}$ induced by k is identical to the topological introduced by thte ν-adic norm. Since k is locally compact, then the closure of $\mathbb{Q}$ in k is the completion of $\mathbb{Q}$ with respect to the ν-adic norm, which is $\mathbb{Q}_\nu$. By Proposition 2.2.3, k is finite-dimensional over $\mathbb{Q}_\nu$. If $\nu = \infty$, then k is either $\mathbb{R}$ or $\mathbb{C}$. Otherwise, if $\nu = p$, then k is a finite extension of a p-adic field $\mathbb{Q}_p$. The positive characteristic case requires a more in depth study of ultrametric fields.

Let $\Gamma = \mathrm{mod}_k(k)$. Then Γ is a discrete subgroup of $\mathbb{R}_+^\times$. The set $\mathfrak{o}_k := \{x \in K| \mathrm{mod}_k(x) \leq 1\}$ is the unique maximal compact subring of k. The set $\{x \in K| \mathrm{mod}_k(x) = 1\}$ is group of units in $\mathfrak{o}_k$. We denote this set by $\mathfrak{o}_k^\times$. One can show that k is a discrete valuation ring and that $\mathfrak{p} = \{x \in K| \mathrm{mod}_k(x) < 1\}$ is the unique maximal ideal. Recall that a uniformizing parameter is an element $\pi \in k^*$ such that $\mathfrak{p} = \pi\mathfrak{o}_k$. It can be shown that any uniformizing parameter π is given as any element in k^* such that $\gamma = \mathrm{mod}_k(\pi)$ is the maximal element of Γ less than 1. Furthermore, the residue field $\mathfrak{o}_K/\mathfrak{p}$ is finite. The following short exact sequence of groups splits:

$$1 \longrightarrow \mathfrak{o}_K^\times \to k^* \xrightarrow{\mathrm{mod}_k} \Gamma \longrightarrow 1.$$

As such, every element $a \in k^*$ can be uniquely represented as $u\pi^n$ for some $u \in \mathfrak{o}_K^\times$ and $n \in \mathbb{Z}$. In such a case, we say a has order n.

Proposition 2.2.4. *Assume k is an ultrametric local field. The the following assertions hold:*

(i) Let $\{a_j\}_{j\geq 0}$ be a sequence in k such that $\lim a_j = 0$. Then

$$\sum_{j=0}^{\infty} a_j$$

converges in k.

(ii) Let R be a fixed set of coset representatives of $\mathfrak{o}_K/\mathfrak{p}$ that includes 0. Let $a \in k^*$ have order n. Then

$$a = \sum_{i=n}^{\infty} a_i \pi^i,$$

where $a_i \in R$.

Proof. See Ramakrishnan and Valenza [24], Chapter 4, Proposition 4-17. $\square$

Let k be a ultrametric local field. Let p be the unique prime such that $\text{mod}_k(p) < 1$. Let $q =$Card$(\mathfrak{o}_k/\mathfrak{p})$. Since $p\mathbf{1}_k \in \mathfrak{p}$, then the characteristic of $\mathfrak{o}_k/\mathfrak{p}$ is p, and hence $q = p^r$ for some positive integer r. Since $\mathfrak{o}_K$ is compact, then it is of finite Haar measure. Furthermore, $\mathfrak{o}_K$ is a disjoint union of q additive translates of $\mathfrak{p}$. Therefore, $\mu(\mathfrak{o}_K) = q\mu(\mathfrak{p}) = q\mu(\pi\mathfrak{o}_k)$, which implies $\text{mod}_k(\pi) = q^{-1}$. We call q the module of k. It can be shown that for $a \in \mathfrak{o}_k^\times$, $\lim a^{q^n}$ exists. For $a \in \mathfrak{p}$, $\lim a^{q^n} = 0$. We define $\omega(a) = \lim a^{q^n}$ for $a \in \mathfrak{o}_k$. It is clear that $\omega(ab) = \omega(a)\omega(b)$, and hence that $\omega(a^n) = \omega(a)^n$. Since $(1+\mathfrak{p})^{p^n} \subseteq 1 + \mathfrak{p}^{n+1}$, then $a \in 1+\mathfrak{p}$, or equivalently $a \equiv 1 \bmod \mathfrak{p}$, implies $\omega(a) = 1$. If $\omega(a) = 1$, then $(a-1)^{q^n} \in \mathfrak{p}$, which implies that $a \in 1+\mathfrak{p}$. Also, note that $a^{q-1} \in 1 + \mathfrak{p}$ for all $a \in \mathfrak{o}_k^\times$, so $\omega(a)^{q-1} = 1$ for all $a \in \mathfrak{o}_k^\times$. Let $b \in \mathfrak{o}_k^\times$ be chosen so that its projection generates $(\mathfrak{o}_k/\mathfrak{p})^*$. Put $\nu = \omega(b)$. For any integer n

$$\nu^n = 1 \Leftrightarrow \omega(b^n) = 1 \Leftrightarrow b^n \in 1+\mathfrak{p} \Leftrightarrow n \equiv 0 (\text{mod} q-1).$$

Therefore, ν generates a cyclic group of order $q-1$. Define M^* to be the group of roots of unity in k of order prime to p. Then it can be shown that ω induces an isomorphism $(\mathfrak{o}_k/\mathfrak{p})^* \to M^*$. See Ramakrishnan and Valenza [24], Chapter 4, Proposition 4-19. That is, for all $a \in M^*$, $\omega(a) = a$. Hence $M = M^* \cup \{0\}$ is a complete set of coset representatives for $(\mathfrak{o}_k/\mathfrak{p})$, and the polynomial $x^{q-1} - 1$ splits in k. Let $\overline{M}$ be the algebraic closure of $\mathbb{F}_p$ in k. Let a be a nonzero element in $\overline{M}$. Then a lies in some finite extension of $\mathbb{F}_p$ and so, $a^{p^m - 1} = 1$ for some $m \geq 1$. Thus, the order of a has no factor of p, and $a \in M$. Therefore, M is the algebraic closure of $\mathbb{F}_p$ in k.

Proposition 2.2.5. *Assume that k is an ultrametric local field. Then the following assertions hold:*

(i) Every element of $\mathfrak{p}^n$ with $n \in \mathbb{Z}$ is uniquely expressible as

$$\sum_{j \geq n}^{\infty} a_j \pi^j \ \ (a_j \in M).$$

(ii) $\langle M, + \rangle$ is a subgroup of $\langle k, + \rangle$, and hence a field, if an only if char(k) is positive.

Proof. Part (i) follows directly from Proposition 2.2.4 since M constitutes a complete set of coset representatives for $\mathfrak{o}_K/\mathfrak{p}$. If char$(k)$ is positive, then M is a field. If M is closed under addition, then it must have torsion because there is an injection of M into $\mathfrak{o}_k/\mathfrak{p}$. Hence, k has positive characteristic. □

Let k have positive characteristic. By the above proposition, every element of k can be expressed uniquely as a power series in π with coefficients in M. If k is of positive characteristic, then M is a field and the assignment $\pi \mapsto x$ induces an isomorphism from k to $M((x))$, the field of formal power series in the indeterminate x with coefficients in M. This completes the proof of Theorem 2.2.1.

□

Let K be a number field. Recall that a place of K is defined to be an equivalence classes of non-trivial absolute values on K. Places are either archimedean or non-archimedean (ultrametric). We say a place ν of K lies over a place p of $\mathbb{Q}$ if ν restricts to p on $\mathbb{Q}$. Since K is separable, then $K = \mathbb{Q}(\alpha)$ for some $\alpha \in \overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}_p}$. Let $m_\alpha(x)$ be the minimal polynomial of α over $\mathbb{Q}$ and suppose that $m_\alpha(x) = \prod_{j=1}^{r} m_{\alpha,j}(x)$ is the irreducible factorization of $m_\alpha(x)$ in $\mathbb{Q}_p[x]$. For each j, fix a root α_j of $m_{\alpha,j}(x)$. The following assertions hold:

(i) $K_\nu = \mathbb{Q}_p(\beta)$, where β is a root of $m_\alpha(x)$. As such K_ν is a finite separable extension of $\mathbb{Q}_p$.

(ii) The places ν that lie over p are in bijective correspondence with the embeddings of K into $\overline{\mathbb{Q}_p}$ induced by the assignments $\alpha \mapsto \alpha_j$.

See Ramakrishnan and Valenza [24],Chapter 4, Proposition 4-31 for a proof of this fact.

As such, our analysis of local fields has helped us to understand the completions of of global fields. It can be shown that $\mathfrak{o}_K = \cap_{\mathfrak{p}} \mathfrak{o}_{\mathfrak{p}}$, where $\mathfrak{o}_K$ is the ring of integers in K and $\mathfrak{o}_{\mathfrak{p}} = \{x \in K_\nu : |x|_\nu \leq 1\}$.

We will end this chapter with the following important result.

Proposition 2.2.6. *The algebraic isomorphism* $K \otimes_F F_u \to M = \prod_{\nu|u} K_\nu$ *is in fact a topological isomorphism.*

Proof. See Neukirch [23], Chapter II, Section 8, Proposition 8.3 for a proof of this fact. □

CHAPTER 3
Restricted Direct Topology and the Adeles and Ideles

In this chapter, we will study the topology, dual group, and Haar measure of the restricted direct product. Furthermore, we will develop tools for both the integration and the Fourier transform of functions defined on the restricted direct topology. After developing the theory for the general restricted product in the first section, we will explore in the second section how the construction is used in the number theoretic context. Namely, we will introduce the additive topological group of adeles of a global field K, denoted $\mathbb{A}_K$, and multiplicative topological group of ideles of a global field K, denoted $\mathbb{I}_K$; these will be used extensively in Tate's thesis. While $\mathbb{A}_K^*$ is isomorphic to $\mathbb{I}_K$ as a group, the two are not topologically isomorphic. The adeles and ideles will enable us to do harmonic analysis on a global number field K. In the Pontryagin duality section of chapter 1, specifically Proposition 1.3.4, we showed that the dual group of a discrete group is a compact group. As an example, $\mathbb{Z}$ lies discretely in $\mathbb{R}$ and the dual group of $\mathbb{Z}$ is $\mathbb{R}/\mathbb{Z} \cong S^1$, which is a compact group of $\mathbb{C}$. In a similar fashion, we will see that K embeds discretely in $\mathbb{A}_K$ and is, furthermore, a co-compact subgroup of $\mathbb{A}_K$. In Chapter 4 and specifically the "Adelic Schwartz-Bruhat Functions and the Riemann-Roch Theorem for Number Fields" section, we will prove that the dual group of K is $\mathbb{A}_K/K$. As such, we will obtain the fascinating result that the character group of $\mathbb{Q}$ is $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$. The multiplicative group K^* embeds discretely in $\mathbb{I}_K$, but not co-compactly. Nevertheless, the idele-class group of K ($C_K = \mathbb{I}_K/K^*$) is important because the traditional ideal class group and ray class groups of K are quotients of C_K. Although the quotient group C_K is not compact, we will introduce the subgroup $C_K^1 = \mathbb{I}_K^1/K^*$, which is compact. The multiplicative group $\mathbb{I}_K^1$ is the group of ideles of absolute value 1, where the absolute value on $\mathbb{I}_K$, denoted $|\cdot|_{\mathbb{A}_K}$, is the product of normalized absolute values over the places of K. In order to show that the quotient $\mathbb{I}_K^1/K^*$ makes sense, we will show that all the elements of K^* have absolute value 1. We will also show that $C_K \cong C_K^1 \times \mathbb{R}_+^\times$ for a number field K. This isomorphism will lead to factorization of quasi-characters on C_K, which are sometimes called idele-class characters or Hecke Characters.

3.1 Restricted Direct Topology

Definition 3.1.1. Let $J = \{\nu\}$ be a set of indices for which we are given G_ν, a locally compact group, and let J_∞ be a fixed finite subset of J such that for each $\nu \notin J_\infty$ we are given a compact open subgroup $H_\nu \leq G_\nu$. From Proposition 1.1.21, we know that H_ν is a closed subgroup of G_ν. The *restricted direct product* of G_v with respect to H_v is defined by

$$\prod_{\nu \in J}{}' G_\nu = \{(x_\nu) : x_\nu \in G_\nu \text{ with } x_\nu \in H_\nu \text{ for all but finitely many } \nu\}.$$

We will denote by G the restricted direct product of G_ν with respect to the H_ν. The restricted direct product is a subset of the set-theoretic direct product of the G_v and a subgroup of the group-theoretic product of G_v. The restricted direct product lies somewhere in between the group direct product and the group direct sum (all but finitely many entries are the identity) The topology, which we will call the restricted direct topology on G, **is not** equivalent to the product topology. However, the restricted direct topology will turn out to be quite natural and, furthermore, will induce an equivalent topology on a subgroup $G_s \leq \prod'_{\nu \in J} G_\nu$, which we will define shortly as the product topology on G_S. This fact will enable us to conclude that the restricted direct product is locally compact. Local compactness is essential.

Since the restricted direct product is clearly a group with respect to the componentwise group operation, in order to define the topology it suffices, by homogeneity, to specify a neighborhood base of the identity. We define the neighborhood base at the identity to be:

$$B = \{\prod N_\nu : N_\nu \text{ a neighborhood of } 1 \in G_\nu \text{ and } N_\nu = H_\nu \text{ for all but finitely many } \nu\}.$$

For any $S \subseteq J$, which necessarily contains J_∞, define G_S by

$$G_s = \prod_{\nu \in S} G_\nu \times \prod_{\nu \notin S} H_\nu.$$

We have assumed $J_\infty \subseteq S$ because we have not necessarily required that there exists a H_ν for those $\nu \in J_\infty$. By Proposition 1.1.22, $G_s = \prod_{\nu \in S} G_\nu \times \prod_{\nu \notin S} H_\nu$ is locally compact

Proposition 3.1.2. *G_S is an open subgroup of G and the product topology on G_S is identical to the subspace topology induced by restricted direct topology defined above.*

Proof. Since G_ν and H_ν are open neighborhoods of the identity, then G_S is an open subgroup of G in the restricted direct topology. A neighborhood basis of the identity of the product topology of G_S consists of sets of the form $\prod_{\nu \in S} V_\nu \times \prod_{\nu \notin S} X_\nu$ where V_ν is an open neighborhood of the identity in G_ν, where X_ν is an open neighborhood of the identity in H_ν, and where $X_\nu = H_\nu$ for all but finitely many indices. Such a set also is clearly open in the subspace topology of G_S induced by the restricted direct topology on G. A neighborhood basis of the identity of the subspace topology of G_S, induced by the restricted direct topology on G, consists of sets of the form $\prod_\nu N_\nu \cap G_S$, where N_ν is a neighborhood of the identity in G_ν and $N_\nu = H_\nu$ for all but finitely many ν. Let U be a basic open neighborhood of the identity in the subspace restricted direct topology on G_S. Let S' be the set of indices such that $N_\nu \neq H_\nu$. Then

$$\begin{aligned} U = \prod_\nu N_\nu \cap G_S &= \left(\prod_{\nu \in S'} N_\nu \times \prod_{\nu \notin S'} H_\nu\right) \cap \left(\prod_{\nu \in S} G_\nu \times \prod_{\nu \notin S} H_\nu\right) \\ &= \left(\prod_{\nu \in S \cap S'} N_\nu \times \prod_{\nu \in S \cap S'^c} H_\nu\right) \times \left(\prod_{\nu \in S' \cap S^c} N_\nu \cap H_\nu \times \prod_{\nu \notin S'} H_\nu\right). \end{aligned}$$

For all $\nu \in S' \cap S^c$, the neighborhood $N_\nu \cap H_\nu$ of the identity in H_ν is an open neighborhood of the identity in H_ν because N_ν is an open neighborhood of the identity in G_ν. Also, H_ν is an open subset of G_ν containing the identity, so for all $\nu \in S \cap S'^c$ we have that H_ν is an open neighborhood of the identity in G_ν. As such, U is open in the product topology of G_S. $\square$

Corollary 3.1.3. *G is locally compact.*

Proof. Since G_S is locally compact in the product topology, then it is locally compact in the restricted direct topology, since the two topologies are equivalent. Furthermore, every $x \in G$

is contained in some G_S for an appropriate set S containing J_∞. Therefore, every element $x \in G$ admits a compact neighborhood via its inclusion in some appropriate G_S. $\square$

Proposition 3.1.4. *A subset Y of G has compact closure if and only if $Y \subseteq \prod K_\nu$, for some family of compact subsets $K_\nu \subseteq G_v$, such that $K_\nu = H_\nu$ for all but finitely many indices ν.*

Proof. First, assume that Y has compact closure. Let K be the closure of Y. Since subsets of the form G_S cover G and thus K, then a finite number of them cover K, say $S_1, S_2, \ldots, S_n$. Let $S' = \cup_{i=1}^{n} S_i$. Then $G_{S'}$ covers K. Let ρ_ν denote the projection of G onto G_ν. The projection map only need be continuous in the product topology. Since K is a subset of $G_{S'}$, which has two equivalent topologies– one of which is the product topology, for which ρ_v is continuous–then $\rho_\nu(K)$ is compact in G_ν, and $\rho_\nu(K) = H_\nu$ for all but finitely many indices ν. Let $K_v = \rho_\nu(K)$. Therefore, K, and hence Y, is contained in $\prod K_\nu$. Now, assume that $Y \subseteq \prod K_\nu$ for K_ν. Let C be the closure of Y, which is necessarily the smallest closed set containing Y. Since $\prod K_\nu$ is a closed set containing Y, then $C \subseteq \prod K_\nu$, which then implies that C is compact. $\square$

There exists a topological embedding of $G_\nu \longrightarrow G$ given by

$$x \longmapsto (\ldots, 1, 1, x, 1, 1, \ldots),$$

where the x is in the νth component. Let $S = \{\nu\}$ and consider the open, and hence closed, subgroup G_S of G. The image of G_ν under this embedding lies in G_S and is a closed subgroup of G_S in the product topology, and hence the restricted direct topology. Therefore, G_ν can be identified with a closed subgroup of G.

In the algebraic number theory setting, which we will discuss formally in the next section, we will define the adeles of a global field K, denoted $\mathbb{A}_K$, to be the restricted direct product of the locally compact and complete groups K_ν, corresponding to the completion of a global field K at the νth place of K, with respect to $\mathfrak{o}_\nu$, the compact-open ring of integers of K_ν. We will define the ideles of a global field K, denoted $\mathbb{I}_K$, to be the restricted direct product of the locally compact and complete groups K_ν^*, corresponding to the multiplicative group of the completion of a global field K at the νth place of K, with respect to $\mathfrak{o}_\nu^\times$, the compact-open group of units of $\mathfrak{o}_\nu$. As such, by the above argument, for all places ν of K, K_ν can be identified with a closed subgroup of $\mathbb{A}_K$, and K_ν^* can be identified with a closed subgroup of $\mathbb{I}_K$.

3.1.1 Restricted Direct Quasi-Characters and Dual Group

Now we will investigate the group of quasi-characters, $\mathrm{Hom}_{\mathrm{Cont}}(G, \mathbb{C}^\times)$, of the restricted direct product G. For $y \in G$, let y_ν be the projection onto the factor G_ν, which may be identified with a closed subgroup of G.

Lemma 3.1.5. *Let $\chi \in \mathrm{Hom}_{Cont}(G, \mathbb{C}^\times)$. Then χ is trivial on all but finitely many H_ν. Therefore, for $y \in G$, $\chi(y_\nu) = 1$ for all but finitely many ν, and*

$$\chi(y) = \prod_\nu \chi(y_\nu).$$

Proof. Let U be a neighborhood of 1 in $\mathbb{C}^\times$ that contains no subgroups of $\mathbb{C}^\times$ besides the trivial subgroup. Since χ is continuous, then there exists an open neighborhood, V, of the identity of G, such that $\chi(V) \subseteq U$. We know that open neighborhoods of the identity in the restricted direct topology are sets of the form $\prod_\nu N_\nu$, where N_ν is a neighborhood of the identity in G_ν, and where $N_\nu = H_\nu$ for all ν lying outside some finite subset S containing J_∞. Let $V = \prod_\nu N_\nu = \prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} H_\nu$. Then

$$\chi\left(\prod_{\nu \notin S} H_\nu\right) \subseteq U,$$

where we are identifying $\prod_{\nu \in S} 1 \times \prod_{\nu \notin S} H_\nu$ with $\prod_{\nu \notin S} H_\nu$, which is contained in V. Since $\prod_{\nu \notin S} H_\nu$ is a subgroup of G and χ is a homomorphism, then $\chi(\prod_{\nu \notin S} H_\nu)$ is a subgroup of U. Therefore,

$$\chi\left(\prod_{\nu \notin S} H_\nu\right) = \{1\},$$

since the only subgroup of U is the trivial subgroup. Hence, $\chi(H_\nu) = \{1\}$ for all $\nu \notin S$. Given any $y \in G$, we can factor y into $y_1 y_2 y_3$, where y_1 is a finite product of the projections of y that lies outside any H_ν, and where y_2 is a finite product of the projections of y that lie in some H_ν for $\nu \in S$, and where y_3 is a product of the projections of y, all of which lie in H_ν for $\nu \notin S$. Therefore, χ is trivial on all but finitely many projections of y and $\chi(y) = \prod_\nu \chi(y_\nu)$. $\square$

Lemma 3.1.6. *For each ν let $\chi_\nu \in \mathrm{Hom}_{Cont}(G_\nu, \mathbb{C}^\times)$ and $\chi_\nu|_{H_\nu} = 1$ for all but finitely many indices ν. Then we have that $\chi = \prod_\nu \chi_\nu \in \mathrm{Hom}_{Cont}(G, \mathbb{C}^\times)$.*

Proof. Let S be a finite set of indices such that $\chi_\nu|_{H_\nu} = 1$ for all $\nu \notin S$. Let m be the cardinality of S. Since $y = (y_\nu)$, where $y_\nu \in H_\nu$ for all but finitely many ν and $\chi_\nu|_{H_\nu} = 1$ for all ν outside S, then the product $\prod_\nu \chi_\nu$ is a well-defined quasi-character. Let U be a neighborhood of the 1 in $\mathbb{C}^\times$. By Proposition 1.1.9 we can choose a neighborhood V of the identity in $\mathbb{C}^\times$ so that $V^{(m)} \subseteq U$. Since χ_ν is a continuous quasi-character of G_ν, then for each $\nu \in S$, there exists a neighborhood N_ν of the identity in G_ν such that $\chi_\nu(N_\nu) \subseteq V$. Then

$$\prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} H_\nu$$

is a neighborhood of the identity in G such that

$$\chi(\prod_{\nu \in S} N_\nu \times \prod_{\nu \notin S} H_\nu) = \prod_{\nu \in S} \chi_\nu(N_\nu) \prod_{\nu \notin S} \chi_\nu(H_\nu) = \prod_{\nu \in S} \chi_\nu(N_\nu) \subseteq V^{(m)} \subseteq U.$$

Therefore, χ is continuous. $\square$

Restricting our attention to abelian groups G_ν, let us consider the dual group, $\mathrm{Hom}_{\mathrm{Cont}}(G, S^1)$, of G. By Proposition 1.3.4, we know that the dual group of G is locally compact because G is locally compact. In the following theorem we will prove that the dual group of a restricted direct product of G_ν, with respect to H_ν, is the restricted direct product of $\hat{G_\nu}$ with respect to $K(G_\nu, H_\nu)$, which is the set of characters of G_ν that restrict to the identity on H_ν.

Theorem 3.1.7. *Let G be the restricted direct product of locally compact abelian groups G_ν with respect to compact-open subgroups H_ν. As topological groups, we have that*

$$\hat{G} \cong {\prod}' \hat{G}_\nu,$$

where the restricted direct product on the right is taken with respect to subgroups defined by

$$K(G_\nu, H_\nu) = \{\chi_\nu \in \hat{G}_\nu : \chi_\nu|_{H_\nu} = 1\}$$

for $\nu \notin J_\infty$. This subgroup traditionally is denoted $H_\nu^\perp$.

Proof. We will begin by showing that $K(G_\nu, H_\nu)$ is a compact-open subgroup of $\hat{G}_\nu$. It is clear that $K(G_\nu, H_\nu)$ is a subgroup of G_ν. Let U be a neighborhood of 1 in $\mathbb{C}^\times$ that contains no other subgroup besides the trivial subgroup. Recalling the definition of the compact-open topology on the dual of an abelian topological group 1.1, consider the neighborhood of the trivial character on G_ν defined by

$$W(H_\nu, U) = \{\chi \in \hat{G}_\nu : \chi(H_\nu) \subseteq U\}.$$

Since $\chi(H_\nu)$ is a subgroup of U, then $\chi(H_\nu) = \{1\}$, and hence

$$W(H_\nu, U) = K(G_\nu, H_\nu).$$

This shows that $K(G_\nu, H_\nu)$ is an open subgroup of $\hat{G}_\nu$. As such, $K(G_\nu, H_\nu)$ is also a closed subgroup of $\hat{G}_\nu$ (Proposition 1.1.9). If $\chi \in K(G_\nu, H_\nu)$, then χ factors through the quotient group G_ν/H_ν. In the first chapter, specifically in Proposition 1.1.13, we established that for a topological group G, and a normal subgroup H of G, the quotient space G/H is discrete if H is open. Since G_ν is abelian and H_ν is open, then G_ν/H_ν is a discrete group. Let $\rho_{H_\nu} : G_\nu \to G_\nu/H_\nu$ be the projection map, which is both **continuous and open** by Proposition 1.1.13. Let $\chi \in K(G_\nu, H_\nu)$. Then the following diagram commutes:

$$\begin{tikzcd}
G_\nu \arrow[rr, "\chi"] \arrow[dr, "\rho_\nu"'] & & S^1 \\
& G_\nu/H_\nu \arrow[ur, "\chi \circ \rho_\nu"'] &
\end{tikzcd}$$

The map $\chi \mapsto \chi \circ \rho_{H_\nu}$ defines a topological isomorphism between $K(G_\nu, H_\nu)$ and $\widehat{G_\nu/H_\nu}$. By Proposition 1.3.4, we have that $\widehat{G_\nu/H_\nu}$ is compact, since G_ν/H_ν is discrete. Therefore, $K(G_\nu, H_\nu)$ is a compact subgroup of $\hat{G}_\nu$.

Now it makes sense to define the restricted direct product of $\hat{G}_\nu$ with respect to $K(G_\nu, H_\nu)$ for $\nu \notin J_\infty$. Consider the mapping

$$\begin{aligned}\prod{}' \hat{G}_\nu &\xrightarrow{\phi} \hat{G} \\ (\chi_\nu) &\mapsto \prod \chi_\nu.\end{aligned}$$

By the previous two lemmas we know that ϕ is an algebraic isomorphism. We only need to show that the map ϕ is bicontinuous at the identity. Let U be a neighborhood of 1 in $\mathbb{C}^\times$ and let K be a compact neighborhood of the identity in G. By Proposition 3.1.4, $K = \prod K_\nu$, where K_ν is a compact neighborhood of the identity of G_ν, and where $K_\nu = H_\nu$ for all but finitely many indices ν. Let χ in $W(K, U)$; that is, $\chi(\prod K_\nu) \subseteq U$. Now we will construct an open neighborhood, N, of the trivial character in the restricted direct product of the dual groups such that $\phi(N) \subseteq W(K, U)$. By the first lemma above, we know that $\chi(y_\nu) = 1$ for all but finitely many ν. As such, let S be the set of indices such that $\chi|_{K_\nu} \neq 1$, and let $m = \mathrm{Card}(s)$. Let V be a neighborhood of 1 in $\mathbb{C}^\times$ such that $V^{(m)} \subseteq U$. Since $W(K_\nu, V)$ is an open neighborhood of the trivial character in $\hat{G}_\nu$, then the set

$$N = \prod_{\nu \in S} W(K_\nu, V) \times \prod_{\nu \notin S} K(G_\nu, H_\nu)$$

is an open neighborhood of the identity in $\prod' \hat{G}_\nu$. If $(\chi_\nu) \in N$, then

$$\phi((\chi_\nu))(K) = \phi((\chi_\nu))(\prod K_\nu) = \prod \chi_\nu(K_\nu) \subseteq \prod_{\nu \in S} V = V^{(m)} \subseteq U.$$

Thus, we have proved that ϕ is continuous. To show that ϕ is open, let us pick a basic open neighborhood of the identity N in $\prod' \hat{G}_\nu$. As such, there exists a finite set S' of places and an open set U of 1 in $\mathbb{C}^\times$ such that $N = \prod_{\nu \in S'} W(K_\nu, U) \times \prod_{\nu \notin S'} K(G_\nu, H_\nu)$. Then $W(\prod K_\nu, U) \subseteq \phi(N)$, where $W(\prod K_\nu, U)$ is the open neighborhood of the trivial character in $\hat{G}$ such that $\chi(\prod K_\nu) \subseteq U$. Therefore, ϕ is an algebraic isomorphism and is both open and continuous, making ϕ a topological isomorphism. □

The above two lemmas and the theorem about the dual group of the restricted direct product will be used in three ways in Tate's Thesis. First, Lemma 3.1.6 will be used to construct the standard non-trivial adelic character, ψ_K, of a number field K. We will define the character ψ_K to be the product $\prod_\nu \psi_\nu$, where ψ_ν is the standard non-trivial character of the additive group of the completion of K at the νth place. The standard non-trivial characters ψ_ν satisfy the property that $\psi_\nu|_{\mathfrak{o}_\nu} = 1$ for all but finitely many ν of K. Furthermore, we will see that ψ_K will be trivial on the diagonal embedding of K in $\mathbb{A}_K$. Second, Theorem 3.1.7 will be used to show, in combination with the fact that $K_\nu \cong \hat{K_\nu}$ (i.e. local field duality) for all places ν of K and in combination with the existence of the non-trivial character ψ_K on the adeles, that the map $\alpha_{\psi_K} : a \mapsto \psi_K(a\cdot)$, from $\mathbb{A}_K$ to $\widehat{\mathbb{A}_K}$, is an algebraic and topological isomorphism. Additionally, we will show that the map $\alpha_{\psi_K} : k \mapsto \psi_K(k\cdot)$, from K to $\widehat{\mathbb{A}_K/K}$, is an algebraic and topological isomorphism. The existence of the standard character, ψ_K, on $\mathbb{A}_K \cong \widehat{\mathbb{A}_K}$ of a number field K, will be essential in introducing the Fourier transform of "nice" functions defined on the adeles and, ultimately, in proving the Poisson summation formula and its useful extension, the Rieman-Roch theorem. The Rieman-Roch theorem is the main tool used in proving the meromorphic continuation and functional equation of the global zeta function. Lastly, Lemma 3.1.5 will be applied to factor a quasi-character $\chi \in \mathrm{Hom}_{\mathrm{Cont}}(\mathbb{I}_K, \mathbb{C}^\times)$ that is trivial on K^* (i.e. an idele-class character or Hecke character) as a product of local quasi-characters $\chi_\nu \in \mathrm{Hom}_{\mathrm{Cont}}(K_\nu^*, \mathbb{C}^\times)$. This will enable us to define the L-function of χ as a product over its local versions, $L(\chi_\nu)$.

3.1.2 Restricted Direct Integration and Self-Dual Measure

Again, let G be the restricted direct product of locally compact groups G_ν with respect to compact-open subgroups H_ν. Since G is locally compact, then G admits a Haar measure. However, like the characters of G, we would like to have some way of defining a Haar measure on G in terms of Haar measures on G_ν. This brings us to the following proposition.

Proposition 3.1.8. *Let dg_ν denote a left (right) Haar measure on G_ν normalized so that*

$$\int_{H_\nu} dg_\nu = 1$$

for almost all $\nu \notin J_\infty$. Recall that a Haar measure is necessarily finite on compact sets (Proposition 1.2.4), so we may normalize the Haar measure as such. Then there is a unique *left (respectively, right) Haar measure* dg *on* G *such that for each finite set of indices* S *containing* J_∞*, the restriction of* dg_s *of* dg *to* G_S *is precisely the product measure. We will write* $dg = \prod_\nu dg_\nu$ *for this measure.*

Proof. Let S be an arbitrary set containing J_∞ and define dg_S to be the product of the measures dg_ν for all ν. By the normalization of dg_ν and the fact that S is finite, then the compact group $\prod_{\nu \notin S} H_\nu$ has finite measure with respect to dg_S. As such, dg_S is a Haar measure on G_S. Indeed, the product measure dg_S is a radon measure and, furthermore, is invariant under the componentwise group operation because each of the dg_ν is invariant under the group operation. See Chapter 7, Theorem 7.28, in Folland's Real Analysis [12]. Let $T \supseteq S$ be a larger set of indices. Clearly, $G_S \leq G_T$. Then, by construction, we have that dg_S coincides with the restriction of dg_T to the subgroup G_S. Since G is locally compact, then G admits a Haar measure, dg, which is unique up to a constant. Furthermore, the restriction of dg to G_S is also a Haar measure on G_S. As such, we can pick any finite set S of indices containing J_∞, and choose the Haar measure dg of G, such that dg restricts to dg_S. For any $T \supseteq S$, the measure dg restricts to the product measure dg_T on G_T because of the above remark about dg_S coinciding with the restriction of dg_T. Let S' be a set of indices containing J_∞. Then dg, constructed relative to dg_S, uniquely picks out the product measure on $G_{S \cup S'}$, and hence on $dg_{S'}$. Therefore, dg is independent of the S chosen and is unique. $\square$

Now that we have established the existence of a well-defined Haar measure dg on G in terms of Haar measures dg_ν on G_ν, we would like to have a way of integrating and taking the Fourier transform functions defined on G. In the following proposition, we should pay special attention to functions of the form $\prod_\nu f_\nu$, as these types of functions will play an important role in the proof of the functional equation and analytic continuation of the Hecke L-function. In the second proposition, we will construct the dual measure $d\chi$ to dg on $\hat{G}$ such that the Fourier inversion theorem holds. The Fourier inverion theorem is a key ingredient in proving both the Poisson summation formula and the Riemann-Roch theorem.

Proposition 3.1.9.

(i) Let f be an integrable function on G. Then

$$\int_G f(g) = \lim_S \int_{G_S} f(g_s)dg,$$

where the limit is taken over larger and larger S. If f is only assumed to be continuous, then the above identity holds, but then we must accept that the integral may take infinite values.

(ii) Let S_0 denote the finite set of indices containing both J_∞ and the set of indices for which $\mathrm{Vol}(H_\nu, dg_\nu) \neq 1$. Suppose that for each index ν, we are given a continuous and integrable function f_ν on G_ν, such that $f_\nu|_{H_\nu} = 1$ for all ν outside some finite set S_1. Then for $g = (g_\nu) \in G$ we can define the function

$$f(g) = \prod_\nu f_\nu(g_\nu).$$

The function f is well-defined and continuous on G. Furthermore, if S is any finite set of indices including S_0 and S_1, then we have

$$\int_{G_S} f(g)dg = \prod_{\nu \in S} \left(\int_{G_\nu} f_\nu(g_\nu)dg_\nu \right).$$

Furthermore, if

$$\prod_\nu \left(\int_{G_\nu} |f_\nu(g_\nu)|dg_\nu \right) \left(= \lim_S \prod_{\nu \in S} \left(\int_{G_\nu} |f_\nu(g_\nu)|dg_\nu \right) \right) < \infty$$

then

$$\int_G f(g)dg = \prod_\nu \left(\int_{G_\nu} f_\nu(g_\nu)dg_\nu \right)$$

and $f \in L^1(G)$.

(iii) Let $\{f_\nu\}$ and f be as they were in the previous part, but with the added constraint of f_ν being a characteristic function of H_ν for all $\nu \notin S_1$. Then $f \in L^1(G)$ and, in abelian case, the Fourier transform of f is given by

$$\hat{f}(g) = \prod_\nu \hat{f}_\nu(g_\nu).$$

If we additionally assume that $\hat{f}_\nu \in L^1(\hat{G}_\nu)$ for all ν, then $\hat{f} \in L^1(\hat{G})$. Recall that $\mathfrak{B}(G)$ is the set of functions such that f is continuous, $f \in L^1(G)$, and $\hat{f} \in L^1(G)$. Moreover, this is the set of functions for which the Fourier inversion theorem holds. Therefore, if we assume $f_\nu \in \mathfrak{B}(G_\nu)$ for all ν, and both $\mathrm{Vol}(H_\nu, dg_\nu) = 1$ and $f_\nu = \mathbf{1}_{H_\nu}$ for all but finitely many ν, then $f \in \mathfrak{B}(G)$.

Proof. (i) From basic integration theory, we know that

$$\int_G f(g)dg = \lim_K \int_K f(g)dg,$$

where the limit is taken over larger and larger compact sets K of G. Since any compact set K is contained in some G_S, then we may take the limit over larger and larger S, and hence G_S instead. So long as f is continuous, then the identity holds.

(ii) By our choice of f_ν, we know that $f_\nu|_{H_\nu} = 1$ for all $\nu \notin S_0$. For $g = (g_\nu) \in G$, we know there exists a set S_g such that $g_\nu \in H_\nu$ for all $\nu \notin S_g$. Therefore,

$$\prod_\nu f_\nu(g_\nu) = \prod_{\nu \in S_0 \cap S_g} f_\nu(g_\nu) \prod_{\nu \in S_0 \cap S_g^c} f_\nu(g_\nu)$$

is a finite product for all $g \in G$, and so f is well-defined. Recall that a neighborhood base of the identity of G consists of sets of the form $\prod N_\nu \times \prod H_\nu$, where N_ν is a neighborhood of the identity in G_ν, and where the first product is over a finite number of indices. We may, however, take the basis to consist of sets of the above form with the added restriction that the first product necessarily includes the indices for which $f_\nu|_{H_\nu} \neq 1$. In this way, locally, we may identify f with the finite product of continuous functions f_ν. As such, f is continuous. Now let S be a set of indices containing S_0 and S_1. Then $f_\nu|_{H_\nu} = 1$ and $\mathrm{Vol}(H_\nu, dg_\nu) \neq 1$ for all $\nu \notin S$. Further, dg_S is nothing more than the product measure on G_S. Therefore,

$$\int_{G_S} f(g_S)dg = \int_{G_S} f(g_S)dg_S = \prod_{\nu \in S}\left(\int_{G_\nu} f_\nu(g_\nu)dg_\nu\right)\prod_{\nu \notin S}\left(\int_{H_\nu} f_\nu(g_\nu)dg_\nu\right) = \prod_{\nu \in S}\left(\int_{G_\nu} f_\nu(g_\nu)dg_\nu\right),$$

since $f_\nu \in L(G_\nu)$ for all ν, and since the product over $\nu \notin S$ is 1. According to part (i), since $|f|$ is continuous, then $|f|$ is integrable if and only if

$$\lim_S \int_{G_S} |f(g_S)|dg = \lim_S \int_{G_S} |f(g_S)|dg_S < \infty.$$

Therefore, in combination with the above equality, applied to $|f(g)| = \prod_\nu |f(g_\nu)|$, we have that the limit exists if and only if

$$\lim_S \int_{G_S} |f(g_S)| dg_S = \lim_S \prod_{\nu \in S} \left(\int_{G_\nu} |f_\nu(g_\nu)| dg_\nu \right) = \prod_{\text{all } \nu} \left(\int_{G_\nu} |f_\nu(g_\nu)| dg_\nu \right) < \infty.$$

This completes the proof of part (ii).

(iii) Since $f = \mathbf{1}_{H_\nu}$ for all $\nu \notin S_1$ and since $\mathrm{Vol}(H_\nu, dg_\nu) = 1$ for $\nu \notin S_0$, then for $S = S_0 \cup S_1$ we have

$$\prod_{\text{all } \nu} \left(\int_{G_\nu} |f_\nu(g_\nu)| dg_\nu \right) = \prod_{\nu \in S} \left(\int_{G_\nu} |f_\nu(g_\nu)| dg_\nu \right).$$

The latter integral is finite because $f_\nu \in L(G_\nu)$ for all ν. Therefore, by part (ii), we obtain that $f \in L^1(G)$. If G is abelian, then let $\chi = (\chi_\nu) \in \hat{G}$ (Theorem 3.1.7). Define $h_\nu = f_\nu \overline{\chi_\nu}$ and $h = \prod_\nu h_\nu$. Since $\chi_\nu|_{H_\nu} = 1$ and $f = \mathbf{1}_{H_\nu}$ for all but finitely may ν, then $h_\nu = \mathbf{1}_{H_\nu}$ for all but finitely many ν. Furthermore, since χ_ν is a unitary continuous character, then h_ν is a continuous and absolutely integrable function on G_ν. Therefore, we have that

$$\prod_{\text{all } \nu} \left(\int_{G_\nu} |h_\nu(g_\nu)| dg_\nu \right) = \prod_{\nu \in S} \left(\int_{G_\nu} |f_\nu(g_\nu)| dg_\nu \right) < \infty.$$

As such, applying part (ii) to h, we have that

$$\hat{f}(g) = \int_G f(g)\overline{\chi}(g) dg = \prod_\nu \left(\int_{G_\nu} f_\nu(g_\nu)\overline{\chi_\nu}(g) dg \right) = \prod_\nu \hat{f}_\nu(g_\nu).$$

Now suppose that $\hat{f}_\nu \in L^1(G_\nu)$. In order to show that $\hat{f} \in L^1(G)$, we need to show that $\hat{f}_\nu$ is a characteristic function for all but finitely many ν. For each ν, let $d\chi_\nu = \widehat{dg_\nu}$ denote the dual measure, as in the Fourier inversion theorem, to dg_ν on $\hat{G}_\nu$. Let $\nu \notin S_1$ so that $f_\nu = \mathbf{1}_{H_\nu}$. Then

$$\hat{f}_\nu(\chi_\nu) = \int_{G_\nu} \mathbf{1}_{H_\nu}(g_\nu)\overline{\chi_\nu}(g_\nu) dg_\nu = \int_{H_\nu} \overline{\chi_\nu}(g_\nu) dg_\nu.$$

If $\chi_\nu|_{H_\nu} = 1$, then $\hat{f}_\nu(\chi_\nu) = \mathrm{Vol}(H_\nu, dg_\nu)$. Suppose $\chi_\nu|_{H_\nu} \neq 1$. Let $g' \in H_\nu$ be chosen such that $\chi_\nu(g') \neq 1$. Writing the group action of G_ν multiplicatively, we have that $g' \cdot H_\nu = H_\nu$, since left multiplication by a group element is an automorphism. By the translation invariance of the Haar measure dg_ν and the fact that χ_ν is a homomorphism, we have

$$\int_{H_\nu} \overline{\chi_\nu}(g_\nu) dg_\nu = \int_{g' \cdot H_\nu} \overline{\chi_\nu}(g' \cdot g_\nu) dg_\nu = \overline{\chi_\nu}(g') \int_{H_\nu} \overline{\chi_\nu}(g_\nu) dg_\nu.$$

Since H_ν is compact and χ_ν is unitary, then

$$\left|\int_{H_\nu} \overline{\chi_\nu}(g_\nu)dg_\nu\right| \leq \int_{H_\nu} dg_\nu = \mathrm{Vol}(H_\nu, dg_\nu) < \infty,$$

because the Haar measure is finite on compact sets. Finally, since $\overline{\chi_\nu}(g') \neq 1$, then $\hat{f}_\nu(\chi_\nu) = 0$ for all $\chi_\nu|_{H_\nu} \neq 1$. Let us denote the set of characters of G such that $\chi_\nu|_{H_\nu} \neq 1$ by $H_\nu^\perp$. We previously denoted the set $H_\nu^\perp$ by $K(G_\nu, H_\nu)$. As such,

$$\hat{f}_\nu(\chi_\nu) = \mathrm{Vol}(H_\nu, dg_\nu)\mathbf{1}_{H_\nu^\perp}(\chi_\nu).$$

Since $\mathrm{Vol}(H_\nu, dg_\nu) = 1$ for all $\nu \notin S_0$, then $\hat{f}_\nu$ is a characteristic function of $\hat{H}_\nu^\perp$ for all $\nu \notin S = S_1 \cup S_0$. By Theorem 3.1.7, we have that $\hat{G} \cong \prod_\nu' \hat{G}_\nu$, where the restricted direct product is taken with respect to the compact-open subgroup $H_\nu^\perp$. Also, since $\hat{f}_\nu$ is both continuous and absolutely integrable, then

$$\prod_{\text{all } \nu}\left(\int_{\hat{G}_\nu} |\hat{f}_\nu(\chi_\nu)|d\chi_\nu\right) = \prod_{\nu\in S}\left(\int_{\hat{G}_\nu} |f_\nu(\chi_\nu)|d\chi_\nu\right) < \infty$$

, and hence $\hat{f} \in L^1(\hat{G})$. Therefore, since $f = \prod_\nu f_\nu$ is continuous, and $f \in L^1(G)$, and $\hat{f} \in L^1(\hat{G})$, then $f \in \mathfrak{B}(G)$.

□

We would now like to construct the measure that is dual to $dg = \prod_\nu dg_\nu$ in the sense that is defined by the Fourier inversion theorem. Again, we will assume that the dg_ν are normalized so that $\mathrm{Vol}(H_\nu, dg_\nu) = 1$ for all but finitely many ν. As in part (iii) of the above proposition, for each ν, let $d\chi_\nu = \widehat{dg_\nu}$ denote the dual measure to dg_ν on $\hat{G}_\nu$. Also, let f_ν be as it is in part (iii) of the above theorem. That is, assume $f_\nu \in \mathfrak{B}(G_\nu)$ for all ν and $f_\nu = \mathbf{1}_{H_\nu}$ for all but finitely many ν. Let us fix a ν such that $f_\nu = \mathbf{1}_{H_\nu}$. In part (iii), we showed that $\hat{f}_\nu = \mathrm{Vol}(H_\nu, dg_\nu)\mathbf{1}_{H_\nu^\perp}$. Implicity identifying $\hat{\hat{G}}_\nu$ with G_ν by Pontryagin duality, we obtain

$$\hat{\hat{f}}_\nu(g_\nu) = \mathrm{Vol}(H_\nu, dg_\nu)\int_{\hat{G}_\nu} \mathbf{1}_{H_\nu^\perp}(\chi_\nu)\chi_\nu(g_\nu)d\chi_\nu = \mathrm{Vol}(H_\nu, dg_\nu)\int_{H_\nu^\perp} \chi_\nu(g_\nu)d\chi_\nu.$$

In the proof of self-duality of the restricted direct product, we proved that $H_\nu^\perp$ is a compact-open, and hence closed, subgroup of $\hat{G}$. Replicating the argument in part (iii) together with the fact that $H^{\perp}$ is compact (i.e. orthogonality of characters), we have that

$$\hat{\hat{f}}_{\nu}(g_{\nu}) = \mathrm{Vol}(H_{\nu}, dg_{\nu})\mathrm{Vol}(H_{\nu}^{\perp}, d\chi_{\nu})\mathbf{1}_{(H_{\nu}^{\perp})^{\perp}}.$$

On the other hand, the Fourier inversion theorem tells us that $\hat{\hat{f}}_{\nu}(g) = f_{\nu}(g^{-1})$. Note that we need $f_{\nu} \in \mathfrak{B}(G_{\nu})$ for the theorem to hold. Therefore,

$$\mathrm{Vol}(H_{\nu}, dg_{\nu})\mathrm{Vol}(H_{\nu}^{\perp}, d\chi_{\nu})\mathbf{1}_{(H_{\nu}^{\perp})^{\perp}}(g) = \mathbf{1}_{H_{\nu}}(g^{-1}) = \mathbf{1}_{H_{\nu}}(g),$$

because H_{ν} is a subgroup. Since $(H_{\nu}^{\perp})^{\perp} = H_{\nu}$, then we have the relation

$$\mathrm{Vol}(H_{\nu}, dg_{\nu})\mathrm{Vol}(H_{\nu}^{\perp}, d\chi_{\nu}) = 1.$$

Since we have assumed that $\mathrm{Vol}(H_{\nu}, dg_{\nu}) = 1$ for all but finitely many ν, then $\mathrm{Vol}(H_{\nu}^{\perp}, d\chi_{\nu}) = 1$ for all but finitely many ν. In this way, we can define $d\chi = \widehat{dg}$, which brings us to our next proposition.

Proposition 3.1.10. *The measure $d\chi = \prod_{\nu} d\chi_{\nu}$, where $d\chi_{\nu} = \widehat{dg_{\nu}}$, is dual the measure $dg = \prod_{\nu} dg_{\nu}$. Therefore,*

$$f(g) = \int_{\hat{G}} \hat{f}(\chi)\chi(g)d\chi,$$

for all $f \in \mathfrak{B}(G)$.

Proof. Since $\mathrm{Vol}(H_{\nu}, dg_{\nu}) = 1$ for all but finitely many ν, then we have that $d\chi = \prod_{\nu} d\chi_{\nu}$ is a Haar measure on $\hat{\hat{G}}$ by Proposition 3.1.8 and Proposition 3.1.7. Since a Haar measure is unique up to a constant, then it suffices to check duality for a given product of functions. That is, we would like to show that the normalization factor is 1. We have already computed the Fourier transform for the set of functions such that $f = \prod_{\nu} f_{\nu}$, where $f_{\nu} \in \mathfrak{B}(G_{\nu})$, and where $f_{\nu} = \mathbf{1}_{H_{\nu}}$ for all but finitely many ν. Such functions are necessarily in $\mathfrak{B}(G)$ by part (iii) of the above proposition. Let $g = (g_{\nu}) \in G$. By part (iii) of the above proposition, we have that

$$\int_{\hat{G}} \hat{f}(\chi)\chi(g)d\chi = \prod_{\nu} \int_{\hat{G}_{\nu}} \hat{f}_{\nu}(\chi_{\nu})\chi_{\nu}(g_{\nu})d\chi_{\nu}.$$

Since $d\chi_\nu$ is the dual measure to dg_ν, then

$$f_\nu(g_\nu) = \int\limits_{\hat{G_\nu}} \hat{f}_\nu(\chi_\nu)\chi_\nu(g_\nu)d\chi_\nu.$$

Therefore,

$$\int\limits_{\hat{G}} \hat{f}(\chi)\chi(g)d\chi = \prod_\nu f_\nu(g_\nu) = f(g).$$

This completes the proof. $\square$

3.2 Adeles and Ideles

Let K be a number field. Let K_ν be the completion of K at the νth place of K. The restricted direct product of K_ν, under addition, with respect to $\mathfrak{o}_\nu$, is called the *adele group* of K, and is denoted $\mathbb{A}_K$. We set $J_\infty = \{\nu : \nu \text{ an infinite place of } K\}$ because when completing K at the infinite places, there does not exist an open compact subgroup $\mathfrak{o}_\nu$. Note that K_ν is an abelian locally compact group and $\mathfrak{o}_K$ is a compact-open subgroup of K_ν for all finite places ν of K. Every element of K is divisible by finitely many prime ideals, and hence the embedding of K into K_ν for all ν lies in $\mathfrak{o}_\nu$ for all but finitely many places. Therefore, K embeds diagonally into $\mathbb{A}_K$:

$$K \to \mathbb{A}_K$$
$$x \mapsto (x, x, x, \ldots).$$

The *idele group*, denoted $\mathbb{I}_K$, is the restricted direct product of K_ν^*, as a multiplicative group, with respect to $\mathfrak{o}_\nu^\times$, an open compact subgroup of K_ν^*. Since every element of K^* is locally an integer, and hence a unit for all but finitely many places, K^* diagonally embeds into $\mathbb{I}_K$:

$$K^* \to \mathbb{I}_K$$
$$x \mapsto (x, x, x, \ldots).$$

By construction, $\mathbb{A}_K$ is an additive group. However, if we define componentwise multiplication on $\mathbb{A}_K$, then $\mathbb{A}_K$ is a ring because multiplication in K_ν is closed. Every element is not unit, though, so it makes sense to consider $\mathbb{A}_K^\times$. The group of units of $\mathbb{A}_K$ is the restricted direct product of K_ν^* with respect to $\mathfrak{o}_\nu^\times$, and hence $\mathbb{I}_K \cong \mathbb{A}_K^\times$. Said otherwise, $\mathbb{I}_K$ embeds into the adele group via the isomorphism with the group of units. Unfortunately, the embedding is not a topological embedding. To see that the above algebraic isomorphism is not always a topological embedding, consider $K = \mathbb{Q}$. The restricted direct topology on $\mathbb{I}_\mathbb{Q}$ is, in fact, stronger than the subspace topology induced by the restricted direct topology on the adeles. In order to show this, we must find an open set in the restricted direct topology on $\mathbb{I}_\mathbb{Q}$ that is not an open set in the subspace topology. In the case $K = \mathbb{Q}$, the completion with respect to the infinite prime (absolute value), of which there is only one, is just $\mathbb{R}$. The characterization of primes or places of $\mathbb{Q}$ formally is called Ostrowski's theorem. A neighborhood base of the multiplicative identity in the subspace topology, induced by the restricted direct topology on the adeles, of $\mathbb{I}_\mathbb{Q}$ consists of sets of the form

$$\left(\prod_{p\in S} N_p \times \prod_{p\notin S} \mathbb{Z}_p\right) \cap \mathbb{I}_\mathbb{Q},$$

where S is any finite set of primes that contains the infinite prime and where N_p are neighborhoods of the multiplicative identity in $\mathbb{Q}_p$. For any choice of N_p and S we can always find a point $x = (x_p) \in \prod_{p\in S} N_p \times \prod_{p\notin S} \mathbb{Z}_p$ such that $x_q \in \mathbb{Z}_q - \mathbb{Z}_q^\times$ for some $q \notin S$ and $x_p \in \mathbb{Z}_p^\times$ for all but finitely many p. Then $x \in \mathbb{I}_\mathbb{Q}$ because $x_p \in \mathbb{Z}_p^\times$ for all but finitely many p. In the restricted direct topology of $\mathbb{I}_\mathbb{Q}$, the set

$$U = \mathbb{R}^\times \times \prod_{p\notin I_\infty} \mathbb{Z}_p^\times$$

is an open neighborhood of the multiplicative identity because $\mathbb{R}^\times$ is an open neighborhood of the multiplicative identity and because the remaining part of the product is a product of $\mathbb{Z}_p^\times$ for all finite primes. Since $(x_p) \notin U$, then

$$\left(\prod_{p\in S} N_p \times \prod_{p\notin S} \mathbb{Z}_p\right) \cap \mathbb{I}_\mathbb{Q} \not\subset \mathbb{R}^\times \times \prod_{p\notin I_\infty} \mathbb{Z}_p$$

for all S and N_p. However, every open neighborhood of the identity in the restricted direct topology of $\mathbb{I}_{\mathbb{Q}}$ is a set of the form

$$\prod_{p \in S} N_p^* \times \prod_{p \notin S} \mathbb{Z}_p^\times,$$

where N_p^* is an open neighborhood of the multiplicative identity in $\mathbb{Q}_p^*$ and where S is some finite set of primes containing the infinite prime. Clearly,

$$\prod_{p \in S} N_p^* \times \prod_{p \notin S} \mathbb{Z}_p^\times = \left(\prod_{p \in S} N_p^* \times \prod_{p \notin S} \mathbb{Z}_p \right) \cap \mathbb{I}_{\mathbb{Q}}$$

is open in the subspace topology because every open neighborhood of the multiplicative identity N_p^* in $\mathbb{Q}_p^*$ is an open neighborhood of the multiplicative identity in $\mathbb{Q}_p$. Consequently, the restricted direct topology on $\mathbb{I}_{\mathbb{Q}}$ is stronger than the subspace topology induced by $\mathbb{A}_{\mathbb{Q}}$.

This, in fact, is something that arises often in the study topological rings. More generally, for any topological ring R, the group of units R^* is not always a topological group with respect to the subspace topology induced by the topology of R. This is because multiplicative inversion may not be continuous. However, the following is an example of where R^* is a topological group in the induced subspace topology. Take $R = \mathbb{Z}_p$ with $R^* = \mathbb{Z}_p^\times$. Then for $a, b \in \mathbb{Z}_p^\times$

$$|\frac{1}{a} - \frac{1}{b}|_p = |\frac{b-a}{ab}|_p = |b-a|_p,$$

which implies that inversion is continuous. In other words, sometimes R^* is a topological group and other times it is not. In order to deal with this problem, we embed R^* in $R \times R$ in the following way and endow R^* with the induced subspace product topology:

$$\phi : R^* \hookrightarrow R \times R$$

$$x \mapsto \left(x, \frac{1}{x} \right).$$

One can prove that this a topological embedding. But in our case, we are presented with a slightly more complicated scenario. The ideles have their own topology, the restricted direct topology of $K_\nu^\times$ with respect to $\mathfrak{o}_\nu^\times$. The ideles algebraically identify with the unit group of the adeles. We have seen that the restricted direct topology on the ideles is not equivalent to the adelic induced topology, at least in the case $K = \mathbb{Q}$. We proceed with the above "fix" to make the ideles, viewed as the unit group of the adeles, a topological group. Fortunately, the restricted direct topology on the ideles coincides with topology induced by embedding of $\mathbb{I}_K$ into $\mathbb{A}_K \times \mathbb{A}_K$ given above.

Proposition 3.2.1. *$\mathbb{I}_{\mathbb{K}}$ is a topological isomorphism onto its image in $\mathbb{A}_K^2$ under the map*

$$\begin{aligned} \phi : \mathbb{I}_{\mathbb{K}} &\longrightarrow \mathbb{A}_K^2 \\ x &\mapsto \left(x, \frac{1}{x}\right) \end{aligned}$$

Proof. Let U be an open set of the multiplicative identity in the image of ϕ. Then $U = V \cap \phi(\mathbb{I}_K)$, where V is an open neighborhood of the multiplicative identity in $\mathbb{A}_K^2$ with the product topology. A neighborhood basis of $\mathbb{A}_K^2$ consists of sets of the form

$$\prod_{\nu \in S} N_\nu \times \prod_{\nu \in S^c} \mathfrak{o}_\upsilon \times \prod_{\nu \in T} M_\nu \times \prod_{\nu \in T^c} \mathfrak{o}_\upsilon,$$

where S and T are both finite sets of places containing the infinite places and where N_ν and M_ν are neighborhoods of the multiplicative identity in K_ν. Let V be such a set. Let $\overline{N}_\nu = N_\nu \cap K_\nu^*$ and $\overline{M}_\nu = M_\nu \cap K_\nu^*$. Also, let $\overline{N}_\nu^o = N_\nu \cap \mathfrak{o}_\nu \cap K_\nu^* = N_\nu \cap \mathfrak{o}_\nu^\times$ and $\overline{M}_\nu^o = M_\nu \cap \mathfrak{o}_\nu^\times$. Note that $\overline{N}_\nu$, $\overline{N}_\nu^o$, $\overline{M}_\nu$, and $\overline{M}_\nu^o$ are open in K_ν^*. As such, $\overline{N}_\nu^{-1}$, $\overline{N}_\nu^{o\,-1}$, $\overline{M}_\nu^{-1}$, and $\overline{M}_\nu^{o\,-1}$ are well defined and open in K_ν^*, since K_ν is a topological field. Then

$$\begin{aligned} V \cap \phi(\mathbb{I}_K) = &\prod_{\nu \in S \cap T} \left(\overline{N}_\nu \cap \overline{M}_\nu^{-1}\right) \times \prod_{\nu \in S-(S\cap T)} \overline{N}_\nu^o \times \prod_{\nu \in T-(S\cap T)} \overline{M}_\nu^{o\,-1} \times \prod_{\nu \in (S\cup T)^c} \mathfrak{o}_\nu^\times \\ &\times \prod_{\nu \in S \cap T} \left(\overline{N}_\nu^{-1} \cap \overline{M}_\nu\right) \times \prod_{\nu \in S-(S\cap T)} \overline{N}_\nu^{o\,-1} \times \prod_{\nu \in T-(S\cap T)} \overline{M}_\nu^{o} \times \prod_{\nu \in (S\cup T)^c} \mathfrak{o}_\nu^\times. \end{aligned}$$

Furthermore,

$$\phi^{-1}\left(V \cap \phi(\mathbb{I}_K)\right) = \prod_{\nu \in S \cap T} \left(\overline{N}_\nu \cap \overline{M}_\nu^{-1}\right) \times \prod_{\nu \in S-(S\cap T)} \overline{N}_\nu^o \times \prod_{\nu \in T-(S\cap T)} \overline{M}_\nu^{o\,-1} \times \prod_{(S\cup T)^c} \mathfrak{o}_\nu^\times,$$

which is clearly open in the restricted direct topology on $\mathbb{I}_K$. Now, let U be an open neighborhood of the multiplicative identity in the restricted direct topology on $\mathbb{I}_K$. Without loss of generality we may take

$$U = \prod_{\nu \in S} N_\nu \times \prod_{\nu \in I-S} \mathfrak{o}_\nu^\times,$$

where S is a finite set of places containing the infinite places and where N_ν are neighborhoods of the multiplicative identity in K_ν^*. Consider the image of U under the mapping ϕ:

$$\begin{aligned}\phi(U) &= \prod_{\nu\in S} N_\nu \times \prod_{\nu\in S^c} \mathfrak{o}_\nu^\times \times \prod_{\nu\in S} N_\nu^{-1} \times \prod_{\nu\in S^c} \mathfrak{o}_\nu^\times \\ &= \left(\prod_{\nu\in S} N_\nu \times \prod_{\nu\in S^c} \mathfrak{o}_\nu \times \prod_{\nu\in S} N_\nu^{-1} \times \prod_{\nu\in S^c} \mathfrak{o}_\nu\right) \cap \phi(\mathbb{I}_K).\end{aligned}$$

Since N_ν and N_ν^{-1} are open neighborhoods of the identity in K_ν^*, and hence in K_ν, then the set on the left-hand side of the intersection is open in $\mathbb{A}_K^2$, making the entire set open in the subspace topology of $\phi(\mathbb{I}_K)$. Therefore, ϕ is bi-continuous and hence an algebraic and topological isomorphism of $\mathbb{I}_K$ onto its image. □

Define the subgroup $\mathbb{A}_{J_\infty}$ of $\mathbb{A}_K$ to be

$$\mathbb{A}_{J_\infty} := \{x = (x_\nu) \in \mathbb{A}_K : x_\nu \in \mathfrak{o}_\nu \text{ for all } \nu \notin J_\infty\}.$$

Hence forth, denote this subgroup by $\mathbb{A}_\infty$. The following is called the Approximation Theorem.

Proposition 3.2.2. *For every global field K, we have both*

$$\mathbb{A}_K = K + A_\infty \quad \text{and} \quad K \cap \mathbb{A}_\infty = \mathfrak{o}_K.$$

Proof. As seen above, K embeds diagonally into $\mathbb{A}_K$. In order to prove the above proposition, we must show that for every element $x \in \mathbb{A}_K$, there exists an element $k \in K$ such that $x - k$ has an absolute value less than or equal to 1 for all finite places; otherwise said, $x - k$ is locally an integer for all finite places. Let ν be a finite place of K and let $\mathfrak{p}_\nu$ be a prime ideal of $\mathfrak{o}_K$ corresponding to ν. Suppose $\mathfrak{p}_\nu$ lies above the rational prime p_ν. Let $x = (x_\nu) \in \mathbb{A}_K$. For all ν, there exists some positive integer m_ν such that $|p_\nu^{m_\nu} x_\nu|_{\mathfrak{p}_\nu} = 1 \;\Leftrightarrow\; p_\nu^{m_\nu} x_\nu \in \mathfrak{o}_\nu$. Since $x \in \mathbb{A}_K$ is locally not an integer at only a finite number of places, then we may find a rational integer m–which we are implicitly diagonally embedding into $\mathbb{A}_K$–such that all finite (place) components of mx lie in the ring of integers. Say $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ are the prime ideals that divide m in $\mathfrak{o}_K$. By construction, the set of aforementioned primes must contain the set of primes for which the corresponding component of x in $\mathbb{A}_K$ failed to be an integral. Let $e_j > 0$ represent the power of the prime $\mathfrak{p}_j$ appearing in the unique factorization of the ideal m in $\mathfrak{o}_K$. That is,

$$m = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}^{e_n}.$$

Applying the Chinese remainder theorem, we can find a $\lambda \in \mathfrak{o}_k$ such that

$$mx_j = \lambda (\text{mod } \mathfrak{p}_j^{e'_j}) \quad j = \{1, 2, \ldots n\},$$

where x_j is the $\mathfrak{p}_j$th component of x in the adeles, and $e'_j \geq e_j$. Note that we are using the fact that prime ideals remain comaximal at higher powers. Let $k = \lambda m$. Then $x - k = m^{-1}(mx - \lambda)$ is integral at $\mathfrak{p}_1 \cdots \mathfrak{p}_n$ and all other finite primes. Recall that one definition of the ring of integers of a global field K is

$$\mathfrak{o}_K = \bigcap_{v \text{ finite}} \mathfrak{o}_v.$$

Also, by definition, $\mathbb{A}_\infty$ consists of all elements of the adeles that are locally an integer at all finite places. So, $K \cap \mathbb{A}_\infty = \mathfrak{o}_K$. This completes the proof. □

For $K = \mathbb{Q}$ the approximation theorem implies the following corollary.

Corollary 3.2.3.

$$\mathbb{A}_\mathbb{Q} = \mathbb{Q} + \mathbb{A}_\infty = \mathbb{Q} + \left(\mathbb{R} \times \prod_{p \text{ prime}} \mathbb{Z}_p \right)$$

and $\mathbb{Q} \cap \mathbb{A}_\infty = \mathbb{Z}$.

Lemma 3.2.4. *Let E/K be a finite extension and fix a K-basis $\{u_1, \ldots, u_n\}$ of E. Then the map*

$$\alpha : \prod_{j=1}^{n} \mathbb{A}_K \to \mathbb{E}$$
$$((x_{\nu,j})_\nu)_j \mapsto \sum_j u_j (x_{\nu,j})_\nu$$

is an isomorphism of topological groups.

Proof. See Ramakrishnan and Valenza [24], Chapter 5, Section 3, Lemma 5-10. □

The following theorem will establish that K is a discrete, co-compact subgroup of $\mathbb{A}_K$. In Chapter 4, we will construct the standard non-trivial character ψ_K on $\mathbb{A}_K$ by taking the product of the standard non-trivial characters ψ_ν on K_ν. We also will show that ψ_K is trivial on K. This fact was mentioned previously in the restricted direct product section. Furthermore, we will show that the map β_{ψ_K} from K to $\widehat{\mathbb{A}_K/K}$, given by $a \to \psi_K(a\cdot)$, is an isomorphism. In showing that this map is surjective, we will need to use the fact that $\mathbb{A}_K/K$ is compact and hence that $\widehat{\mathbb{A}_K/K}$ is discrete (Proposition 1.3.4).

Proposition 3.2.5. *K is a discrete, co-compact subgroup of $\mathbb{A}_K$.*

Proof. Let $[K : \mathbb{Q}] = n$. Using the preceding lemma, we see that $\prod_{i=1}^n \mathbb{A}_\mathbb{Q}$ is isomorphic to $\mathbb{A}_K$ as topological groups. Furthermore, $\prod_{i=1}^n \mathbb{Q}$ is isomorphic to K as topological groups, endowing both with the discrete topology. As such, $\mathbb{A}_K/K$ is isomorphic to $\prod_{i=1}^n \mathbb{A}_\mathbb{Q}/\mathbb{Q}$ as topological groups. It will suffice to show that $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is compact by Tychonoff's thoerem. Similarly, it will suffice to show that $\mathbb{Q}$ is discrete in $\mathbb{A}_\mathbb{Q}$. Let ∞ be the real place of $\mathbb{Q}$. Note that $\mathfrak{o}_\mathbb{Q} = \mathbb{Z}$. We will show that the subset C of $\mathbb{A}_\mathbb{Q}$, given by

$$C = \{x = (x_p) \in \mathbb{A}_\mathbb{Q} : |x_\infty|_\infty \leq 1/2 \quad \text{and} \quad |x_p|_p \leq 1 \ \forall p\},$$

is a compact fundamental domain for $\mathbb{Q}$ in $\mathbb{A}_\mathbb{Q}$. Note that $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x_p|_p \leq 1\}$. If we can show that $C \cap \mathbb{Q} = \{0\}$ and $\mathbb{A}_\mathbb{Q} = C + \mathbb{Q}$, then we are done. The $C \cap \mathbb{Q} = \{0\}$ condition guarantees that there are no repeats and the $\mathbb{A}_\mathbb{Q} = C + \mathbb{Q}$ condition shows that $\mathbb{A}_\mathbb{Q}$ is covered by translates of K by elements in C. Suppose that $x \in C \cap \mathbb{Q}$. Since every integer is divisible by only finitely many primes and since $x \in \mathbb{Z}_p$ for all p, then $x = p_1^{n_1} \cdots p_r^{n_r}$, where the $n_i, i = 1, \ldots, r$ are positive integers. However, since x has absolute value less than $1/2$, then $x = 0$. Consequently, $C \cap \mathbb{Q} = \{0\}$.

Let $y = (y_p) \in \mathbb{A}_\mathbb{Q}$. Then $y_p \in \mathbb{Z}_p$ for all but finitely many p. Let S be the finite set of primes containing both the infinite prime and those primes for which $y_p \notin \mathbb{Z}_p$. We want to show that for all $p \in S$, there exists an element $u(p) \in \mathbb{Q}$ such that $y - u(p) \in \mathbb{Z}_p$ and such that $u(p) \in \mathbb{Z}_q$ for all $q \neq p$. Let $u(p)$ be the fractional part of $y_p \in \mathbb{Q}_p$–the polar part of the p-adic series representation. Then $u(p) = \frac{a}{p^m}$ for some $m > 0$ and for some $a \in \mathbb{Z}$. Therefore, by construction, we have that $y_p - u(p) \in \mathbb{Z}_p$. Furthermore, since $a \in \mathbb{Z}$ and $(p, q) = 1$ for all $q \neq p$, then $u(p) = \frac{a}{p^m} \in \mathbb{Z}_q$ for all $q \neq p$. Letting $\delta = \sum_{p \in S} u(p)$, we see that $y - \delta \in \mathbb{Z}_p$ for all rational primes p. Let δ' be the nearest integer to $(y_\infty - \delta)$. Then $|y_\infty - \delta - \delta'|_\infty \leq 1/2$. It is also clear that since δ' is an integer, then $\delta' \in \mathbb{Z}_p$ for all rational primes p. Then we have

$$|y_p - \delta - \delta'|_p \leq \max\{|y_\nu - \delta|_p, |\delta'|_p\} \leq 1 \ \ \forall\, p \ \text{ and } \ |y_\infty - \delta - \delta'|_\infty \leq 1/2,$$

which implies that $x = y - \delta - \delta' \in C$. Since $\delta - \delta' \in \mathbb{Q}$, then $y = x + \delta + \delta' \in C + \mathbb{Q}$. This completes the proof. $\square$

Proposition 3.2.6. *There exists an isomorphism of topological groups*

$$\mathbb{A}_\mathbb{Q}/\mathbb{Q} \cong \lim_{\leftarrow} \mathbb{R}/n\mathbb{Z}$$

Proof. See Ramakrishnan and Valenza [24], Chapter 5, Section 3, Proposition 5-12. $\square$

Proposition 3.2.7. *The group K^* embeds discretely in $\mathbb{I}_K$.*

Proof. Recall from Proposition 3.2.1 that $\phi : \mathbb{I}_K \to \mathbb{A}_K^2$, defined by $x \mapsto (x, \frac{1}{x})$, yields a topological isomorphism of $\mathbb{I}_K$ onto its image under ϕ. We know from Proposition 3.2.5 that K embeds discretely into $\mathbb{A}_K$. As such, $K \times K$ embeds discretely into $\mathbb{A}_K \times \mathbb{A}_K$, which implies that $K^* \times K^*$ embeds discretely in $\phi(\mathbb{I}_K)$. This completes the proof. $\square$

Definition 3.2.8. We define the idele-class group to be $\mathbb{I}_K/K^*$ and we denote it by C_K.

Now we want to define an absolute value $|\cdot|_{\mathbb{A}_K}$ on $\mathbb{I}_K$ as the product of the local absolute values $|\cdot|_\nu$ over the places ν of K. More specifically, we want to choose the absolute values on the completions K_ν, such that for any idele x, we have that $|x|_{\mathbb{A}_K} = \prod_\nu |x|_\nu$ is precisely the module of the automorphism $y \mapsto xy$, defined on the locally compact abelian group $\mathbb{A}_K$. Regardless of the measure chosen for $\mathbb{A}_K$, the module of automorphism is the same. Indeed, let μ be a Haar measure on $\mathbb{A}_K$. Then the module of automorphism of $y \in \mathbb{I}_K$ is defined to be $\mathrm{mod}_{\mathbb{A}_K}(y) = \mu(y \cdot M)/\mu(M)$ where M is any Borel set with $0 < \mu(M) < \infty$. Since the Haar measure is unique up to a constant, if we chose another measure μ', then $\mu' = c\mu$ for some positive real c. As such, $\mu'(y \cdot M)/\mu'(M) = \mu(y \cdot M)/\mu'(M) = \mathrm{mod}_{\mathbb{A}_K}(y)$. As one would expect, the absolute values on the completions that will induce an absolute value on $\mathbb{A}_K$ with the module of automorphism property are exactly the absolute values with the module of automorphism property.

Definition 3.2.9. Let F be a local field of characteristic zero. We define the *normalized* absolute value on F as follows:

(i) If $F = \mathbb{R}$, then let $|\cdot|_F$ be the standard absolute value.

(ii) If $F = \mathbb{C}$, then let $|\cdot|_F$ be the square of the standard absolute value.

(iii) If F is non-Archimedean, then let $|\cdot|_F$ be such that $|\pi_F|_F = \frac{1}{q}$, where π_F is the uniformizing parameter of F, and q is the order of the residue field $\mathfrak{o}_F/\pi_F\mathfrak{o}_F$.

In order to show that these normalized absolute values are the module of automorphism of the Haar measure, it suffices to pick any measure on F and any measurable set M. However, we might as well pick measures that will be of use to us in the following chapter. In Chapter 4 contained below, we will show that for any non-trivial additive character ψ of F, the map $\alpha_\psi : F \mapsto \hat{F}$, defined by $a \mapsto \psi(a\cdot)$, is a topological group isomorphism. In this special case, the Fourier transform of a function $f \in \mathfrak{B}(F)$ can be identified with a function on F, rather than on $\hat{F}$. That is, if we wish to evaluate the Fourier transform of f at an additive character χ, we first find the $a \in F$ such that our fixed choice of ψ satisfies $\chi = \psi(a\cdot)$; we then can evaluate at $\hat{f}$ at χ by integrating against $\psi(a\cdot)$. As such, we want to pick the measures on the local fields such that the Fourier inversion theorem holds: $\hat{\hat{f}}(x) = f(-x)$ for all $f \in \mathfrak{B}(F)$. However, the choice of measure will depend on the fixed choice of ψ that one uses when defining the Fourier transform. In Chapter 4, at some points, especially when constructing the adelic character ψ_K, we will consider specific non-trivial additive characters on local fields. Recall the requirement in Proposition 3.1.6 for constructing such a character; the local characters must be trivial on H_ν for all but finitely many ν. The non-Archimedean local additive characters that we will construct satisfy this requirement, as they are are trivial on $\mathfrak{o}_\nu$ for all but finitely many places ν of K. The measures that we define below for local fields are precisely the *self-dual measures* to the non-trivial additive characters that we construct for local fields. See Proposition 4.2.4

Definition 3.2.10.

(i) If $F = \mathbb{R}$, then let dx be the standard Lesbesgue measure.

(ii) IF $F = \mathbb{C}$, then let dx be twice the standard Lebesgue measure.

(iii) If F is non-Archimedean, then let dx be such that $\mathrm{Vol}(\mathfrak{o}_F, dx) = N(\mathfrak{D}_F)^{-1/2}$, where $\mathfrak{D}_F$ denotes the different of F. Since the different is a fractional ideal of F, then $\mathfrak{D}_F = \mathfrak{p}^d$ for some integer d, where $\mathfrak{p}$ is the unique prime of F. As such, $N(\mathfrak{D}_F)^{-1/2} = N(\mathfrak{p})^{-d/2} = q^{-d/2}$ where q is the order of the residue field $\mathfrak{o}_F/\mathfrak{p}\mathfrak{o}_F$.

Proposition 3.2.11. *Let $|\cdot|_F$ be the normalized absolute value of F. If μ is a Haar measure on F, then*

$$\frac{\mu(y \cdot M)}{\mu(M)} = |y|_F,$$

for any $y \in F^\times$ and for any measurable set M with $0 < \mu(M) < \infty$.

Proof. As mentioned above, it suffices to consider any Haar measure μ and any set measurable set M. Let us choose the dx in Definition 3.2.10 and $M = \mathfrak{o}_F$, where $\mathfrak{o}_\mathbb{R} = [-1, 1]$, $\mathfrak{o}_\mathbb{C} = S^1$, and $\mathfrak{o}_F$ is the ring of integers for F non-Archimedean. We will write $\mathrm{Vol}(X, dx)$ for the measure of a measurable set X with respect to dx. That is,

$$\mathrm{Vol}(X, dx) = \int_F \mathbf{1}_X dx,$$

where $\mathbf{1}_X$ is the characteristic function of X. We will proceed case by case.

(i) If $F = \mathbb{R}$, then

$$\mathrm{Vol}(y \cdot [-1, 1], dx) = \mathrm{Vol}([-y, y], dx) = \int_{-y}^{y} dx = \begin{cases} 2y & \text{for } y \geq 0 \\ -2y & \text{for } y < 0. \end{cases}$$

Therefore,

$$\mathrm{Vol}(y \cdot [-1, 1], dx) = 2|y|_\mathbb{R} = \mathrm{Vol}([-1, 1], dx)|y|_\mathbb{R}.$$

(ii) Let $|y|_{st}$ be the standard Lebesgue measure on $\mathbb{C}$. Then

$$\begin{aligned} \mathrm{Vol}(y \cdot S^1, dx) &= \int_{y \cdot S^1} dx = \int_0^{2\pi} \int_0^{|y|_{st}} 2r dr d\theta = \int_0^{2\pi} \int_0^{|y|_{st}} r^2 d\theta = \\ &= 2\pi |y|_{st}^2 = \mathrm{Vol}(S^1, dx)|y|_{st}^2 = \mathrm{Vol}(S^1, dx)|y|_\mathbb{C}. \end{aligned}$$

(iii) Let R be the residue system of $\mathfrak{o}_F/y\mathfrak{o}_F$ in $\mathfrak{o}_F$ for F non-Archimedean. Applying the translation invariance of the Haar measure, we obtain

$$[\mathfrak{o}_F : y\mathfrak{o}_F]\,\mathrm{Vol}(y\mathfrak{o}_F, dx) = [\mathfrak{o}_F : y\mathfrak{o}_F]\int\limits_F \chi_{y\mathfrak{o}_F}(x)dx = \sum_{x'\in R}\int\limits_{x'+y\mathfrak{o}_F} dx = \int\limits_{\mathfrak{o}_F} dx = \mathrm{Vol}(\mathfrak{o}_F, dx).$$

Therefore,

$$\mathrm{Vol}(y\mathfrak{o}_F, dx) = [\mathfrak{o}_F : y\mathfrak{o}_F]^{-1}\,\mathrm{Vol}(\mathfrak{o}_F, dx) = |y|_F \mathrm{Vol}(\mathfrak{o}_F, dx).$$

$\square$

For the finite places ν of K, we have $\mathrm{Vol}(\mathfrak{o}_\nu, dx_\nu) = N(\mathfrak{D}_{K_\nu})^{-1/2}$. It can be shown that the inverse different is trivial for all but finitely many places ν. See Neukrich [23], *Algebraic Number Theory*, page 195. Therefore, $\mathrm{Vol}(\mathfrak{o}_\nu, dx_\nu) = 1$ for all but finitely many finite places ν of K because $\mathfrak{D}_{K_\nu} = \mathfrak{o}_\nu$ for all but finitely many finite places ν of K. Therefore, by Proposition 3.1.8, there is a unique Haar measure dx on $\mathbb{A}_K$ such that for each finite set S of places of K, necessarily containing the infinite places, the restriction dx_S of dx to

$$\mathbb{A}_{K,S} = \prod_{\nu\in S} K_\nu \times \prod_{\nu\notin S} \mathfrak{o}_\nu$$

is precisely the product measure $dx_S = \prod_{\nu\in S} dx_\nu$. We write $dx = \prod_\nu dx_\nu$ for the Haar measure on $\mathbb{A}_K$.

Definition 3.2.12. Let K be a number field and let K_ν be the completion at a place ν of K. Let dx_ν be the corresponding measures on K_ν as given in Definition 3.2.10. As in Proposition 3.1.8, we set $dx = \prod_\nu dx_\nu$ to be the Haar measure on $\mathbb{A}_K$.

Proposition 3.2.13. *For every $\alpha = (\alpha_\nu) \in \mathbb{I}_K$, let $|\alpha|_{\mathbb{A}_K} = \prod_\nu |\alpha_\nu|_\nu$, where $|\cdot|_\nu$ are defined as in Definition 3.2.9. Note that the product on the right is finite because $\alpha_\nu \in \mathfrak{o}_\nu^\times$ for all but finitely many places. If μ is a Haar measure on $\mathbb{A}_K$, then*

$$\frac{\mu(\alpha\cdot M)}{\mu(M)} = |\alpha|_{\mathbb{A}_K}$$

for any $\alpha \in \mathbb{I}_K$ and for any measurable set M with $0 < \mu(M) < \infty$.

Proof. Let B be a compact set of $\mathbb{A}_K$. By Proposition 3.1.4, compact sets in $\mathbb{A}_K$ are of the form $\prod_\nu B_\nu$, where B_ν is a compact set for all ν and where $B_\nu = \mathfrak{o}_\nu$ for all but finitely many ν. Let $dx = \prod_\nu dx_\nu$ be as in Definition 3.2.12. Again, it suffices to pick any Haar measure and any measurable set with finite nonzero measure. Then by Proposition 3.1.9 and Proposition 3.2.11, we have that

$$\mathrm{Vol}(\alpha \cdot B, dx) = \int\limits_{\alpha \cdot B} dx = \prod_\nu \int\limits_{\alpha_\nu \cdot B_\nu} dx = \prod_\nu |\alpha_\nu|_\nu \int\limits_{K_\nu} dx = |\alpha|_{\mathbb{A}_K} \int\limits_B dx = |\alpha|_{\mathbb{A}_K} \mathrm{Vol}(B, dx).$$

This completes the proof. □

In addition to Proposition 3.2.11, the normalized absolute values in Definition 3.2.9 satisfy another very important property. Let l/k be finite extension of fields. If one fixes a basis of l over k, then we know that every endomorphism of l as a k-vector space is uniquely representable as a matrix with entries in k. Since l is a field, every element x of l defines an endomorphism ρ_x of l as a k-vector space via multiplication. This formally is called the regular representation. The norm of x, $N_{l/k}(x)$, is defined to be the determinant of the matrix representation of ρ_x. Note that the determinant is independent of the basis chosen for l. Since the determinant is multiplicative, then the norm is multiplicative. For a separable extension, we have that $N_{l/k}(x) = \prod_\sigma \sigma(x)$, where the product runs over the k-embeddings of l. See Neukirch [23], Chapter 1, Section 2 for more information about the norm.

Lemma 3.2.14. *Let l/k be a finite extension of local fields. Then for all $x \in l$, we have*

$$|x|_l = |N_{l/k}(x)|_k.$$

Proof. Let $l = \mathbb{C}$ and $k = \mathbb{R}$. Let $x \in \mathbb{C}$. Let us fix the basis $e = \{1, i\}$ of $\mathbb{C}$ over $\mathbb{R}$. Then $x = a + bi$ for some unique choice of $a, b \in \mathbb{R}$. Further, $x \cdot 1 = a + bi = [a \;\; b]_e^t$ and $x \cdot i = -b + ai = [-b \;\; a]_e^t$. As such, the matrix representation of the endomorphism ρ_x, defined by $z \to x \cdot z$, written with respect to the basis e, is given by

$$[\rho_x]_e = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}_e$$

Then $N_{\mathbb{C}/\mathbb{R}}(x) = \det(\rho_x) = \det([\rho_x]_e) = a^2 + b^2$. By definition, we have $|x|_{\mathbb{C}} = |x|_{\mathrm{st}}^2 = a^2 + b^2$, where $|\cdot|_{\mathrm{st}}$ is the standard absolute value on $\mathbb{C}$. On the other hand, $|N_{\mathbb{C}/\mathbb{R}}(x)|_{\mathbb{R}} = |a^2 + b^2|_{\mathbb{R}} = a^2 + b^2$. Therefore, $|x|_{\mathbb{C}} = |N_{\mathbb{C}/\mathbb{R}}(x)|_{\mathbb{R}}$.

Now, let k be non-Archimedean with the uniformizing parameter π_k. Let π_l be the uniformizing parameter of l. Let $q_k = [\mathfrak{o}_k : \pi_k \mathfrak{o}_k]$ and $q_l = [\mathfrak{o}_l : \pi_l \mathfrak{o}_l]$. Every element in l can be written uniquely in the form $u\pi_l^m$ for some $m \in \mathbb{Z}$ and some $u \in \mathfrak{o}_l^\times$. Since the norm is multiplicative, then $N_{l/k}(u) = N_{l/k}(u^n) = N_{l/k}(u)^n$, which implies that $N_{l/k}(u) \in \mathfrak{o}_K^\times$. Therefore, $|u\pi_l^m|_l = |u|_l|\pi_l^m|_l = |\pi_l|_l^m$ and $|N_{l/k}(u\pi_l^m)|_k = |N_{l/k}(u)N_{l/k}^m(\pi_l)|_k = |N_{l/k}(\pi_l)|_k^m$. As such, it suffices to pick $x = \pi_l$. Let e be the ramification index of l/k; e is determined by the relation $\pi_k = v\pi_l^e$ for some $v \in \mathfrak{o}_l^\times$. Let f be the residual degree of l/k. That is, f is determined by the relation $q_1 = q^f$. From Proposition 2.2.3 and Proposition 3.2.11 we have that

$$|\pi_k|_l = \mathrm{mod}_l(\pi_K) = \mathrm{mod}_k(\pi_K)^n = |\pi_K|_k^n = q^{-n},$$

where $n = [l : k]$. On the other hand, by our choice of e and f, we have that

$$|\pi_k|_l = |\pi_l^e|_l = q_l^{-e} = q^{-ef},$$

which yields the relation $n = ef$. Since the uniformizing parameter is only unique up to a unit in the ring of integers, then we can replace π_k with $v^{-1}\pi_k$, so that $\pi_k = \pi_l^e$. Since $\pi_l^e \in k$, then

$$N_{l/k}(\pi_l^e) = \pi_k^n,$$

, and hence

$$|N_{l/k}(\pi_l^e)|_k = |\pi_k^n|_k = \frac{1}{q^n} = \frac{1}{q^{ef}}.$$

But since $|N_{l/k}(\pi_l^e)|_k = |N_{l/k}(\pi_l)|_k^e$, then $|N_{l/k}(\pi_l)|_k = \frac{1}{q^f}$. On the other hand, by definition, we have that

$$|\pi_l|_l = \frac{1}{q_1} = \frac{1}{q^f}.$$

This completes the proof. □

Theorem 3.2.15. *Let K be a number field. Then:*

(i) For every $x \in K^*$ we have $|x|_{\mathbb{A}_K} = 1$. This is typically referred to as Artin's product formula*.*

(ii) The absolute value map $|\cdot|_{\mathbb{A}_K}$ is surjective.

Proof. (i) Let E/K be a finite field extension. Let $\mathfrak{P}_E$ and $\mathfrak{P}_K$ denote the set of places of E and K, respectively. By the lemma above, for all $x \in E$ we have

$$|x|_{\mathbb{A}_E} = \prod_{\nu \in \mathfrak{P}_E} |x_\nu|_\nu = \prod_{u \in \mathfrak{P}_K} \prod_{\substack{\nu \in \mathfrak{P}_E \\ \nu | u}} |x|_\nu = \prod_{u \in \mathfrak{P}_K} \prod_{\substack{\nu \in \mathfrak{P}_E \\ \nu | u}} |N_{E_\nu/K_u}|_u.$$

Recall from Proposition 2.2.6 that $E \otimes_K K_u \cong \prod_{\nu|u} E_\nu$. Therefore,

$$\prod_{\nu|u} N_{E_\nu/K_u}(x) = N_{E/K}(x),$$

which implies that

$$|x|_{\mathbb{A}_E} = \prod_u |N_{E/K}(x)|_u.$$

As such, it suffices to take $K = \mathbb{Q}$. Every rational x can be expressed as $x = p_1^{e_1} \cdots p_r^{e_r}$ for some unique rational primes $p_1, \ldots, p_r$, and integers $e_1, \ldots, e_r$. Also, note that $|-1|_\nu = 1$ for all ν. Since the absolute value is multiplicative, then it suffices to consider one rational prime p. Then p has non-trivial absolute value at two places, the place corresponding to p and the infinite place. Then we have that $|p|_{\mathbb{A}_\mathbb{Q}} = |p|_\infty \cdot |p|_p = p \cdot \frac{1}{p} = 1$, which completes the proof of part (i).

(ii) Let K be a number field. Let ν be any infinite place of K. Let $t \in \mathbb{R}_+^\times$. If ν is a real place, then let x be the adele with t in the νth component and with 1's elsewhere. If ν is a complex place, then let x be the adele with $\sqrt{t}$ in the νth component and with 1's elsewhere. In either case, $|x|_\infty = |t|_\nu = t$, which proves that $|\cdot|_{\mathbb{A}_K}$ is surjective.

$\square$

Since $|\cdot|_{\mathbb{A}_K}$ is a continuous and surjective map from $\mathbb{I}_K$ to $\mathbb{R}_+^\times$ with $K^* \subset \mathrm{Ker}(|\cdot|_{\mathbb{A}_K})$, then the quotient group $C_K = \mathbb{I}_K/K^*$ **cannot** be compact.

Definition 3.2.16. Let K be an algebraic number field. We define the *ideles of norm one* to be

$$\mathbb{I}_K^1 := \mathrm{Ker}(|\cdot|_{\mathbb{A}_K}).$$

. As K^* is a subgroup of $\mathbb{I}_K^1$ by the above theorem, we define the *norm-one idele-class group* to be the quotient group $C_K^1 := \mathbb{I}_K^1/K^*$.

The above theorem implies that the following sequence is short exact:

$$1 \to C_K^1 = \mathbb{I}_K^1/K^* \xrightarrow{\text{inc}} C_K = \mathbb{I}_K/K^* \xrightarrow{|\cdot|_{\mathbb{A}_K}} \mathbb{R}^\times \to 1.$$

In fact, this short exact sequence splits. To see this, fix ν, an infinite place of K, and define the map $R : \mathbb{R}_+^\times \to \mathbb{I}_K/K^*$ by $t \mapsto (t, 1, 1, \ldots)$, where t is in the νth component if ν is a real place, and where $t \mapsto (\sqrt{t}, 1, 1, \ldots)$ if ν is a complex place. Then $R \circ |\cdot|_{\mathbb{A}_K} = \mathrm{id}_{\mathbb{R}_+^\times}$ and hence

$$C_K^1 = C_K \times \mathbb{R}_+^\times.$$

We will conclude this section with a proof that C_K^1 is compact.

Theorem 3.2.17. *Let K be a number field. The quotient group $C_K^1 = \mathbb{I}_K^1/K^*$ is compact.*

Proof. In Theorem 3.2.5, we showed that K is a discrete, co-compact subgroup of $\mathbb{A}_K$. Therefore, there exists a compact subset Φ of $\mathbb{A}_K$ such that $\mathbb{A}_K = K + \Phi$. Let μ be any Haar measure on the locally compact group$\mathbb{A}_K$. Since Φ is compact, then $\mu(\phi) < \infty$. Pick a compact subset Z of $\mathbb{A}_K$ such that $\mu(Z) > \mu(\Phi)$ and define the sets Z_1 and Z_2 as follows:

$$Z_1 = \{z_1 - z_2 : z_1, z_2 \in Z\} \quad \text{and} \quad Z_2 = \{z_1 z_2 : z_1, z_2 \in Z_1\}.$$

The set Z_1 is compact because it is the continuous image of $Z \times Z \subset \mathbb{A}_K \times \mathbb{A}_K$ in $\mathbb{A}_K$ under the subtraction mapping. The set Z_2 is compact because it is the continuous image of $Z_1 \times Z_1 \subset \mathbb{A}_K \times \mathbb{A}_K$ in $\mathbb{A}_K$ under the multiplication mapping. Since K is discrete in $\mathbb{A}_K$, then $K \cap Z_2$ is finite and contains nonzero elements; for example, $y_1 \cdots y_r$. Set

$$\Psi = \bigcup_{j=1}^{r} \phi^{-1}\left(\{u, y_j^{-1} v) : u, v \in Z_1\}\right),$$

where ϕ is the embedding of $\mathbb{I}_K$ into $\mathbb{A}_K \times \mathbb{A}_K$. Note that Ψ is the finite union of inverse images of compact sets under ϕ. We proved in Proposition 3.2.1 that ϕ is a topological isomorphism of $\mathbb{I}_K$ onto its image. Therefore, Ψ is a compact set of $\mathbb{I}_K$. If we can show that $\mathbb{I}_K^1 \subseteq K^*\Psi$, then we are done.

We showed in Proposition 3.2.13 that $|y|_{\mathbb{A}_K}$ is the module of automorphism of $\mathbb{A}_K$, given by multiplication by y. If $x \in \mathbb{I}_K^1$, then left multiplication by x does not change the volume of compact sets with respect to the Haar measure. Specifically, the compact sets xZ and $x^{-1}Z$ have the same volume as Z.

Claim 3.2.18. Since $\mu(Z) > \mu(\Phi)$, then we claim there exist elements $z_1, z_2, z_3, z_4 \in Z$, with $z_1 \neq z_2$ and $z_3 \neq z_4$, such that $\alpha = x(z_1 - z_2)$ and $\beta = x^{-1}(z_3 - z_4)$ are both in K.

Proof. Let us first show that there exist distinct elements $r_1, r_2 \in Z$ such that $r_1 - r_2 \in K$. By Proposition 3.2.5, the sets $k + \Phi$, $k \in K$ are pairwise disjoint and cover $\mathbb{A}_K$. As such, the sets $Z \cap (k + \Phi)$, $k \in K$, are a disjoint cover of Φ. Therefore,

$$\mu(Z) = \sum_{k \in K} \mu\left(Z \cap (k + \Phi)\right).$$

Since μ is a Haar measure, then μ is translation invariant and hence

$$\mu\left(Z \cap (k + \Phi)\right) = \mu\left((-k + Z) \cap \Phi\right).$$

Suppose, by contradiction, that the sets $(-k + Z) \cap \Phi$, $k \in K$ were disjoint. Then we would obtain

$$\mu(Z) = \sum_{k \in K} \mu\left((-k + Z) \cap \Phi\right).$$

However, the right-hand side is certainly less than or equal to $\mu(\Phi)$. But this contradicts the assumption that $\mu(Z) > \mu(\Phi)$. As such, there exist distinct elements $k_1, k_2 \in K$ such that $(-k_1 + Z) \cap (-k_2 + Z) \cap \Phi \neq \emptyset$. Consequently, there exits elements $r_1, r_2 \in Z$ such that $-k_1 + r_1 = -k_2 + r_2$. If $r_1 = r_2$, then $-k_1 = -k_2$, which is a contradiction. Therefore, there exit distinct elements $r_1, r_2 \in Z$ such that $r_1 - r_2 = -k_2 + k_1 \in K$. Since the compact sets xZ and $x^{-1}Z$ have the same volume as Z, then $\mu(xZ) > \mu(\Phi)$ and $\mu(x^{-1}Z) > \mu(\Phi)$. By what we have just shown, there exist $z_1, z_2, z_3, z_4 \in Z$, with $z_1 \neq z_2$ and $z_3 \neq z_4$, such that $\alpha = x(z_1 - z_2)$ and $\beta = x^{-1}(z_3 - z_4)$ are both in K □

Now we return to showing that $\mathbb{I}_K^1 \subseteq K^*\Psi$. By the claim, we have that $\alpha\beta = (z_1 - z_2)(z_3 - z_4) \in K^* \cap Z_2 = \{y_1, \cdots, y_r\}$. Thus, there exists some y_j, $j = 1, \cdots, r$, such that $(z_1 - z_2)(z_3 - z_4)y_j^{-1} = 1$. Then

$$\phi(x\beta) = \phi(z_3 - z_4) = (z_3 - z_4, (z_3 - z_4)^{-1}) = (z_3 - z_4, (z_1 - z_2)y_j^{-1} \in Z_1 \times Z_1 y_j^{-1}.$$

Therefore, $x\beta \in \Psi$, which completes the proof. □

CHAPTER 4
Tate's Thesis

In this treatment of Tate's thesis, we have followed the presentation of Ramakrishnan and Valenza [24], while also referring to Tate [27], Koch [16], Lang [19], and Kudla [18] for some details and ideas. Unfortunately, function fields will not be treated in this exposition of Tate's Thesis. Please see the exercises in Chapter 7 of Ramakrishnan and Valenza [24] and Section 18 in Chapter 7 of Koch [16]. In addition, the distributional approach to Tate's thesis, originally presented by A. Weil in 1966 [29], will not be included. However, we recommend that the reader consult the article written by Kudla [18] in Chapter 6 of *Introduction to the Langlands Program* for an introduction to the distributional approach to Tate's Thesis. Kudla's article is an excellent foray into automorphic forms and representations.

4.1 Local Quasi-Characters and their Associated Local L-factors

Let F be a local field and let $|\cdot|_F$ be the normalized absolute value, as defined in Definition 3.2.9. The unit group $F^\times$ of a local field F is the direct product of $\mathfrak{o}_F^\times \times V(F)$, where $\mathfrak{o}_F^\times$ is a subgroup of of $F^\times$ of elements of absolute value 1 and

$$V(F) := \{y \in \mathbb{R}_+^\times : y = |x|_F, \text{ for some } x \in F^\times\}.$$

If $F = \mathbb{R}$, then $\mathfrak{o}_F^\times = \{\pm 1\}$ and $V(F) = \mathbb{R}_+^\times$. If $F = \mathbb{C}$, then $\mathfrak{o}_F^\times = S^1$ and $V(F) = \mathbb{R}_+^\times$. In the non-Archimedean case, $\mathfrak{o}_F^\times$ is the group of units in the ring of integers of F, and $V(F) = q^{\mathbb{Z}}$, where q is the order of the residue field $\mathfrak{o}_F/\mathfrak{p}\mathfrak{o}_F$ for $\mathfrak{p}$ the unique prime ideal of F. Therefore, for F archimedean, every $x \in F^\times$ can be written uniquely in the form $x = \tilde{x}\rho$, where $\tilde{x} \in \mathfrak{o}_F^\times$ and $\rho > 0$. Also, for F non-archimedean, if we fix a uniformizing parameter π, then every $x \in F^\times$ can be written uniquely in the form $x = \tilde{x}\pi^{\nu(x)}$, where $\tilde{x} \in \mathfrak{o}_F^\times$. Let $\tilde{\cdot}$ denote the continuous homomorphism from $F^\times$ to $\mathfrak{o}_F^\times$.

The following two propositions will prove that any quasi-character $\chi \in \mathrm{Hom}_{\mathrm{cont}}\left(F^{\times}, \mathbb{C}^{\times}\right)$ factors into the product $\chi = \tilde{\chi}\,|\cdot|^{s}$, where $\tilde{\chi}$ is the pullback of a unitary character (image in S^1) on $\mathfrak{o}_F^{\times} \subseteq F^{\times}$ and $s \in \mathbb{C}$. Note that $\tilde{\chi}$ is uniquely defined by the restriction of χ onto the first component of the product factorization of $F^{\times}$. We note that since $\mathfrak{o}_F^{\times}$ is compact, then the set of continuous quasi-characters of $\mathfrak{o}_F^{\times}$ must in fact be unitary; the quasi-characters of $V(F)$ are of the form $t \mapsto t^s$ for some $s \in \mathbb{C}$. Also, only the real part of s is uniquely determined in the above factorization. We call $\Re(s)$ the exponent of χ.

Definition 4.1.1. A $\chi \in \mathrm{Hom}_{\mathrm{cont}}\left(F^{\times}, \mathbb{C}^{\times}\right)$ is *unramified* if it is trivial on the group of units $\mathfrak{o}_F^{\times}$ of F.

Proposition 4.1.2. *For every unramified quasi-character χ of $F^{\times}$ there exists a complex number s such that $\chi(\alpha) = |\alpha|_F^s$ for $\alpha \in F^{\times}$.*

Proof. It is clear that $|\alpha|_F^s = 1$ for all $\alpha \in \mathfrak{o}_F^{\times}$ and that $|\alpha\beta|_F^s = |\alpha|_F^s|\beta|_F^s$. Furthermore, $|\cdot|_F$ is continuous since the topology of the local field is compatible with $|\cdot|_F$. Hence, $|\cdot|_F^s : F^{\times} \to \mathbb{C}^{\times}$ is continuous since the composition of continuous maps is continuous. Therefore, $|\cdot|_F^s$ is an unramified quasi-character. Let χ be an unramified quasi-character. As such, χ factors through the projection $F^{\times} \to V(F)$ defined by $x \mapsto |x|_F$. That is, $\chi(x) = \chi'(|x|_F)$, where $\chi' : V(F) \to \mathbb{C}^{\times}$ is a continuous homomorphism. Identifying $\mathbb{C}^{\times}$ with $\mathbb{R}_+^{\times} \times S^1$ via the map $z = re^{it} \mapsto (r, e^{it})$, we can decompose χ' into two components: $\chi'_r : V(F) \to \mathbb{R}^{\times}$ and $\chi'_u : V(F) \to S^1$. However, we will need this decomposition only for the Archimedean case.

In the case that F is Archimedean, we have $V(F) = \mathbb{R}_+^{\times}$. Let $d\chi'_r$ be the "differential" of χ'_r; that is, $d\chi'_r(t) = \log \chi'_r(e^t)$. Since the differential is a linear map, then it must be equivalent to a multiplication by a real number σ. Exponentiating, we obtain $\chi'_r(|x|_F) = |x|^{\sigma}$. Applying again the "differential argument", we obtain $\chi'_u(|x|_F) = |x|^{it}$ for some real number t. Therefore, $\chi(x) = |x|_F^s$ for some $s \in \mathbb{C}$.

If F is non-Archimedean, we have $V(F) = q^{\mathbb{Z}}$, and hence $\chi' : q^{\mathbb{Z}} \to \mathbb{C}^{\times}$. Therefore, χ' is completely determined by its value on q, and $\chi'(q) = q^s$ for the complex number $s = \frac{log(\chi'(q))}{\log(q)}$, which is determined up to an integer multiple of $2\pi i/\log(q)$. Consequently,

$$\chi(x) = \chi'(|x|_F) = |x|_F^s.$$

$\square$

Proposition 4.1.3. *Every quasi-character χ of $F^\times$ has the form*

$$\chi(x) = \tilde{\chi}(\tilde{x})|x|_F^s,$$

where $\tilde{\chi}$ is a unitary character of $\mathfrak{o}_F^\times$, $\tilde{x}$ is the continuous homomorphism of $F^\times$ to $\mathfrak{o}_F^\times$, and $s \in \mathbb{C}$. The real part of s is uniquely determined by the quasi-character, but the imaginary part of s is not, since $|\cdot|^{i\tau}$ for $\tau \in \mathbb{R}$ is a unitary character. We denote by σ the real part of s and call it the exponent of χ.

Proof. Certainly $\tilde{\chi}(\tilde{x})|\cdot|_F^s$ is a quasi-character because the product of two quasi-characters is a quasi-character. Conversely, let χ be a quasi-character and denote by $\tilde{\chi}$ the restriction of χ to $\mathfrak{o}_F^\times$. Since $\mathfrak{o}_F^\times$ is compact and $\tilde{\chi}$ is a continuous homomorphism of $\mathfrak{o}_F^\times$ into $\mathbb{C}^\times$, then $\tilde{\chi}(\mathfrak{o}_F^\times)$ is a compact subgroup of $\mathbb{C}^\times$ and hence is contained in S^1. Therefore, $\tilde{\chi}$ is an actual character of $\mathfrak{o}_F^\times$. The continuous homomorphism defined by $x \mapsto \chi(x)\tilde{\chi}(\tilde{x})^{-1}$ is, by definition, an unramified quasi-character of $F^\times$. From the previous proposition we have that $\chi(x)\tilde{\chi}^{-1}(\tilde{x}) = |x|_F^s$ for some $s \in \mathbb{C}$. $\square$

Remark 4.1.4. Note that a unitary character on $F^\times$ is not the same as a unitary character on $\mathfrak{o}_F^\times$. Indeed, $|\cdot|_F^{it}$ is a unitary character on F for all $t \in \mathbb{R}$. We always denote unitary multiplicative characters on F^times by $\tilde{\chi}$, regardless of whether there is an associated χ in question. If χ is a quasi-character, then $\tilde{\chi}$ is the pullback of a unitary character on $\mathfrak{o}_F^\times$ as in the proposition above. Other times, we may simply say to consider a unitary character $\tilde{\chi}$, when there is no "larger" χ of which to speak. Furthermore, if we analyze the characters $\tilde{\chi}$ of $\mathfrak{o}_F^\times$, then we can completely specify the quasi-characters of $F^\times$. We will do so case by case.

(i) Let $F = \mathbb{R}$. Since $\mathfrak{o}_\mathbb{R}^\times = \{\pm 1\}$, then $\tilde{\chi}(\tilde{x}) = \tilde{x}^n$ with $n = 0, 1$. If $n = 0$, then $\tilde{\chi} = 1$. Otherwise, if $n = 1$, then $\tilde{\chi} = \operatorname{sgn}$. In sum, a quasi-character of $\mathbb{R}$ is either of the form $|\cdot|^s$ or $|\cdot|^s \operatorname{sgn}$.

(ii) Let $F = \mathbb{C}$. We have that $\mathfrak{o}_\mathbb{C}^\times = S^1$. In Proposition 1.3.1 part (ii), we showed that the dual group of S^1 is isomorphic to $\mathbb{Z}$. Therefore, $\tilde{\chi}(\tilde{x}) = \tilde{x}^n$ for some $n \in \mathbb{Z}$. Identifying $\mathbb{C}^\times$ with $\mathbb{R}_+^\times \times S^1$, every quasi-character of $\mathbb{C}$ takes the form

$$\chi_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}.$$

(iii) Finally, let F be non-Archimedean and $\mathfrak{p}$ be the unique prime ideal in F. Since $\tilde{\chi}$ is continuous and unitary, then for sufficiently large $n \in \mathbb{N}$, the subgroup $1 + \mathfrak{p}^n$ is mapped by $\tilde{\chi}$ to a a neighborhood of 1 in S^1. However, there are no small subgroups of S^1. Therefore, there exists an $n \in \mathbb{N}$ such that $\tilde{\chi}(1 + \mathfrak{p}^n) = \{1\}$. For the smallest n with this property, we call $\mathfrak{p}^n$ the *conductor* of $\tilde{\chi}$. If $\tilde{\chi}$ is trivial ($n = 0$), then we say the conductor is $\mathfrak{p}^0 = \mathfrak{o}_F^\times$. Consequently, $\tilde{\chi}$ is induced by a character on the finite group $\mathfrak{o}_F^\times/(1+\mathfrak{p}^n)$. Since $\mathfrak{o}_F^\times/(1+\mathfrak{p}^n)$ is a finite abelian group, then the character group is isomorphic to the group itself, and hence is finite. Although in the proof of the local functional equation we will denote quasi-characters by $\chi_{s,n}$, where $\chi_{s,n} = |\cdot|_F^s\tilde{\chi}$ and where $\tilde{\chi}$ is unitary character of conductor $\mathfrak{p}^n$, we must realize that often there is more than one unitary character of conductor $\mathfrak{p}^n$.

Two quasi-characters are called *equivalent* if their quotient is an unramified quasi-character. This relation certainly is reflexive, transitive, and symmetric, and hence an equivalence relation. Each equivalence class is isomorphic to the space of unramified quasi-characters. We now will describe the space of quasi-characters with the aforementioned equivalence relation for the three types of local fields.

(i) If $F = \mathbb{R}$, then he space of quasi-characters is a pair of complex-planes.

(ii) If $F = \mathbb{C}$, then the space of quasi-characters is a countable set of complex planes indexed by the integers.

(iii) If F is non-Archimedean, then the space of quasi-characters is a countable set of cylinders

$$\{s \in \mathbb{C} : s \cong s' \text{ if } s - s' = m\frac{2\pi i}{\log(q)},\ m' \in \mathbb{Z}\}.$$

Define the Gamma function by the integral

$$\Gamma(z) = \int_0^\infty e^{-z}t^{z-1}dt.$$

This integral(function) converges absolutely and is analytic for $\Re(z) > 0$. The Gamma function can also be defined by the Euler product

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left(\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)\right).$$

Since $1/z \neq 0$ for $\Re(z) > 0$, and the limit as $n \to \infty$ of both $\left(1+\frac{1}{n}\right)^z$ and $\left(1+\frac{z}{n}\right)$ is 1, then the Γ function is never 0 for $\Re(z) > 0$. From this product definition, it is routine to show that $\Gamma(z)$ satisfies the following functional equation

$$\Gamma(z+1) = z\Gamma(z).$$

Using this functional equation, one can show that $\Gamma(z)$ can be meromorphically continued to the entire complex plane with simple poles at $z = -n$, for $n \in \mathbb{N} \cup \{0\}$, with residues $(-1)^n/n!$. See Lang [20], Chapter XV, for an introduction to the Gamma function.

Definition 4.1.5. Let F be a local field and let $\chi \in \mathrm{Hom}_{\mathrm{cont}}(F^\times, \mathbb{C}^\times)$.

(i) If $F = \mathbb{C}$, then let

$$L(\chi_{s,n}) = \Gamma_{\mathbb{C}}(s + \frac{|n|}{2}) = (2\pi)^{-(s+\frac{|n|}{2})}\Gamma(s + \frac{|n|}{2}). \tag{4.1}$$

(ii) If $F = \mathbb{R}$ and $\chi = \tilde{\chi}|\cdot|^s$, then let

$$L(\chi) = \begin{cases} \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2) & \text{if } \tilde{\chi} = 1 \\ \Gamma_{\mathbb{R}}(s+1) & \text{if } \tilde{\chi} = \mathrm{sgn}. \end{cases} \tag{4.2}$$

(iii) If F is non-Archimedean, then let

$$L(\chi) = \begin{cases} (1 - \chi(\pi_F))^{-1} & \text{if } \chi \text{ is unramified ,} \\ 1 & \text{otherwise,} \end{cases} \tag{4.3}$$

where π_F is the uniformizing parameter, a generator of the unique maximal ideal, $\mathfrak{p}$ of F. Note that since χ is unramified, then $\chi(\mathfrak{o}_F) = 1$, which implies that $\chi(\pi_F)$ is well-defined.

Remark 4.1.6. We have seen that each equivalence class of quasi-characters is a surface that is isomorphic to the whole complex plane, or a quotient group of the complex plane. Therefore, it is reasonable to say that $L(\chi)$, for a given local field F, is a function on the domain of quasi-characters of F. In this way, it makes sense to say that $L(\chi)$ is a meromorphic, nonzero, function of $s \in \mathbb{C}$. That is, on each equivalence class of quasi-characters, $L(\chi)$ is a meromorphic function of $s \in \mathbb{C}$.

Given any quasi-character χ of $F^\times$ and a complex number s, the product $\chi|\cdot|_F^s$ is also a character. And we write $L(s,\chi)$ for $L(\chi|\cdot|_F^s)$. We define the *shifted dual* of χ to be

$$\check{\chi} = \chi^{-1}|\cdot|_F$$

so that

$$L((\chi|\check{\cdot}|_F^s)) = L(\chi^{-1}|\cdot|_F^{-s}|\cdot|_F) = L(\chi^{-1}|\cdot|_F^{1-s}) = L(1-s,\chi^{-1}).$$

4.2 Local Additive Characters and the Self-Duality of Local Fields

In order the prove the self-duality of local fields, we will need to establish the existence of a non-trivial additive character. We will now construct the standard non-trivial additive characters for each of the local fields.

(i) ($F=\mathbb{R}$) Let $\psi(x) = e^{-2\pi i x}$. We have $\psi(x) \neq 1$ if and only if $x \in \mathbb{R} - \mathbb{Z}$. Furthermore, $\psi(x+y) = e^{-2\pi i(x+y)} = e^{2\pi i x}e^{-2\pi i y} = \psi(x)\psi(y)$ and $|\psi(x)| = \psi(x)\overline{\psi(x)} = 1$. Clearly, ψ is continuous.

(ii) ($F=\mathbb{C}$) Set $\psi(x) = e^{-2\pi i \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x)}$, where $\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x) = x+\overline{x} = \Re(x)$. We have $\psi(x) \neq 1$ if and only if $\Re(x) \notin \mathbb{Z}$. It can be verified easily that ψ is a continuous homomorphism of $\mathbb{C}$ into S^1.

(iii) (F non-Archimedean). First, we will define a non-trivial character on $\mathbb{Q}_p$, for some rational prime p, and then use the trace map, which is additive, to define a character on a finite extension of $\mathbb{Q}_p$. Define ψ_p on $\mathbb{Q}_p$ by the following composition:

$$\psi_p = [\mathbb{Q}_p \xrightarrow{\text{can.}} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{2\pi i(\cdot)}} S^1].$$

Recall that every $x \in \mathbb{Q}_p$ can be represented in the form

$$x = x_{-r}p^{-r} + x_{1-r}p^{1-r} + \cdots + x_{-1}p^{-1} + x_0 + x_1 p + \cdots = \frac{\sum_{j=-r}^{-1} x_j p^{j+r}}{p^r} + \sum_{i=0}^{\infty} x_i p^i \quad (4.4)$$

with $x_n \in \mathbb{Z}$ and $0 \leq x_n \leq p-1$. Also, $x \in \mathbb{Z}_p$ if and only if $\alpha_{-r} = 0$ whenever $r > 0$. Setting $a := \sum_{j=-r}^{-1} x_j p^{j+r}$ and $b := \sum_{i=0}^{\infty} x_i p^i$, we have

$$x = \frac{a}{p^r} + b \quad \text{with } a \in \mathbb{Z},\ b \in \mathbb{Z}_p,\ r \in \mathbb{N} \cup \{0\}.$$

We say that $\frac{a}{p^r}$ the fractional part of x. Since the class $\frac{a}{p^r} + \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$ is independent of the choice of a and r, then there exists a well-defined homeomorphism λ of $\mathbb{Q}_p$ into $\mathbb{Q}/\mathbb{Z}$ given by

$$\lambda(x) = \frac{a}{p^r} + \mathbb{Z}.$$

We set

$$\psi_p(x) = e^{2\pi i \lambda(x)}.$$

Since $\lambda(x) \in \mathbb{Z}$ if and only if $x \in \mathbb{Z}_p$, then $\psi_p(x) = 1$ if and only if $x \in \mathbb{Z}_p$. As such, $\psi_p(x+y) = \psi_p(x)$ for all $y \in \mathbb{Z}_p$, making ψ_p locally constant.

If $\alpha \in \mathbb{Q}$, then there is a unique expansion of the form

$$\alpha = \sum_p \frac{a_p}{p^{\nu_p}} + b,$$

where $a_p, \nu_p, b \in \mathbb{Z}$ and $a_p = 0$ for all but finitely many primes. Indeed, let the fraction $\frac{a_p}{p^{\nu_p}}$ be the fractional part of α in the p-adic numbers. Then consider the difference $\alpha - \sum_p \frac{a_p}{p^{\nu_p}}$. For $p \neq q$, $\frac{a_p}{p^{\nu_p}} \in \mathbb{Z}_q$, while $\alpha - \frac{a_q}{p^{\nu_q}} \in \mathbb{Z}_q$, since $\frac{a_q}{p^{\nu_q}}$ is the polar part of the q-adic expansion of α. Consequently,

$$\alpha - \sum_p \frac{a_p}{p^{\nu_p}} = \alpha - \frac{a_q}{p^{\nu_q}} - \sum_{p \neq q} \frac{a_p}{p^{\nu_p}} \in \mathbb{Z}_q$$

for all primes q, and thus $\alpha - \sum_p \frac{a_p}{p^{\nu_p}} \in \mathbb{Z}$. We are explicitly realizing $\mathbb{Q}/\mathbb{Z}$ as the direct sum of its p-power torsion subgroups. That is, $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p$. The character ψ_p fits into the following commutative diagram

$$\begin{tikzcd}
\mathbb{Q}_p/\mathbb{Z}_p \arrow[d, "inc"'] \arrow[dr, "\psi_p"] & \\
 & S^1 \\
\mathbb{Q}/\mathbb{Z} \arrow[ur, "r \mapsto e^{2\pi r}"'] &
\end{tikzcd}$$

Unlike $e^{2\pi x}$ for $x \in \mathbb{R}$, the function $\psi_p : x \to e^{2\pi i \lambda(x)}$ does not take all values in S^1, rather it takes exactly the pth power roots of unity.

For any finite extension F of $\mathbb{Q}_p$, we set $\psi(x) = \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(x))$, where ψ_p is the standard additive character on $\mathbb{Q}_p$. Recall that the trace of an element x in F is defined to be the trace of the endomorphism $y \mapsto xy$ of the finite-dimensional $\mathbb{Q}_p$-vector space F. It can be shown that the trace is a non-degenerate bilinear form. See Neukirch [23], Chapter 1, Section 2. The *conductor* of an additive-character of a non-Archimedean local field is defined to be $\mathfrak{p}^m$ where $\mathfrak{p}$ is the unique prime ideal of F and

$$m = \inf\{r \in \mathbb{Z} : \psi|_{\mathfrak{p}^r} = 1\}.$$

Since ψ is continuous and takes the value 1 at 0, then m is finite. With this in mind, let us define the subset $\mathfrak{o}'_F$ of F, called the dual of $\mathfrak{o}_F$, by

$$\mathfrak{o}'_F := \{x \in F : \mathrm{tr}_{F/\mathbb{Q}_p}(x \cdot \mathfrak{o}_F) \subseteq \mathbb{Z}_p\}.$$

Let $x, y \in \mathfrak{o}'_F$. Then

$$\mathrm{tr}_{F/\mathbb{Q}_p}((x+y) \cdot \mathfrak{o}_F) = \mathrm{tr}_{F/\mathbb{Q}_p}(x \cdot \mathfrak{o}_F + y \cdot \mathfrak{o}_F) = \mathrm{tr}_{F/\mathbb{Q}_p}(x \cdot \mathfrak{o}_F) + \mathrm{tr}_{F/\mathbb{Q}_p}(y \cdot \mathfrak{o}_F) \in \mathbb{Z}_p,$$

which implies that $\mathfrak{o}'_F$ is a subgroup of F. Furthermore, if $w \in \mathbb{Z}_p$, then

$$\mathrm{tr}_{F/\mathbb{Q}_p}(wx \cdot \mathfrak{o}_F) = \mathrm{tr}_{F/\mathbb{Q}_p}(x \cdot w\mathfrak{o}_F) = \mathrm{tr}_{F/\mathbb{Q}_p}(x \cdot \mathfrak{o}_F) \in \mathbb{Z}_p,$$

which implies that $wx \in \mathfrak{o}'_F$. These two facts prove that $\mathfrak{o}'_F$ is a $\mathbb{Z}_p$ sub-module of F. Since F is a local field, then

$$\mathfrak{o}'_F = \pi_F^{-d}\mathfrak{o}_F,$$

where π_F is a uniformizing parameter of the unique prime $\mathfrak{p}$ of F. The *different* of a local field, denoted $\mathfrak{D}_F$, is defined to be $\mathfrak{o}'^{-1}_F$. In this notation, we say the conductor of ψ is $\mathfrak{D}_F^{-1} = \pi_F^{-d}\mathfrak{o}_F$.

Armed with the existence of non-trivial additive characters on local fields, except for the function field case, which we are not considering, we will now provide a proof of self-duality.

Theorem 4.2.1. *Let ψ be a fixed nontrivial unitary additive character of the locally compact field F. The existence of such a character was shown above. For each $a \in F$, define $\psi_a : F \to S^1$ by $\psi_a(x) = \psi(ax)$. Then the map $\alpha_\psi : F \to \hat{F}$ given by $a \mapsto \psi_a$ is a topological group isomorphism.*

Proof. We first will show that α_ψ is a well-defined map and, furthermore, is an injective group homomorphism of F into its Pontryagin dual $\hat{F}$. We have

$$\psi_a(x+y) = \psi(a(x+y)) = \psi(ax+ay) = \psi(ax)\cdot\psi(ay) = \psi_a(x)\cdot\psi_a(y)$$

and $|\psi_a(x)| = 1$ for all $x, y \in F$ because ψ a homomorphism of F into S^1. Since left multiplication by a is a continuous map from F into itself, and ψ is a continuous map of F into S^1, then ψ_a is continuous, and thus a unitary character. Consequently, α_ψ is a well defined map. By definition, $\alpha_\psi(a+b) = \psi_{a+b}$. For all $x \in F$, we have

$$\psi_{a+b}(x) = \psi((a+b)x) = \psi(ax+bx) = \psi(ax)\cdot\psi(bx) = \psi_a(x)\cdot\psi_b(x).$$

Hence $\alpha_\psi(a+b) = \alpha_\psi(a)\cdot\alpha_\psi(b)$, which proves that α_ψ is a homomorphism of groups. If ψ_a is trivial, then $\psi_a(x) = 1 \Leftrightarrow \psi(ax) = 1$ for all $x \in F$. However, since left multiplication by a nonzero element of F is an automorphism of the field F viewed as an additive group, then ψ_a is trivial only if $a = 0$ because ψ was assumed to be non-trivial. Conversely, if $a = 0$, then ψ_a is trivial. As a consequence, α_ψ is an injective group homomorphism of F into $\hat{F}$.

The topology on the dual group $\hat{F}$ is the compact-open topology. A neighborhood base of trivial character is given by

$$W(C,V) = \{\chi : \chi(C) \subseteq V\}$$

where C is a compact set of F and V is an open neighborhood of the identity of S^1. A neighborhood base of the identity of S^1 is given by $V_\epsilon := \{s \in S^1 : |s-1| < \epsilon\}$ for $\epsilon \in \mathbb{R}$, $\epsilon > 0$. Furthermore, it suffices to consider compact sets of C of the form $C_m := \{x \in F : |x|_F \leq m\}$ for $m \in \mathbb{R}$ and $m > 0$. In other words, we can reformulate the neighborhood base of the trivial character as

$$W(C_m, \epsilon) = \{\chi : |\chi(x) - 1| < \epsilon \text{ for } x \in C_m\}.$$

By simplifying the topology of the dual group $\hat{F}$ we can simplify the proof of bi-continuity of α_ψ.

Since ψ is continuous, then for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\psi(x) - 1| < \epsilon$ whenever $|x|_F < \delta$. In order to show continuity of the group homomorphism α_ψ we must show that for all $\{W(\epsilon, C_m)\}_{\epsilon, m \in \mathbb{R}_+^\times}$ there exists an open neighborhood U of 0 in F with $\alpha_\psi(U) \subseteq W(\epsilon, C_m)$. The set

$$U = \{y \in F : |y|_F < \delta/m\}$$

is open. For all $y \in U$ we have that

$$|\alpha_\psi(y)(x) - 1| = |\psi(yx) - 1| < \epsilon$$

for all $x \in C_m$ because $|yx|_F = |y|_F|x|_F < \delta$. Therefore, $\alpha_\psi(U) \subseteq W(\epsilon, C_m)$, which implies that α_ψ is a continuous injective group homomorphism.

To show that α_ψ^{-1} is a continuous map of $\alpha_\psi(F)$ onto F, we need to show that for all $\delta > 0$ there exist an $\epsilon > 0$ and $m > 0$ such that $|\alpha_\psi^{-1}(\chi)|_F < \delta$ for all $\chi \in \alpha_\psi(F) \cap W(\epsilon, C_m)$. Since ψ is not trivial, then there exists $x_0 \in F$ with $\psi(x_0) \neq 1$. For a given $\delta > 0$ set

$$\epsilon = |\psi(x_0) - 1|, \quad m = \frac{|x_0|_F}{\delta}.$$

Let $\chi \in \alpha_\psi(F) \cap W(\epsilon, C_m)$. For $y \in F$ with $\alpha_\psi(y) \in W(\epsilon, C_m)$ we have that $|\psi(yx) - 1| < |\psi(x_0) - 1|$ for all $x \in C_m$, or, equivalently, if $x \in F$ with $|x|_F < \frac{|x_0|_F}{\delta}$. It follows that

$x_0 \notin yC_m$, and hence

$$|x_0|_F > |y|_F \frac{|x_0|_F}{\delta} \Leftrightarrow |y|_F < \delta.$$

This proves that α_ψ is a topological isomorphism onto its image.

See Proposition 1.3.4 for a proof that $\hat{F}$ is locally compact. Since α_ψ is an open map, then $\alpha_\psi(F)$ is an open and hence closed subgroup of the locally compact group $\hat{F}$. By Proposition 1.1.21, a closed subgroup of a locally compact group is locally compact in the subspace topology. Let $H = \alpha_\psi(F)$. Then $H^\perp = \{z \in F : \psi(zx) = 1 \text{ for all } x \in F\}$. However, since ψ is non-trivial, then $H^\perp = \{0\}$. The functorial nature of Pontryagin duality, Theorem 1.3.10, tells us that $\hat{F}/H \cong \hat{H^\perp}$, which is trivial. This proves that α_ψ is surjective, completing the proof of self-duality of a local field. $\square$

Let G be a locally compact abelian group G with bi-invariant Haar measure dx. Let $\hat{G}$ be the dual group of G. Note that $\hat{G}$ is locally compact by Proposition 1.3.4. Let $f \in L^1(G)$. Recall that $\hat{f} : \hat{G} \to \mathbb{C}$, the Fourier transform of f, is defined by the formula

$$\hat{f}(\chi) = \int\limits_G f(y)\overline{\chi}(y)dg,$$

for $\chi \in \hat{G}$. Let $\mathfrak{B}(G)$ be the set of functions such that f is continuous, $f \in L^1(G)$, and $\hat{f} \in L^1(\hat{G})$. The Fourier inversion theorem states that there exists a Haar measure $d\chi$ on $\hat{G}$ such that for all $f \in \mathfrak{B}(G)$,

$$f(y) = \int\limits_{\hat{G}} \hat{f}(\chi)\chi(y)d\chi = \hat{\hat{f}}(-y).$$

Note that we have written $-y$, but it could very well be y^{-1} if one is writing the group operation multiplicatively.

Proposition 4.2.2. *Let G be a locally compact abelian group with Haar measure dx, and let $d\chi$ be the dual measure; that is, the measure on the Pontryagin dual $\hat{G}$ relative to which the Fourier inversion formula holds. Suppose that we have an isomorphism $\alpha : G \to \hat{G}$ of topological groups. Then there is a unique measure μ such that $\mu = t \cdot dx$ for some $t \in \mathbb{R}_+^\times$, and μ identifies with its dual measure under α. One calls μ the* self-dual *measure on G relative to the isomorphism α.*

Proof. Identifying $\hat{G}$ with G, we can define the Fourier transform of an $f \in L^1(G)$ by

$$\hat{f}(y) = \int_G f(x)\overline{\alpha(y)}(x)dx,$$

where $\alpha(y)$ is the unique character in $\hat{G}$ associated to $y \in G$. The Fourier inversion theorem asserts the existence of a measure such that $\hat{\hat{f}}(y) = f(-y)$ for all $f \in \mathfrak{B}(G)$ and $y \in G$. Since the Haar measures are unique up to a constant, then the Fourier inversion theorem implies that $\hat{\hat{f}}(y) = \frac{1}{t} \cdot f(-y)$ for some constant t with $\hat{\cdot}$, defined relative to dx. Therefore, if we let $\mu = t \cdot dx$, then μ identifies with the dual measure under α. $\square$

Remark 4.2.3. In this chapter, we will drop the traditional conjugation of the second factor of the integrand. However, do note that the conjugation will reappear in the Fourier inversion formula.

Proposition 4.2.4. *For each local field F, the measure dx in Definition 3.2.10 is self-dual with respect to the standard non-trivial characters constructed above.*

Proof. It suffices to check one function, $f(x)$, by the Fourier inversion theorem.

(i) Pick $f(x) = e^{-\pi x^2}$. We will differentiate the Fourier transform of f and obtain a differential equation. Note that we are justified in bringing the derivative into the integral. See Proposition 4.4.4, part (i), for justification of exchanging the order of the derivative and integral. As such, we have

$$\begin{aligned}
\frac{d}{dy}\hat{f}(y) &= \frac{d}{dy}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi ixy}dx = \int_{-\infty}^{\infty} e^{-\pi x^2}\frac{d}{dy}e^{-2\pi ixy}dx = \\
&= \int_{-\infty}^{\infty} e^{-\pi x^2} - 2\pi ixe^{-2\pi ixy}dx = \int_{-\infty}^{\infty} (-2\pi ixe^{-\pi x^2})e^{-2\pi ixy}dx = \\
&= ie^{-2\pi xy}e^{-\pi x^2}|_{-\infty}^{\infty} + i\int_{-\infty}^{\infty} e^{-\pi x^2}(-2\pi yi)e^{-2\pi ixy}dx = 2\pi y\hat{f}(y).
\end{aligned}$$

Solving the differential equation yields $\hat{f}(y) = Ce^{-\pi y^2}$. We will compute $\hat{f}(0)$ in order to determine C.

$$\begin{aligned}
\hat{f}(0) = &= \int_{-\infty}^{\infty} e^{-\pi x^2} dx = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dxdy} = \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2} r dr d\theta} \\
&= \sqrt{\pi \int_0^{\infty} e^{-\pi u} du} = \sqrt{-e^{-\pi u}|_0^{\infty}} = 1.
\end{aligned}$$

Hence, $C = 1$ and $\hat{f}(y) = e^{-\pi y^2}$. Applying the Fourier transform again yields $\hat{\hat{f}}(y) = f(y) = f(-y)$.

(ii) Pick $f(x) = e^{-2\pi x\overline{x}} = e^{-2\pi((\Re x)^2+(\Im x)^2)}$. Let $\xi = y_1 + iy_2$. Then

$$\begin{aligned}
\hat{f}(\xi) &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(x_1^2+x_2^2)} e^{-4\pi i(x_1y_1 - x_2y_2)} dx_1 dx_2 \\
&= 2\left(\int_{-\infty}^{\infty} e^{-2\pi x_1^2} e^{-4\pi i x_1 y_1} dx_1\right) \cdot \left(\int_{-\infty}^{\infty} e^{-2\pi x_2^2} e^{4\pi i x_2 y_2} dx_2\right) \\
&= 2\left(\int_{-\infty}^{\infty} e^{-\pi(\sqrt{2}x_1)^2} e^{-2\pi i(\sqrt{2}x_1)(\sqrt{2}y_1)} dx_1\right) \cdot \left(\int_{-\infty}^{\infty} e^{-\pi(\sqrt{2}x_2)^2} e^{-2\pi i(\sqrt{2}x_2)(-\sqrt{2}y_2)} dx_2\right).
\end{aligned}$$

Applying the change of variable $u_1 = \sqrt{2}x_1$ and $u_2 = \sqrt{2}x_2$, we obtain

$$\begin{aligned}
\hat{f}(\xi) &= \left(\int_{-\infty}^{\infty} e^{-\pi u_1^2} e^{-2\pi i u_1(\sqrt{2}y_1)} du_1\right) \cdot \left(\int_{-\infty}^{\infty} e^{-\pi u_2^2} e^{-2\pi i u_2(-\sqrt{2}y_2)} du_2\right) \\
&= e^{-\pi(\sqrt{2}y_1)^2} e^{-\pi(-\sqrt{2}y_2)^2} = e^{-2\pi(y_1^2+y_2^2)} = e^{-2\pi\xi\overline{\xi}} = f(\xi).
\end{aligned}$$

Therefore, $\hat{\hat{f}}(x) = f(x) = f(-x)$.

(iii) Let $f(x)$ be the characteristic function of $\mathfrak{o}_F$. Let ψ be the standard non-trivial character. Then,

$$\hat{f}(y) = \int_F f(x)\psi(xy)dx = \int_{\mathfrak{o}_F} \psi(xy)dx.$$

We see that when $x \in \mathfrak{o}_F$, $\psi(xy) = 1$ if and only if $y \in \mathfrak{D}_F^{-1}$. Otherwise, the integral is 0 by orthogonality of characters, since $\mathfrak{o}_F$ is a subgroup. By our choice of measure, we have that

$$\hat{f}(y) = \mathrm{Vol}(\mathfrak{o}_F, dx) = N(\mathfrak{D}_F)^{-1/2} \text{ for } y \in \mathfrak{D}_F^{-1},$$

and 0 elsewhere. Then,

$$\hat{\hat{f}}(x) = \int\limits_{\mathfrak{D}_F^{-1}} N(\mathfrak{D}_F)^{-1/2} \chi(yx) dy.$$

For $y \in \mathfrak{D}_F^{-1}$, $\chi(yx)$ is trivial if and only if $x \in \mathfrak{o}_F$. Otherwise, the integral is 0 by orthogonality of characters, since $\mathfrak{D}_F^{-1}$ is a subgroup. Therefore,

$$\hat{\hat{f}}(x) = N(\mathfrak{D}_F)^{-1/2} \mu(\mathfrak{D}_F^{-1}) = N(\mathfrak{D}_F)^{-1/2} N(\mathfrak{D}_F) \mu(\mathfrak{o}_F) = 1 \text{ for } x \in \mathfrak{o}_F,$$

and 0 otherwise. So, $\hat{\hat{f}}(x) = f(x) = f(-x)$.

$\square$

4.3 The Multiplicative Haar Measure

We would like to construct a measure on the multiplicative group $F^\times$ from an additive measure dx. Since F is locally compact and $F^\times = F - \{0\}$ is closed, then $F^\times$ is locally compact as well. Therefore, $F^\times$ admits a Haar measure, which is unique up to positive real constant. If we construct an invariant functional on $\mathcal{C}_c^+(F^\times) = \{f \in \mathcal{C}_c(F^\times) : f(x) \geq 0 \ \forall x \in F^\times \text{ and } ||f||_u > 0\}$ where $||\cdot||_u$ is the uniform or sup norm, then we can apply the Riesz representation theorem to recover the invariant Radon measure on $F^\times$ corresponding to the functional. Let $|\cdot|_F^{-1}$ be as it was in Proposition 3.2.11. If $g \in \mathcal{C}_c^+(F^\times)$, then $g|\cdot|_F^{-1} \in \mathcal{C}_c^+(F - \{0\})$. This is, in fact, a one-to-one correspondence. Let us define a functional Φ on $\mathcal{C}_c^+(F^\times)$ by

$$\Phi(g) = \int_{F-\{0\}} g(x) \frac{dx}{|x|_F}.$$

This is clearly a positive, non-trivial, linear functional on $\mathcal{C}_c^+(F^\times)$. To show the invariance of Φ we consider the action of L_y on g. That is,

$$\Phi(L_y g) = \int_{F-\{0\}} L_y g(x) \frac{dx}{|x|_F} = \int\limits_{F-\{0\}} g(y^{-1}x) \frac{dx}{|x|_F} = \int\limits_{y^{-1}(F-\{0\})} f(x) \frac{|y|_F dx}{|yx|_F} = \int\limits_{F^\times} f(x) \frac{dx}{|x|_F}.$$

Therefore, Φ is an invariant, positive, non-trivial, linear functional on $\mathcal{C}_c^+(F^\times)$. Applying the Riesz representation theorem, we know that this functional Φ must come from an invariant Radon measure (Haar measure). Let us denote this measure d^*x and write $d^*x = \frac{dx}{|x|_F}$. Since the functions in $\mathcal{C}_c^+$ are dense in L^1, then we can extend, by limits, the one-to-one correspondence between $\mathcal{C}_c^+(F^\times)$ and $\mathcal{C}_c^+(F - \{0\})$ to $L^1(F^\times)$ and $L^1(F - \{0\})$, respectively. We summarize the above in the following proposition.

Proposition 4.3.1. *There is a one-to-one correspondence of $L^1(F^\times)$ and $L^1(F - \{0\})$ given by $g(x) \mapsto g(x)|x|_F^{-1}$, and for these functions we have*

$$\int_{F^\times} g(x) d^*x = \int_{F-\{0\}} g(x) \frac{dx}{|x|_F}.$$

Since the Haar measure is unique up to a constant, then any measure on $F^\times$ is of the form $c dx/|x|_F$ for some positive constant c. For F non-Archimedean, let us consider the multiplicative measure

$$d^*x = \frac{q}{q-1} \frac{dx}{|x|_F},$$

where q is the order of the residue field $\mathfrak{o}_F/\mathfrak{p} \cdot \mathfrak{o}_F$ and $\mathfrak{p}$ is the unique prime in F. This multiplicative measure is the unique measure such that $\mathfrak{o}_F^\times$ has the same measure as the additive measure of $\mathfrak{o}_F$. Let us check this:

$$\begin{aligned} \mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) &= \int\limits_{\mathfrak{o}_F^\times} \frac{q}{q-1} \frac{dx}{|x|_F} = \frac{q}{q-1} \int\limits_{\mathfrak{o}_F^\times} dx = \sum_{n=0}^{\infty} q^{-n} \int\limits_{\mathfrak{o}_F^\times} dx = \sum_{n=0}^{\infty} \int\limits_{\mathfrak{o}_F^\times} q^{-n} dx = \\ &= \sum_{n=0}^{\infty} \int\limits_{\pi_F^n \mathfrak{o}_F^\times} dx = \int\limits_{\cup_{n=0}^{\infty} \pi_F^n \mathfrak{o}_F^\times} dx = \int\limits_{\mathfrak{o}_F} dx = \mathrm{Vol}(\mathfrak{o}_F, dx). \end{aligned}$$

4.4 Local Schwartz-Bruhat Functions

Definition 4.4.1. A complex-valued function f on F is *smooth* if it is $\mathcal{C}^\infty$ for F Archimedean, and is locally constant otherwise. That is, if F is non-Archimedean, we say f is *smooth* if $f(x) = f(x_0)$ for all x sufficiently close to x_0. In the Archimedean case, a *Schwartz function* f on F is a smooth function such that the function, together with all its higher derivatives, vanish at infinity faster than any power of $|x|$. That is, f is a Schwartz function if, for any nonnegative integers N, M,

$$\sup_{x \in F}(1 + |x|)^N |\frac{d^M}{dx^M} f(x)| < \infty.$$

A *Schwartz-Bruhat* function is a Schwartz function if F is Archimedean, and is a smooth function with compact support if F is non-Archimedean. Let $S(F)$ denote the space of Schwartz-Bruhat functions.

If f is a Schwarz-Bruhat function on F Archimedean, then $\frac{d^M}{dx^M} f(x) \in L^p$ for all $p \in [1, \infty]$. Indeed, $|\frac{d^M}{dx^M} f(x)| \leq C_N(1 + |x|)^{-n}$ for all N, and $(1 + |x|)^{-n} \in L^p$ for $N > n/p$.

Examples 4.4.2.

(i) If F is Archimedean, then $f_n(x) = x^n e^{-|x|^2}$ is a Schwartz-Bruhat function for any nonnegative integer n.

(ii) If F is non-Archimedean, then the characteristic functions of compact sets of F are Schwartz-Bruhat. Examples of compact sets of F are $\mathfrak{p}^n$ for n, a non-negative integer, where $\mathfrak{p}$, the unique prime of F.

Proposition 4.4.3. *For every $f \in S(F)$, F non-Archimedean, there exist integers m and n, $-m \leq n$, such that $f(x) = 0$ for $x \notin \mathfrak{p}^{-m}$, and for $x \in \mathfrak{p}^{-m}$, $f(y) = f(x)$ for all $y \in x + \mathfrak{p}^n$.*

Proof. Let $x \in \text{supp}(f)$. Since f is locally constant, then there exists an open neighborhood U_x of x such that $f(U_x) = f(x)$. Moreover, since $\{\mathfrak{p}^n\}_{n \in \mathbb{N}}$ forms a neighborhood basis for $0 \in F$, then by homogeneity, we may take $U_x = x + \mathfrak{p}^{n(x)}$ for some $n(x) \in \mathbb{N}$. Then $\cup_{x \in \text{supp}(f)} U_x$ is an open cover of $\text{supp}(f)$. Since the support of f is compact, then finite number of the U_x cover the support. That is, there exists a finite set of $x_1, \ldots, x_r \in \text{supp}(f)$ such that $\text{supp}(f) \subseteq \cup_{i=1}^r U_{x_i}$. Let $n = \min n(x_i)$. Then $\text{supp}(f) \subseteq \cup_{i=1}^n (x + \mathfrak{p}^n)$. Since the Heine-Borel theorem holds for a non-Archimedean local field, then $\text{supp}(f)$, which is compact, is bounded. See Proposition 1.1.33 for a proof in the p-adic case. Also, every bounded set in F is contained in some $\mathfrak{p}^{-m}$. This completes the proof. □

Proposition 4.4.4. *Let $F = \mathbb{C}$ and $\psi(x)$ and dx be given as above. If $f \in S(F)$ then the following assertions hold for $i = 1, 2$:*

(i)

$$\frac{\partial^p}{\partial y_i^p}\hat{f}(y_1 + iy_2) = (-4\pi i)^p(\widehat{x_i^p f})(y_1 + iy_2)$$

(ii)

$$y_i^p \hat{f}(y_1 + iy_2) = (4\pi i)^{-p}(\widehat{\frac{\partial^p}{\partial x_i^p} f})(y_1 + iy_2)$$

Proof.

(i) Let $h \in \mathbb{R}^\times$. Let $K(y_1, y_2) = \hat{f}(y_1 + iy_2)$. Then

$$\begin{aligned}\frac{K(y_1 + h, y_2) - K(y_1, y_2)}{h} &= \frac{2}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1 + ix_2)(e^{-4\pi i(x_1(y_1+h)-x_2y_2)} - e^{-4\pi i(x_1y_1-x_2y_2)})dx_1dx_2 \\ &= \frac{2}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1 + ix_2)(e^{-4\pi i x_1 h} - 1)e^{-4\pi i(x_1y_1-x_2y_2)}dx_1dx_2.\end{aligned}$$

Applying L'Hospitals, we obtain

$$\lim_{h\to 0}\frac{f(x_1 + ix_2)(e^{-4\pi i x_1 h} - 1)}{h} = -4\pi i x_1 f(x).$$

Since $f \in S(F)$, then $-4\pi i x_1 f(x)$ is integrable. Therefore, the Lebesgue dominated convergence theorem yields

$$\frac{\partial}{\partial y_1}\hat{f}(y_1 + iy_2) = \lim_{h\to 0}\frac{K(y_1 + h, y_2) - K(y_1, y_2)}{h} = -4\pi i x_1 \hat{f}(y_1 + iy_2).$$

Using induction, one obtains

$$\frac{\partial^p}{\partial y_1^p}\hat{f}(y_1 + iy_2) = (-4\pi i)^p(\widehat{x_1^p f})(y_1 + iy_2).$$

The same argument works for $\frac{\partial}{\partial y_2}$, which completes part (i).

(ii) Using integration by parts and that $f(x_1+ix_2)e^{-4\pi i(x_1y_1-x_2y_2)} \to 0$ as $x_1 \to \infty$ for a fixed $x_2, y_1, y_2 \in \mathbb{R}$, one obtains:

$$\begin{aligned} y_1\hat{f}(y_1+iy_2) &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1+ix_2)y_1e^{-4\pi i(x_1y_1-x_2y_2)}dx_1dx_2 \\ &= \frac{2}{4\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial}{\partial x_1}f(x_1+ix_2)e^{-4\pi i(x_1y_1-x_2y_2)}dx_1dx_2 = \\ &= (4\pi i)^{-1}(\widehat{\frac{\partial}{\partial x_1}f})(y_1+iy_2) \end{aligned}$$

Using induction, one obtains

$$y_1^p\hat{f}(y_1+iy_2) = (4\pi i)^{-p}(\widehat{\frac{\partial^p}{\partial x_1^p}f})(y_1+iy_2).$$

The same argument works for y_2.

$\square$

Proposition 4.4.5. *Let $F = \mathbb{R}$ and $\psi(x)$ and dx be given as above. If $f \in S(F)$, then the following assertions hold for $p \in \mathbb{N} \cup \{0\}$:*

(i) $\frac{d^p}{dy^p}\hat{f}(y) = (-2\pi i)^p\widehat{(x^pf)}(y)$

(ii) $y^p\ \hat{f}(y) = (2\pi i)^{-p}\widehat{(\frac{d^p}{dx^p}f)}(y)$.

Proof. The proof is identical to the $F = \mathbb{C}$ case, but one dimension less. $\square$

Theorem 4.4.6. *$S(F)$ is a complex vector space and the Fourier transform maps $S(F)$ into $S(F)$.*

Proof. If $a, b \in \mathbb{C}$ and $f, g \in S(F)$, then $af + bg \in S(F)$. Indeed, for F Archimedean,

$$\sup_{x\in F}(1+|x|)^N|\frac{d^M}{dx^M}(af+bg)(x)| \leq \sup_{x\in F}(1+|x|)^N(|a||\frac{d^M}{dx^M}f(x)|+|b||\frac{d^M}{dx^M}g(x)|) < \infty.$$

If F is non-Archimedean, then $af + bg$ is locally constant and of compact support since f, g are locally constant and of compact support. All of the other properties of vector spaces hold and are easily verified. One sees that

$$|\hat{f}(y)| = |\int_F f(x)\psi(xy)|dx \leq \int_F |f(x)\psi(xy)|dx \leq \int_F |f(x)|dx.$$

Hence, if $f \in S(F)$, F Archimedean, then $\hat{f}$ is bounded. We will now proceed case by case.

(i) $(F = \mathbb{R})$ Since $y^p\hat{f}(y) = (2\pi i)^{-p}\widehat{d^p/dx^p f}(y)$ and $d^p/dx^p f \in S(F)$, then $y^p\hat{f}(y)$ is bounded. Thus, $\hat{f}$ tends rapidly to zero at infinity. Also,

$$\begin{aligned} y^p\frac{d^q}{dy^q}(\hat{f})(y) = (-2\pi i)^q y^p(\widehat{x^q f})(y) &= (-2\pi i)^q(2\pi i)^{-p}(\widehat{\frac{d^q}{dx^q}(x^p f)})(y) \\ &= (-1)^q(2\pi i)^{q-p}(\widehat{\frac{d^q}{dx^q}(x^p f)})(y). \end{aligned}$$

Since $d^q/dx^q(x^pf) \in S(F)$, then $y^p(d^q/dx^q)\hat{f}(y)$ is bounded. Therefore, $\hat{f} \in S(F)$.

(ii) $(F = \mathbb{C})$ Since

$$y_1^p y_2^q \hat{f}(y_1 + iy_2) = (4\pi i)^{-p}(-4\pi i)^{-q}(\widehat{\frac{\partial^p}{\partial x_1^p}\frac{\partial^q}{\partial x_2^q}f})(y_1 + iy_2),$$

then $y_1^p y_2^q \hat{f}(y_1 + iy_2)$ is bounded. Thus, $\hat{f}$ tends rapidly to zero at infinity. Also,

$$\begin{aligned} y_1^p y_2^q \frac{\partial^r}{\partial y_1^r}\frac{\partial^s}{\partial y_2^s}\hat{f}(y_1 + iy_2) &= (-1)^r(4\pi i)^{r+s} y_1^p y_2^q(\widehat{x_1^r x_2^s f})(y_1 + iy_2) \\ &= (-1)^{r+q}(4\pi i)^{r+s-p-q}(\widehat{\frac{\partial^p}{\partial x_1^p}\frac{\partial^q}{\partial x_2^q}(x_1^r x_2^s f)})(y_1 + iy_2). \end{aligned}$$

Since $\frac{\partial^p}{\partial x_1^p}\frac{\partial^q}{\partial x_2^q}(x_1^r x_2^s f) \in S(F)$, then $y_1^p y_2^q \frac{\partial^r}{\partial y_1^r}\frac{\partial^s}{\partial y_2^s}\hat{f}(y_1 + iy_2)$ is bounded. Therefore, $\hat{f} \in S(F)$.

(iii) (F non-Archimedean) We will not fix a specific measure for this case. Although we will fix the additive character $\psi(x) = \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(x))$ with conductor $\mathfrak{p}^{-d}$, the proof will hold for any additive character. In view of Proposition 4.4.3, all $f \in S(F)$ factor through the finite quotient group $\mathfrak{p}^{-m}/\mathfrak{p}^n$, $m, n \in \mathbb{Z}$, $-m \leq n$. Let R denote a residue system of $\mathfrak{p}^{-m}/\mathfrak{p}^n$ in $\mathfrak{p}^{-m}$. Then we see that

$$\hat{f}(y) = \int_F f(x)\psi(xy)dx = \sum_{x' \in R}\int_{x'+\mathfrak{p}^n} f(x)\psi(xy)dx = \sum_{x' \in R} f(x')\int_{x'+\mathfrak{p}^n}\psi(xy)dx.$$

Applying the translation invariance of the Haar measure, we obtain

$$\int_{x'+\mathfrak{p}^n}\psi(xy)dx = \int_{\mathfrak{p}^n}\psi((x'+x)y)dx = \psi(x'y)\int_{\mathfrak{p}^n}\psi(xy)dx.$$

For $y \in \mathfrak{p}^{-n}\mathfrak{D}_F^{-1}$ and $x \in \mathfrak{p}^n$ we have $\psi(xy) = \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(xy)) = 1$, and hence that

$$\int_{\mathfrak{p}^n} \psi(xy)dx = \mathrm{Vol}(\mathfrak{p}^n, dx) = (N(\mathfrak{p}^n))^{-1}\mathrm{Vol}(\mathfrak{o}_F, dx).$$

For $y \notin \mathfrak{p}^{-n}\mathfrak{D}_F^{-1}$ there exists an $x_0 \in \mathfrak{p}^n$ such that $\psi(x_0y) \neq 1$. Applying the translation invariance of the Haar measure, we obtain

$$\int_{\mathfrak{p}^n} \psi(xy)dx = \int_{\mathfrak{p}^n} \psi((x+x_0)y)dx = \psi(x_0y)\int_{\mathfrak{p}^n} \psi(xy)dx$$

, and hence that $\hat{f}(y) = 0$ for $y \notin \mathfrak{p}^{-n}\mathfrak{D}_F^{-1}$. Note that in order to conclude that $\hat{f}(y) = 0$, we needed that $\mathfrak{p}^n$ be compact, and hence of finite measure. Therefore,

$$\int_{\mathfrak{p}^n} \psi(xy)dx = \begin{cases} N(\mathfrak{p})^{-n}\mathrm{Vol}(\mathfrak{o}_F, dx) & \text{for } y \in \mathfrak{p}^{-n}\mathfrak{D}_F^{-1} \\ 0 & \text{for } y \notin \mathfrak{p}^{-n}\mathfrak{D}_F^{-1}. \end{cases}$$

Putting it all together, we obtain

$$\hat{f}(y) = \begin{cases} N(\mathfrak{p})^{-n}\mathrm{Vol}(\mathfrak{o}_F, dx)\sum_{x' \in R} f(x')\psi(x'y) & \text{for } y \in \mathfrak{p}^{-n}\mathfrak{D}_F^{-1} \\ 0 & \text{for } y \notin \mathfrak{p}^{-n}\mathfrak{D}_F^{-1}. \end{cases}$$

For all $x' \in R$ and $y \in \mathfrak{p}^{-n}\mathfrak{D}_F^{-1}$ we have that

$$\begin{aligned} \psi(x'(y + x'^{-1}\mathfrak{D}_F^{-1})) &= \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(x'(y + x'^{-1}\mathfrak{D}_F^{-1})) = \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(x'y) + \mathrm{tr}_{F/\mathbb{Q}_p}(x'x'^{-1}\mathfrak{D}_F^{-1})) = \\ &= \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(x'y) + \mathrm{tr}_{F/\mathbb{Q}_p}(\mathfrak{D}_F^{-1})) = \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(x'y))\psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(\mathfrak{D}_F^{-1})) = \\ &= \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(x'y)) = \psi(x'y). \end{aligned}$$

This proves that ψ is locally constant. Consequently, $\hat{f} \in S(F)$, which completes the proof. □

Corollary 4.4.7. *$S(F) \subseteq \mathfrak{B}(F)$, where $\mathfrak{B}(F)$ is the set of functions such that the Fourier inversion theorem holds.*

Proof. By definition, if $f \in S(F)$, then f is continuous and $f \in L^1(F)$. In the proposition above we proved that $\hat{f} \in S(F)$, which implies that $\hat{f} \in L^1(\hat{F}) = L^1(F)$. This completes the proof. □

Corollary 4.4.8. *The Fourier transform is an isomorphism of $S(F)$.*

Proof. We know that the Fourier transform maps $S(F)$ into $S(F)$ by the theorem above. It is easy to see that the map $f \mapsto \check{f}$, defined by $\check{f}(x) = \hat{f}(-x)$, maps $S(F)$ into $S(F)$. The Fourier inversion formula, or the self-duality of the measure in this case (local fields), asserts that $\check{\cdot}$ and $\hat{\cdot}$ are inverses of each other. Therefore, $\hat{\cdot}$ is an isomorphism. $\square$

4.5 The Meromorphic Continuation and Functional Equation of the Local Zeta Function

Definition 4.5.1. For $f \in S(F)$ and $\chi \in \mathrm{Hom}_{\mathrm{cont}}(F^\times, \mathbb{C}^\times)$, we define the associated *local zeta function* to be

$$Z(f, \chi) = \int_{F^\times} f(x)\chi(x)d^*x.$$

Note that $Z(f, \chi)$ is dependent on the multiplicative measure d^*x. If we fix an additive measure dx and choose $d^*x = dx/|x|_F$ as in Proposition 4.3.1, then $Z(f, \chi)$ is dependent on dx.

Remark 4.5.2. In the same way as Remark 4.1.6, $Z(f, \chi)$ is a function on the domain of quasi-characters of F. Since each equivalence class of quasi-characters is a surface that is isomorphic to either the whole complex plane or a quotient group of the complex plane, then we may speak of the analytic continuation from one subset of an equivalence class to a larger subset. In the next theorem, we first will show that $Z(f, \chi)$ is a holomorphic and absolutely convergent function in the domain of quasi-characters of exponent ($\sigma = \Re(s)$) greater than 1. Furthermore, we will show that it satisfies a functional equation, which thereby yields an analytic continuation of $Z(f, \chi)$ to a function in the domain of quasi-characters of all exponents.

In the following theorem, for F Archimedean we take ψ to be standard non-trivial additive character, and dx to be the Lebesgue measure if $F = \mathbb{R}$ or twice the Lebesgue measure if $F = \mathbb{C}$. Also, set $d^*x = dx/|x|_F$, where $|x|_F$ is the normalized absolute value of F. However, in the style of Ramarkishnan and Valenza [24], for non-Archimedean fields F we will not restrict ourselves to a specific additive character, Haar measure, or multiplicative measure. Nevertheless, for any dx and d^*x one has the relation $d^*x = c\,dx/|x|_F$ for some positive real constant c, since the multiplicative Haar measure is unique up to a constant. Two factors, γ and ϵ, will appear in the theorem below. Both of them will depend, most importantly, on χ, and hence on s, but they will also depend on dx and ψ. As such, we write $\epsilon(\chi, \psi, dx)$ and $\gamma(\chi, \psi, dx)$ to indicate their dependence on χ, ψ, and dx. The pair (ψ, dx) is not required to be self-dual in the proof of this theorem. In the following section, we will show how $\epsilon(\chi, \psi, dx)$ depends on the choice of ψ and dx.

Theorem 4.5.3. *Let $f \in S(F)$, and $\chi = \tilde{\chi}|\cdot|^s$ where $\tilde{\chi}$ is the unitary part of the quasi-character χ. Let $\sigma = \Re(s)$. Then the following statements hold:*

(i) $Z(f, \chi) = Z(f, \tilde{\chi}, s)$ *is holomorphic and absolutely convergent if $\sigma > 0$.*

(ii) *If $0 < \sigma < 1$, then there is a functional equation*

$$Z(\hat{f}, \check{\chi}) = \gamma(\chi, \psi, dx)Z(f, \chi),$$

for some $\gamma(\chi, \psi, dx)$, which is both independent of f and meromorphic as a function of s. Thus, $Z(f, \chi)$ admits a meromorphic continuation to the whole s plane.

(iii) *There exists a factor $\epsilon(\chi, \psi, dx)$ that lies in $\mathbb{C}^\times$ for all s and satisfies the relation*

$$\gamma(\chi, \psi, dx) = \epsilon(\chi, \psi, dx)\frac{L(\check{\chi})}{L(\chi)}.$$

Therefore, the relation

$$L(\chi)Z(\hat{f}, \check{\chi}) = \epsilon(\chi, \psi, dx)L(\check{\chi})Z(f, \chi)$$

illustrates that the poles of $Z(f, \chi)$ are no worse than those of $L(\chi)$, which is independent of f. Furthermore, $L(\chi) = Z(f_0, \chi)$ for some suitable f_0.

Proof. (i) Since $f \in S(F)$, then for F Archimedean we have $|f(x)||x|_F^{\sigma-1} \to 0$ rapidly as $|x| \to \infty$. Let K be a punctured neighborhood of 0 in F. We know that there exists a positive real number C such that $|f(x)| \leq C$ for all $x \in K$. The local zeta function convergence thus is determined by the integrability of $|x|_F^{\sigma-1}$ around zero for any positive σ.

In other words,

$$\begin{aligned}
|Z(f,\chi)| &= \left|\int\limits_{F^\times} f(x)\chi(x)d^*x\right| = \int\limits_{F-\{0\}} |f(x)||x|_F^{\sigma-1}dx \\
&= (\int\limits_{F-K} + \int\limits_{K})|f(x)||x|_F^{\sigma-1}dx \leq M + C\int\limits_{K} |x|_F^{\sigma-1}dx,
\end{aligned}$$

where M is a positive real number. From basic calculus, the integral is finite for $\sigma > 0$. We see that

$$\frac{d}{ds}f(x)\tilde{\chi}(x)|x|_F^s = f(x)\tilde{\chi}(x)\frac{d}{ds}e^{s\log(|x|_F)} = f(x)\tilde{\chi}(x)\log(|x|_F)|x|_F^s,$$

which is continuous and absolutely integrable for $\sigma > 0$. Therefore,

$$\frac{d}{ds}Z(f,\tilde{\chi},s) = \frac{d}{ds}\int\limits_{F^\times} f(x)\tilde{\chi}(x)|x|_F^s d^*x = \int\limits_{F^\times} f(x)\tilde{\chi}(x)\log(|x|_F)|x|_F^s d^*x.$$

We now are left with the non-Archimedean case. Let $d^*x = cdx$. Let q be the order of the residue field $\mathfrak{o}_F/\mathfrak{p}\mathfrak{o}_F$, where $\mathfrak{p}$ is the unique prime of F. Since $f \in S(F)$, then by Lemma 5.16, f factors through the finite quotient group $\mathfrak{p}^{-m}/\mathfrak{p}^n$, $m,n \in \mathbb{Z}$, $-m \leq n$. By linearity and translation invariance of the Haar measure, it suffices to consider $f = \chi_{\mathfrak{p}^n}$. Let π_F be a uniformizing parameter of $\mathfrak{p}$. From

$$\pi_F^n\mathfrak{o}_F - \{0\} = \bigcup_n^\infty \pi_F^k\mathfrak{o}_F^\times$$

and the translation invariance of the multiplicative measure, it follows that

$$\begin{aligned}
|Z(f,\chi)| &= c\int\limits_{F-\{0\}} |f(x)||x|_F^{\sigma-1}dx = c\int\limits_{F-\{0\}} \chi_{(\pi_F^n)}|x|_F^{\sigma-1}dx = c\sum_{k=n}^\infty \int\limits_{\pi_F^k\mathfrak{o}_F^\times} |x|_F^\sigma d^*x = \\
&= \sum_{k=n}^\infty \int\limits_{\mathfrak{o}_F^\times} |\pi_F^k x|_F^\sigma d^*x = \sum_{k=n}^\infty q^{-k\sigma}\int\limits_{\mathfrak{o}_F^\times} d^*x = \frac{q^{-n\sigma}}{1-q^{-\sigma}}\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x),
\end{aligned}$$

which is finite for $\sigma > 0$. Like the Archimedean case, we can apply d/ds to $Z(f,\tilde{\chi},s)$. We have proved part (i).

(ii) We will first prove a lemma.

Lemma 4.5.4. *For all $\chi \in X(F^\times)$ with $0 < \sigma < 1$, we have*

$$Z(f,\chi)Z(\hat{g},\check{\chi}) = Z(\hat{f},\check{\chi})Z(g,\chi).$$

Proof. By part (i), the zeta functions that have $\chi = \tilde{\chi}|\cdot|_F^s$ are well-defined for $\sigma > 0$, where as the zeta functions that have a $\check{\chi} = \tilde{\chi}^{-1}|\cdot|_F^{1-s}$ are well-defined for $\Re(1-s) > 0 \Leftrightarrow 1-\sigma > 0 \Leftrightarrow \sigma < 1$. Recall that since $f, g \in S(F)$, then $\hat{f}, \hat{g} \in S(F)$. Therefore, the above zeta functions are all well-defined for $0 < \sigma < 1$. By definition,

$$Z(f,\chi)Z(\hat{g},\check{\chi}) = \iint\limits_{F^\times \times F^\times} f(x)\chi(x)\hat{g}(y)\chi^{-1}(y)|y|_F d^*x d^*y = \iint\limits_{F^\times \times F^\times} f(x)\hat{g}(y)\chi(xy^{-1})|y|_F d^*x d^*y.$$

The product (Haar) measure on $F^\times \times F^\times$ is the d^*xd^*y, and hence the measure is invariant under the translation $(x,y) \mapsto (x,xy)$. Applying this transformation, the double integral becomes

$$\iint\limits_{F^\times \times F^\times} f(x)\hat{g}(xy)\chi(y^{-1})|xy|_F d^*x d^*y = \int\limits_{F^\times} \left(\int\limits_{F^\times} f(x)\hat{g}(xy)|x|_F d^*x \right) \chi(y^{-1})|y|_F d^*y.$$

The last equality is justified by Fubini's theorem, which is applicable because $f \in S(F)$ and $g \in S(F) \Rightarrow \hat{g} \in S(F)$. From the definition of $\hat{g}$, and Fubini's Theorem, we obtain

$$\int\limits_{F^\times} f(x)\hat{g}(xy)|x|_F d^*x = c\int\limits_{F} g(z) \left(\int\limits_{F} f(x)\phi(xzy)dx \right) dz = \int\limits_{F^\times} g(z)\hat{f}(zy)|z|_F d^*z.$$

Applying the above results and Fubini's theorem, we obtain

$$\begin{aligned} Z(f,\chi)Z(\hat{g},\check{\chi}) &= \int\limits_{F^\times} \left(\int\limits_{F^\times} g(z)\hat{f}(zy)|z|_F d^*z \right) \chi(y^{-1})|y|_F d^*y \\ &= \int\limits_{F^\times \times F^\times} \hat{f}(zy)h(z)\chi(y^{-1})|zy|_F d^*y d^*z \end{aligned}$$

Making the change of variable $y \mapsto z^{-1}y$ yields

$$\begin{aligned}
Z(f,\chi)Z(\hat{g},\check{\chi}) &= \int\limits_{F^\times \times F^\times} \hat{f}(y)h(z)\chi((z^{-1}y)^{-1})|y|_F d^*y d^*z \\
&= \int\limits_{F^\times \times F^\times} \hat{f}(y)h(z)\chi(y^{-1}z)|y|_F d^*y d^*z \\
&= \int\limits_{F^\times \times F^\times} \hat{f}(y)\chi^{-1}(y)|y|_F h(z)\chi(z) d^*y d^*z \\
&= Z(\hat{f},\check{\chi})Z(h,\chi),
\end{aligned}$$

which completes the proof. □

Let us return to the proof of part (ii). Fix a Schwartz function $f_0 \in S(F)$ and put

$$\gamma(\chi,\psi,dx) = \frac{Z(\hat{f}_0,\check{\chi})}{Z(f_0,\chi)}.$$

By the lemma above, γ is independent of the choice of f_0, and hence we obtain the desired result

$$Z(\hat{f},\check{\chi}) = \gamma(\chi,\psi,dx)Z(f,\chi). \tag{4.5}$$

Also, notice that $\gamma(\chi,\psi,dx)$ is independent of the multiplicative measure d^*x chosen. Indeed, since the Haar measure is unique up to a positive real constant, then let $d^\times = t \cdot d^*x$. We see that

$$\gamma(\chi,\psi,dx) = \frac{Z(\hat{f},\check{\chi})}{Z(f,\chi)} = \frac{\int_{F^\times} \hat{f}(x)\check{\chi}(x)d^*x}{\int_{F^\times} f(x)\chi(x)d^*x} = \frac{\int_{F^\times} \hat{f}(x)\check{\chi}(x)td^*x}{\int_{F^\times} f(x)\chi(x)td^*x} = \frac{\int_{F^\times} \hat{f}(x)\check{\chi}(x)d^\times x}{\int_{F^\times} f(x)\chi(x)d^\times x}.$$

As a bi-product of our calculations in part (iii), we will show that $\gamma(\chi,\psi,x)$ is meromorphic as a function of s. As such, since $Z(f,\chi)$ is holomorphic for $\sigma > 0$ and that $Z(\hat{f},\check{\chi})$ is holomorphic for $\sigma < 1$, then equation 4.5 yields a meromorphic continuation of $Z(f,\chi)$ to the entire complex plane.

(iii) In this part, we will choose test functions $f \in S(F)$ for each type of local field of characteristic zero and moreover, for specific equivalence class of quasi-characters, such that

$$Z(f,\chi) = h(f,\chi,\psi,dx)L(\chi)$$

and

$$Z(\hat{f},\chi) = h(\hat{f},\check{\chi},\psi,dx)L(\chi),$$

for some entire nonzero function h that is dependent on f, ψ and dx. If one does not fix the multiplicative measure $d^*x = dx/|x|_F$, then h is also dependent on d^*x. As such, we will have

$$\gamma(\chi,\psi,dx) = \frac{Z(\hat{f},\check{\chi})}{Z(f,\chi)} = \frac{h(\hat{f},\check{\chi},\psi,dx)}{h(f,\chi,\psi,dx)}\frac{L(\check{\chi})}{L(\chi)}.$$

Since $L(\chi)$, $L(\check{\chi})$, and $h(\hat{f},\check{\chi},\psi,dx)/h(f,\chi,\psi,dx)$ are meromorphic as a functions of s, then we will establish that $\gamma(\chi,\psi,dx)$ is a meromorphic as a function of s, and **is not** dependent on f. Finally, we will get that

$$\epsilon(\chi,\psi,dx) = \frac{h(\hat{f},\check{\chi},\psi,dx)}{h(f,\chi,\psi,dx)}.$$

Although $h(\hat{f},\check{\chi},\psi,dx)$ and $h(f,\chi,\psi,dx)$ are dependent on f and are only intermediary results of this proof, they will reappear when we prove the functional equation of the Hecke L-function. More specifically, h will be used to prove that the global L-function $L(s,\tilde{\chi})$, where $\tilde{\chi}$ is a unitary idele-class character, is a meromorphic function of s (Theorem 4.10.4).

We now proceed case by case.

($F = \mathbb{R}$). For this calculation, we take dx be the Lebesgue measure and $\psi(x) = e^{-2\pi i x}$. Recall that every quasi-character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}^\times,\mathbb{C}^\times)$ is either of the form $|\cdot|^s$ or $\mathrm{sgn}\,|\cdot|^s$. Note that we dropped the F from the absolute value because for $\mathbb{R}$, the normalized absolute value is precisely the standard absolute value. These are the two equivalence classes of quasi-characters, both of which are isomorphic to $\mathbb{C}$.

First, consider the equivalence class $\chi = |\cdot|^s$ and pick $f_0(x) = e^{-\pi x^2}$ for this class. The function f_0 is the standard example of a Schwartz function. Then

$$Z(f_0,\chi) = \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^s d^*x = 2\int_0^\infty e^{-\pi x^2}x^{s-1}dx.$$

Applying the change of variable $u = \pi x^2 \Rightarrow du = 2\pi x = 2\pi^{1/2}u^{1/2}$, we obtain

$$Z(f_0,\chi) = \int_0^\infty e^{-u}(u\pi^{-1})^{(s-1)/2}\pi^{-1/2}u^{-1/2}du = \pi^{-s/2}\int_0^\infty e^{-u}u^{s/2-1}du.$$

Since

$$\Gamma(s/2) = \int_0^\infty e^{-u} u^{s/2-1} du,$$

then by the definition of $L(\chi)$ we have that

$$Z(f_0, \chi) = \pi^{-s/2}\Gamma(s/2) = L(\chi) \tag{4.6}$$

for all characters $\chi = |\cdot|^s$. In the proof of self-duality of the measure dx (Proposition 4.2.4), we proved that $\hat{f} = f$. Therefore, by the above argument, we have that

$$Z(\hat{f}_0, \check{\chi}) = L(\check{\chi}).$$

Consequently, for $\chi = |\cdot|^s$, we have

$$\gamma(|\cdot|^s, \psi, dx) = \frac{L(\check{\chi})}{L(\chi)} \text{ and } \epsilon(|\cdot|^s, \psi, dx) = 1. \tag{4.7}$$

Let us now consider the other equivalence class of quasi-characters. That is, $\chi = \operatorname{sgn}|\cdot|^s = \frac{x}{|x|} \cdot |\cdot|^s$. Pick $f_1(x) = xe^{-\pi x^2} \in S(F)$. Then

$$Z(f_1, \chi) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |\cdot|^{s+1} d^*x = \pi^{-(\frac{s+1}{2})}\gamma(\frac{s+1}{2}) = L(\chi), \tag{4.8}$$

where the second to last line follows from the computation done for f_0 above. Using the Fourier transform identity from Proposition 5.1.8 and that $\hat{f}_1 = f_1$, we have

$$\hat{f}_1(y) = (-2\pi i)^{-1} \frac{d}{dy} e^{-\pi y^2} = iye^{-\pi y^2}.$$

Therefore,

$$Z(\hat{f}_1, \check{\chi}) \;=\; i\int_{\mathbb{R}^\times} xe^{-\pi x^2} \cdot \frac{x}{|x|} \cdot |x|^{1-s} d^*x = i\int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^{2-s} d^*x = i\pi^{-(\frac{2-s}{2})}\Gamma(\frac{2-s}{2}).$$

By definition $\check{\chi} = \operatorname{sgn}^{-1}|\cdot|^{1-s} = \operatorname{sgn}|\cdot|^{1-s}$, and hence $L(\check{\chi}) = \pi^{-(\frac{2-s}{2})}\Gamma(\frac{2-s}{2})$. Therefore, $Z(\hat{f}, \check{\chi}) = iL(\check{\chi})$, which implies that

$$\gamma(\operatorname{sgn}|\cdot|^s, \psi, dx) = i\frac{L(\check{\chi})}{L(\chi)} \text{ and } \epsilon(\operatorname{sgn}|\cdot|^s, \psi, dx) = i. \tag{4.9}$$

In addition, we see that for the two equivalence classes of quasi-characters of $F = \mathbb{R}$, denoted solely by the test function choice ($n = 0, 1$), that we have

$$h(f_n, \chi, \psi, dx) = 1 \quad \text{and} \quad h(\hat{f}_0, \check{\chi}, \psi, dx) = 1 \quad \text{and} \quad h(\hat{f}_1, \check{\chi}, \psi, dx) = 1.$$

($F = \mathbb{C}$) We set dx to be twice the standard Lebesgue measure and $\psi(x) = e^{-2\pi i(x+\overline{x})}$. Although it is not conventional to use x for a complex variable, we use it anyway. Every quasi-character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}^\times, \mathbb{C}^\times)$ is of the form $\chi = \chi_{s,n} : re^{i\theta} \mapsto r^s e^{in\theta}$ for some $s \in \mathbb{C}$ and some uniquely defined $n \in \mathbb{Z}$. Only the real part of s, $\sigma(s)$ is uniquely defined. For each equivalence class of quasi-characters, indexed by n , pick

$$f_n(x) = \begin{cases} (2\pi)^{-1}\overline{x}^n e^{-2\pi x\overline{x}} & \text{for } n \geq 0 \\ (2\pi)^{-1} x^{-n} e^{-2\pi x\overline{x}} & \text{for } n < 0. \end{cases}$$

It is clear that $f_n \in S(F)$ for all integers n.

Claim 4.5.5. For all integers n we have the relation $\hat{f}_n(x) = i^{|n|} f_{-n}(x)$.

Proof. Since $f_0(x) = e^{-2\pi z\overline{x}}$, then $\hat{f}_0(x) = f_0(x)$. We proved this in Proposition 4.2.4. Assume the formula is true for some $m \in \mathbb{N}$. That is, assume we have

$$\int_{\mathbb{C}} f_m(x) e^{-2\pi i(x+\overline{x})} dx = i^m f_{-m}(x),$$

and, equivalently, that for $x = y_1 + iy_2$

$$2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - ix_2)^m e^{-2\pi(x_1^2+x_2^2)-4\pi i(x_1y_1 - x_2y_2)} dx_1 dx_2 = i^m (y_1 + iy_2)^m e^{-2\pi(y_1^2+y_2^2)}.$$

Now, let us apply the operator

$$D = \frac{1}{4\pi i}\left(\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}\right)$$

to the above equality. If h is an analytic function, then the Cauchy-Riemann equations imply that

$$\frac{\partial h}{\partial y_1} h(y_1 + iy_2) = -i\frac{\partial h}{\partial y_2} h(y_1 + iy_2) \Leftrightarrow \left(\frac{\partial}{\partial y_1} + i\frac{\partial}{\partial y_2}\right) h(y_1 + iy_2) = 0.$$

Since $(y_1 + iy_2)^m$ is analytic, then

$$\begin{aligned} Di^m(y_1+iy_2)^m e^{-2\pi(y_1^2+y_2^2)} &= i^m(y_1+iy_2)^m \frac{-4\pi}{4\pi i}(y_1+iy_2)e^{-2\pi(y_1^2+y_2^2)} \\ &= i^{m+1}(y_1+iy_2)^{m+1}e^{-2\pi(y_1^2+y_2^2)}. \end{aligned}$$

Therefore, applying D to both sides of the induction hypothesis equation, we obtain

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x_1-ix_2)^{m+1}e^{-2\pi(x_1^2+x_2^2)-4\pi i(x_1y_1-x_2y_2)}dx_1dx_2 = i^{m+1}(y_1+iy_2)^{m+1}e^{-2\pi(y_1^2+y_2^2)},$$

which establishes

$$\widehat{f_{m+1}}(x) = i^{m+1}f_{-(m+1)}(x).$$

We have proved the claim for $n > 0$. By what was just proven we have that $\widehat{f_{-m}}(x) = i^{|m|}f_m(x)$ for $m < 0$. Also, by definition, we have

$$f_{-m}(-x) = (-1)^m f_{-m}(x).$$

Applying the Fourier transform to both sides and using self duality ($\hat{\hat{f}}(x) = f(-x)$), we obtain

$$f_{-m}(-x) = i^{|m|}\hat{f_m}(x) \Leftrightarrow \hat{f_m}(x) = i^{-|m|}f_{-m}(-x) = (-i)^{-|m|}f_{-m}(x) = i^{|m|}f_{-m}(x).$$

This proves the claim. □

In order to compute the zeta function we will make use of polar coordinates. With $x = re^{-i\theta(x)}$, we have

$$f_n(x) = (2\pi)^{-1}r^{|n|}e^{-in\theta(x)}e^{-2\pi r^2} = (2\pi)^{-1}r^{|n|}e^{-2\pi r^2 - in\theta(x)},$$

$$d^*x = \frac{2dxd\overline{x}}{|x|_F} = \frac{2rdrd\theta}{r^2} = \frac{2}{r}drd\theta,$$

and

$$\chi_{s,n}(x) = (x\overline{x})e^{in\theta(x)} = r^{2s}e^{in\theta(x)}.$$

Then

$$\begin{aligned}
Z(f_n,\chi_{s,n}) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} r^{|n|}e^{-2\pi r^2-in\theta(x)}r^{2s}e^{in\theta}\frac{2}{r}drd\theta \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} e^{-2\pi r^2}r^{|n|+2s-1}drd\theta \\
&= (2\pi)^{-(s+\frac{|n|}{2})}2\int_0^{2\pi}\int_0^{\infty} e^{-2\pi r^2}(2\pi r^2)^{s-1+\frac{|n|}{2}}rdrd\theta \\
&= (2\pi)^{-(s+\frac{|n|}{2})}\int_0^{\infty} e^{-2\pi r^2}(2\pi r^2)^{s-1+\frac{|n|}{2}}4\pi rdr.
\end{aligned}$$

Applying the substituion $t = 2\pi r^2 \Rightarrow dt = 4\pi rdr$ we obtain

$$\begin{aligned}
Z(f_n,\chi_{s,n}) &= (2\pi)^{-(s+\frac{|n|}{2})}\int_0^{\infty} e^{-t}t^{s-1+\frac{|n|}{2}}dt \\
&= (2\pi)^{-(s+\frac{|n|}{2})}\Gamma(s+\frac{|n|}{2}) = L(\chi_{s,n}).
\end{aligned} \tag{4.10}$$

Since $\check{\chi_{s,n}} = \chi_{1-s,-n}$ and $\hat{f}_n(x) = i^{|n|}f_{-n}(x)$, then

$$Z(\hat{f}_n,\check{\chi_{s,n}}) = Z(i^{|n|}f_{-n}(x),\chi_{1-s,-n}) = i^{|n|}(2\pi)^{-(1-s+\frac{|n|}{2})}\Gamma(1-s+\frac{|n|}{2}) = i^{|n|}L(\check{\chi_{s,n}}).$$

Therefore,

$$\begin{aligned}
\gamma(\chi_{s,n},\psi,dx) &= i^{|n|}\frac{L(\check{\chi_{s,n}})}{L(\chi_{s,n})} \\
\epsilon(\chi_{s,n},\psi,dx) &= i^{|n|}.
\end{aligned} \tag{4.11}$$

Note that both $h(f_n,\chi,\psi,dx) = 1$ and $h(\hat{f}_n,\check{\chi},\psi,dx) = 1$ for the $n \in \mathbb{Z}$ equivalence classes of quasi-characters of $\mathbb{C}^\times$.

(F non-Archimedean). Let F is a finite extension of $\mathbb{Q}_p$. We will prove only the characteristic zero case. By Proposition 4.1.3, every quasi-character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(F^\times,\mathbb{C}^\times)$ is of the form $\tilde{\chi}|\cdot|_F^s$, where $\tilde{\chi}$ is a unitary character. Let $\mathfrak{p}$ be the unique prime of F. We write U_n for subgroups of $\mathfrak{o}_F^\times$ of the form $1+\mathfrak{p}^n$ with $n \geq 0$. Write $\chi_{s,n}$ for the map

$$x \mapsto |x|_F^s\tilde{\chi}(\tilde{x}),$$

where $\tilde{x} \in \mathfrak{o}_F^\times$ is uniquely defined by the relation $x = \tilde{x}\pi_F^{\nu(x)}$ and where $\mathfrak{p}^n$ is the conductor of $\tilde{\chi}$. Note that χ is said to be unramified if $\chi(\mathfrak{o}_F) = 1$, or, equivalently, if $n = 0$. Recall that s and n do not uniquely determine $\chi_{s,n}$ because often there are many characters of conductor n and because s is determined modulo $\frac{2\pi i}{\log(q)}$, where q is the order of the residue field $\mathfrak{o}_F/\mathfrak{p}\mathfrak{o}_F$.

We defined the standard non-trivial additive character on a local field to be $\psi = \psi_p(\mathrm{tr}_{F/\mathbb{Q}_p}(\cdot))$, where p is the prime lying below $\mathfrak{p}$ and ψ_p is the standard non-trivial additive character on $\mathbb{Q}_p$. In addition, we saw that the conductor of ψ is precisely the inverse different of F, which is $\mathfrak{D}_F^{-1} = \{x \in F : \mathrm{tr}_{F/\mathbb{Q}_p}(x) \subseteq \mathbb{Z}_p\}$. The inverse different is a $\mathbb{Z}_p$-submodule of F and thus has the representation

$$\mathfrak{D}_F^{-1} = \mathfrak{p}^{-d} = \pi_F^{-d}\mathfrak{o}_F,$$

where π_F is a uniformizing parameter. As such, we say the standard non-trivial additive character on F has conductor $\mathfrak{p}^{-d}$. **Although we will not choose the standard non-trivial additive character of F for this calculation, we conveniently will choose an arbitrary additive character, ψ, with conductor $\mathfrak{p}^{-d}$.**

For a given quasi-character $\chi_{s,n}$, and additive character ψ with conductor $\mathfrak{p}^{-d}$, define

$$f_n(x) = \psi(x)\mathbf{1}_{\mathfrak{p}^{-d-n}}(x),$$

where $\mathbf{1}_{\mathfrak{p}^{-d-n}}(x)$ is a characteristic function of $\mathfrak{p}^{-d-n}$.

First, assume $n = 0$, which is the unramified case. Using the fact that $\pi_F^{-d}\mathfrak{o}_F - \{0\} = \cup_{k=-d}^{\infty}\pi_F^k\mathfrak{o}_F^\times$, we obtain

$$\begin{aligned}
Z(f_0, \chi_{s,0}) &= \int_{F^\times} f_0(x)\chi_{s,0}(x)d^*x = \int_{\pi_F^{-d}-\{0\}} |x|_F^s d^*x = \\
&= \sum_{k=-d}^{\infty} \int_{\pi_F^k\mathfrak{o}_F^\times} |x|_F^s d^*x = \sum_{k=-d}^{\infty} q^{-ks}\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) = \\
&= \mathrm{Vol}(\mathfrak{o}_F^\times, d^*x)\frac{q^{ds}}{1-q^{-s}} = q^{ds}\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x)(1 - |\pi_F|_F^s)^{-1} = \\
&= q^{ds}\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x)L(\chi_{s,0}).
\end{aligned} \tag{4.12}$$

Now, let us consider $n > 0$, the ramified case. Then

$$\begin{aligned} Z(f_n, \chi_{s,n}) &= \int\limits_{F^\times} f_n(x)\chi_{s,n}(x)d^*x = \int\limits_{\pi_F^{-d-n}\mathfrak{o}_F - \{0\}} \psi(x)\tilde{\chi}(\tilde{x})|x|_F^s d^*x = \\ &= \sum_{k=-d-n}^{\infty} \int\limits_{\mathfrak{o}_F^\times} \psi(\pi_F^k u)\tilde{\chi}(\widetilde{\pi_F^k u})|\pi_F^k u|_F^s d^*u = \sum_{k=-d-n} q^{-ks} \int\limits_{\mathfrak{o}_F^\times} \psi(\pi_F^k u)\tilde{\chi}(\tilde{u})d^*u, \end{aligned}$$

since $\widetilde{\pi_F^k u} = \tilde{u}$.

Definition 4.5.6. For any multiplicative character $\omega : \mathfrak{o}_F^\times \to S^1$ and any additive character $\lambda : \mathfrak{o}_F \to S^1$, define the associated *Gauss sum* to be

$$g(\omega, \psi) = \int\limits_{\mathfrak{o}_F^\times} \omega(u)\lambda(u)d^*u.$$

As such,

$$Z(f_n, \chi_{s,n}) = \sum_{k=-d-n} q^{-ks} g(\tilde{\chi}, \psi_{\pi_F^k}),$$

where $\psi_{\pi_F^k}(x) = \psi(\pi_F^k x)$. In view of this, we will prove the following lemma about Gauss sums. The generalization of Gauss sums was an important part of Tate's thesis. A Gauss sum will be appear in the epsilon factor for ramified quasi-characters.

Lemma 4.5.7. *Let ω and λ be taken as above with conductors $\mathfrak{p}^n$ and $\mathfrak{p}^r$, respectively. Let $c > 0$ be the number such that $d^*x = cdx$. Then the following statements hold:*

(i) If $r < n$, then $g(\omega, \lambda) = 0$.

(ii) If $r = n = 0$, then $|g(\omega, \lambda)|^2 = \mathrm{Vol}(\mathfrak{o}_F^\times, d^*x)^2$.

(iii) If $r = n$, then $|g(\omega, \lambda)|^2 = c\mathrm{Vol}(\mathfrak{o}_F, dx)\mathrm{Vol}(U_r, d^*x)$.

(iv) If $r > n$, then $|g(\omega, \lambda)|^2 = c\mathrm{Vol}(\mathfrak{o}_F, dx)(\mathrm{Vol}(U_r, d^*x) - q^{-1}\mathrm{Vol}(U_{r-1}, d^*x))$.

Proof. If $r = n = 0$, then $\omega_{\mathfrak{o}_F^\times} = 1$ and $\lambda|_{\mathfrak{o}_F^\times} = 1$. Therefore, we have

$$g(\omega, \psi) = \int\limits_{\mathfrak{o}_F^\times} \omega(u)\lambda(u)d^*u = \mathrm{Vol}(\mathfrak{o}_F^\times, d^*x),$$

, and hence $|g(\omega,\psi)|^2 = \mathrm{Vol}(\mathfrak{o}_F^\times, d^*x)^2$. Let R denote a residue system of $\mathfrak{o}_F^\times/U_r\mathfrak{o}_F^\times$ in $\mathfrak{o}_F^\times$. For $a \in R$ and $1 + b\pi_F^r \in U_r$, we have

$$\lambda(a(1+\pi_F^r b)) = \lambda(a)\lambda(a\pi_F^r b) = \lambda(a)$$

because $\mathfrak{p}^r = \pi_F^r \mathfrak{o}_F$ is the conductor of λ. Then

$$g(\omega,\lambda) = \sum_{a\in R} \lambda(a)\omega(a) \int_{U_r} \omega(u) d^*u. \tag{4.13}$$

If $r < n$, then there exists an element $u_0 \in U_r$ such that $\omega(u_0) \neq 1$. By the translation invariance of the multiplicative Haar measure we obtain

$$\int_{U_r} \omega(u) d^*u = \int_{U_r} \omega(uu_0) d^*u = \omega(u_0) \int_{U_r} \omega(u) d^*u$$

which proves that

$$\int_{U_r} \omega(u) d^*u = 0.$$

This proves part i. Suppose $r \geq n$. Applying the transformation $x = zy$ and translation invariance of the Haar measure, we obtain

$$\begin{aligned} |g(\omega,\lambda)|^2 = \int_{\mathfrak{o}_F^\times} \omega(x)\lambda(x) d^*x \cdot \overline{\int_{\mathfrak{o}_F^\times} \omega(y)\lambda(y) d^*y} &= \int_{\mathfrak{o}_F^\times}\int_{\mathfrak{o}_F^\times} \omega(xy^{-1})\lambda(x-y) d^*x d^*y \\ &= \int_{\mathfrak{o}_F^\times} \omega(z)h(z) d^*z, \end{aligned}$$

where

$$h(z) = \int_{\mathfrak{o}_F^\times} \lambda(y(z-1)) d^*y = c\int_{\mathfrak{o}_F^\times} \lambda(y(z-1)) dy.$$

Since $\mathfrak{o}_F^\times = \mathfrak{o}_F - \mathfrak{p}$, then

$$h(z) = c\int_{\mathfrak{o}_F} \lambda(y(z-1)) dy - c\int_{\mathfrak{p}} \lambda(y(z-1)) dy.$$

For $z - 1 \in \mathfrak{p}^r$ and $y \in \mathfrak{o}_F$, we have that $\lambda(y(z-1)) = 1$. For $z - 1 \in \{\mathfrak{p}^{r-1} - \mathfrak{p}^r\}$ and $y \in \mathfrak{p}$, we have $\lambda(y(z-1)) = 1$. On the other hand, if $z - 1 \in \{\mathfrak{p}^{r-1} - \mathfrak{p}^r\}$, then there exists a $y_0 \in \mathfrak{o}_F^\times \subset \mathfrak{o}_F$ such that $\lambda(y_0(z-1)) \neq 1$. For $z - 1 \notin \mathfrak{p}^{r-1}$ there exists a $y_0 \in \mathfrak{p}$ such that $\lambda(y_0(z-1)) \neq 1$. In sum, we obtain

$$h(z) = \begin{cases} c(1-q^{-1})\mathrm{Vol}(\mathfrak{o}_F, dx) & \text{if } z-1 \in \mathfrak{p}^r \\ -cq^{-1}\mathrm{Vol}(\mathfrak{o}_F, dx) & \text{if } z-1 \in \{\mathfrak{p}^{r-1} - \mathfrak{p}^r\} \\ 0 & \text{otherwise .} \end{cases}$$

Consequently, if $r > 0$, then

$$\begin{aligned} |g(\omega,\lambda)|^2 &= \int\limits_{\{U_{r-1}-U_r\}} \omega(z)h(z)d^*z + \int\limits_{U_r} \omega(z)h(z)d^*z \\ &= c\mathrm{Vol}(\mathfrak{o}_F, dx)\left(-q^{-1} \int\limits_{\{U_{r-1}-U_r\}} \omega(z)d^*z + (1-q^{-1})\int\limits_{U_r} \omega(z)d^*z\right) \\ &= c\mathrm{Vol}(\mathfrak{o}_F, dx)\left(-q^{-1}\int\limits_{U_{r-1}} \omega(z)d^*z + \mathrm{Vol}(U_r, d^*z),\right) \end{aligned}$$

where q is the order of the residue field $\mathfrak{o}_F/\mathfrak{p}\mathfrak{o}_F$. More specifically, if $r = n > 0$, then

$$|g(\omega,\lambda)|^2 = c\mathrm{Vol}(\mathfrak{o}_F, dx)\mathrm{Vol}(U_r, d^*x)$$

because $\omega(z)$ is non-trivial on $U_{r-1} = U_{n-1}$ (orthogonality of characters), whereas if $r > n$, then

$$|g(\omega,\lambda)|^2 = c\mathrm{Vol}(\mathfrak{o}_F, dx)(\mathrm{Vol}(U_r, d^*z) - q^{-1}\mathrm{Vol}(U_{r-1}, d^*x)).$$

In the case $r = 0 (= n)$, we obtain

$$\begin{aligned} |g(\omega,\lambda)|^2 &= \int\limits_{U_r} \omega(z)h(z)d^*z = c(1-q^{-1})\mathrm{Vol}(\mathfrak{o}_F, dx)\int\limits_{\mathfrak{o}_F^\times} \omega(z)d^*z = \\ &= c(1-q^{-1})\mathrm{Vol}(\mathfrak{o}_F, dx)\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) \end{aligned}$$

Since $c(1-q^{-1})\mathrm{Vol}(\mathfrak{o}_F, dx)\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) = \mathrm{Vol}(\mathfrak{o}_F^\times, d^*x)$, then $|g(\omega,\lambda)|^2 = \mathrm{Vol}(\mathfrak{o}_F^\times, d^*x)^2$, which is the same result we obtained by the direct calculation in the beginning of the proof. Indeed,

$$\begin{aligned} c(1-q^{-1})\mathrm{Vol}(\mathfrak{o}_F, dx) &= c(1-q^{-1})\int_{\mathfrak{o}_F} dx = c(1-q^{-1})\sum_{m=0}^{\infty}\int_{\pi_F^m\mathfrak{o}_F^\times} dx = \\ &= c(1-q^{-1})\sum_{m=0}^{\infty}\int_{\mathfrak{o}_F^\times}|\pi_F^m|_F dx = c(1-q^{-1})\sum_{m=0}^{\infty} q^{-m}\int_{\mathfrak{o}_F^\times} dx = \\ &= \int_{\mathfrak{o}_F^\times} c\,dx = \int_{\mathfrak{o}_F^\times} c\frac{dx}{|x|_F} = \mathrm{Vol}(\mathfrak{o}_F^\times, d^*x). \end{aligned}$$

This completes the lemma. □

We now will return to the computation of $Z(f_n, \chi_{s,n})$ for $n > 0$. Since the conductor of ψ is $\mathfrak{p}^{-d}$, then the conductor of ψ_{π_k} is $(\mathfrak{p}^{-d-k})$. Before the lemma we determined that

$$Z(f_n, \chi_{s,n}) = \sum_{k=-d-n} q^{-ks} g(\tilde{\chi}, \psi_{\pi_F^k}),$$

where $\tilde{\chi}$ has conductor $\mathfrak{p}^n$. Note that $-d-k < n \Leftrightarrow k > -d-n$. Consequently, from part (i) of the above lemma, we have that $g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) = 0$ for all $k > -d-n$. Therefore,

$$Z(f_n, \chi_{s,n}) = q^{(d+n)s} g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}).$$

Since both $\tilde{\chi}$ and $\psi_{\pi_F^{-d-n}}$ have conductor $\mathfrak{p}^n$, then from part (ii) and (iii) of the above lemma we see that $g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) \neq 0$. As such, $Z(f_n, \chi_{s,n})$ is essentially an exponential function with neither zeros nor poles. Recall that for $n > 0$, $L(\chi_{s,n})$ is 1 because $\chi_{s,n}$ is not ramified. Thus,

$$Z(f_n, \chi_{s,n}) = q^{(d+n)s} g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) L(\chi_{s,n}). \tag{4.14}$$

Let us compute the Fourier transform of our test function f, so that we can determine $Z(\hat{f}, \check{\chi_{s,n}})$.

Lemma 4.5.8. *For $n = 0$, we have $\hat{f}_0(y) = \mathrm{Vol}(\mathfrak{p}^{-d}, dx)\mathbf{1}_{\mathfrak{o}_F}(y)$, where $\mathbf{1}_{\mathfrak{o}_F}(y)$ is the characteristic function of $\mathfrak{o}_F$. For $n > 0$ we have $\hat{f}_n(y) = \mathrm{Vol}(\mathfrak{p}^{-d-n}, dx)\mathbf{1}_{\mathfrak{p}^n-1}(y)$, where $\mathbf{1}_{\mathfrak{p}^n-1}(y)$ is the characteristic function of $\mathfrak{p}^n - 1$.*

Proof. By definition,

$$\hat{f}_n(y) = \int\limits_F f_n(x)\psi(xy)dx = \int\limits_{\mathfrak{p}^{-d-n}} \psi(x)\psi(xy) = \int\limits_{\mathfrak{p}^{-d-n}} \psi(x(y+1))dx.$$

First, let $n = 0$. The conductor of ψ is $\mathfrak{p}^{-d}$. For $y \notin \mathfrak{o}_F$, we have that $\psi(x(y+1))$ is non-trivial for some $x \in \mathfrak{p}^{-d}$, hence $\hat{f}(y) = 0$ by orthogonality of characters. For $y \in \mathfrak{o}_F$, we have that $\psi(x(y+1)) = 1$ for all $x \in \mathfrak{p}^{-d}$, hence $\hat{f}(y) = \mathrm{Vol}(\mathfrak{p}^{-d}, dx)$. Let $n > 0$. For $y \notin \mathfrak{p}^n - 1$, we have that $\psi(x(y+1))$ is non-trivial for some $x \in \mathfrak{p}^{-d-n}$, hence $\hat{f}(y) = 0$ by orthogonality of characters. For $y \in \mathfrak{p}^n - 1$, we have that $\psi(x(y+1)) = 1$, hence $\hat{f}(y) = \mathrm{Vol}(\mathfrak{p}^{-d-n}, dx)$. $\square$

We have computed $Z(f, \chi_{s,n})$ for the unramified and ramified case, and now will now compute $Z(\hat{f}, \check{\chi_{s,n}})$ for both the unramified and ramified case. Let $n = 0$. Using the above lemma and the fact that $\mathfrak{o}_F - \{0\} = \cup_{k=0}^{\infty} \pi^k \mathfrak{o}_F^{\times}$, we obtain

$$\begin{aligned}
Z(\hat{f}_0, \check{\chi_{s,0}}) &= \int\limits_{F^\times} f_0(y)\check{\chi_{s,0}}(y)d^*y = \mathrm{Vol}(\mathfrak{p}^{-d}, dx) \int\limits_{\mathfrak{o}_F - \{0\}} \check{\chi_{s,0}}(y)d^*y = \\
&= \mathrm{Vol}(\mathfrak{p}^{-d}, dx) \sum_{k=0}^{\infty} \int\limits_{\pi^k \mathfrak{o}_F^\times} |y|_F^{1-s} d^*y = \mathrm{Vol}(\mathfrak{p}^{-d}, dx)\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) \sum_{k=0}^{\infty} q^{-k(1-s)} = \\
&= \mathrm{Vol}(\mathfrak{p}^{-d}, dx)\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) \frac{1}{1 - q^{-(1-s)}} = \mathrm{Vol}(\mathfrak{p}^{-d}, dx)\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) \frac{1}{1 - \check{\chi_{s,0}}(\pi_F)} \\
&= \mathrm{Vol}(\mathfrak{p}^{-d}, dx)\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) L(\check{\chi_{s,0}}) = q^d \mathrm{Vol}(\mathfrak{o}_F, dx)\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) L(\check{\chi_{s,0}}).
\end{aligned}$$

Consequently, from equation (4.12), we have that

$$\gamma(\chi_{s,0}, \psi, dx) = q^{-d(s-1)} \mathrm{Vol}(\mathfrak{o}_F, dx) \frac{L(\check{\chi_{s,0}})}{L(\chi_{s,0})}$$

and

$$\epsilon(\chi_{s,0}, \psi, dx) = q^{-d(s-1)} \mathrm{Vol}(\mathfrak{o}_F, dx). \tag{4.15}$$

Now consider $n > 0$. By definition, $\check{\chi_{s,n}} = \chi_{s,n}^{-1} |\cdot|_F = \tilde{\chi}^{-1} |\cdot|_F^{1-s}$. Since $\tilde{\chi}$ is unitary, then $\tilde{\chi}^{-1} = \overline{\tilde{\chi}}$. Note that the conductor of $\check{\chi_{s,n}}$, which is just the conductor of $\tilde{\chi}^{-1}$, is also n. Using this fact, the above lemma, and that $L(\chi_{s,n}) = 1$, we have

$$\begin{aligned}
Z(\hat{f}_n, \check{\chi_{s,n}}) &= \mathrm{Vol}(\mathfrak{p}^{-d-n}, dx) \int\limits_{\mathfrak{p}^n - 1} \overline{\tilde{\chi}}(\tilde{y}) |y|_F^{1-s} d^* y \\
&= \mathrm{Vol}(\mathfrak{p}^{-d-n}, dx) \int\limits_{\mathfrak{p}^n - 1} \overline{\tilde{\chi}}(y) d^* y \\
&= \mathrm{Vol}(\mathfrak{p}^{-d-n}, dx) \int\limits_{1+\mathfrak{p}^n} \overline{\tilde{\chi}}(-y) d^* y \\
&= q^{d+n} \mathrm{Vol}(\mathfrak{o}_F, dx) \overline{\tilde{\chi}}(-1) \int\limits_{1+\mathfrak{p}^n} \overline{\tilde{\chi}}(y) d^* y \\
&= q^{d+n} \mathrm{Vol}(\mathfrak{o}_F, dx) \mathrm{Vol}(1+\mathfrak{p}^n, d^*x) \tilde{\chi}(-1) L(\chi_{s,n}).
\end{aligned}$$

If $n > 0$, then by applying the translation invariance of the Haar measure, we have

$$\mathrm{Vol}(U_n, d^*x) = \int\limits_{U_n} d^*x = c \int\limits_{(1+\mathfrak{p}^n)-\{0\}} |x|_F^{-1} dx = c \int\limits_{\mathfrak{p}^n} dx = c\mathrm{Vol}(\mathfrak{p}^n, dx) = cq^{-n} \mathrm{Vol}(\mathfrak{o}_F, dx).$$

As such,

$$Z(\hat{f}_n, \check{\chi_{s,n}}) = cq^d \mathrm{Vol}^2(\mathfrak{o}_F, dx) \tilde{\chi}(-1) L(\chi_{s,n}).$$

Therefore, we have

$$\epsilon(\psi_{s,n}, \psi, dx) = \gamma(\chi_{s,n}, \psi, dx) = \frac{cq^d q^{-(d+n)s} \mathrm{Vol}^2(\mathfrak{o}_F, dx) \tilde{\chi}(-1)}{g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})}.$$

Applying the translation invariance of the Haar measure d^*u, we obtain

$$\begin{aligned}
\overline{g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})} &= \overline{\int\limits_{\mathfrak{o}_F^\times} \tilde{\chi}(u) \psi(\pi_F^{-d-n} u) d^* u} = \int\limits_{\mathfrak{o}_F^\times} \overline{\tilde{\chi}}(u) \psi(-\pi_F^{-d-n} u) d^* u = \\
&= \tilde{\chi}(-1) \int\limits_{\mathfrak{o}_F^\times} \overline{\tilde{\chi}}(u) \psi(\pi_F^{-d-n} u) d^* u = \tilde{\chi}(-1) g(\overline{\tilde{\chi}}, \psi_{\pi_F^{-d-n}}).
\end{aligned}$$

Since the conductor of $\tilde{\chi}$ and ψ_{-d-n} is $\mathfrak{p}^n$, then

$$g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) \overline{g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})} = c\mathrm{Vol}(\mathfrak{o}_F, dx) \mathrm{Vol}(1+\mathfrak{p}^n, d^*x) = c^2 q^{-n} \mathrm{Vol}^2(\mathfrak{o}_F, dx).$$

Consequently, for $n > 0$ we have

$$\begin{aligned}
\epsilon(\psi_{s,n}, \psi, dx) &= \frac{cq^d q^{-(d+n)s}\mathrm{Vol}^2(\mathfrak{o}_F, dx)\tilde{\chi}(-1)}{g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})} \frac{\tilde{\chi}(-1)g(\overline{\tilde{\chi}}, \psi_{\pi_F^{-d-n}})}{\overline{g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})}} \\
&= \frac{cq^d q^{-(d+n)s}\mathrm{Vol}^2(\mathfrak{o}_F, dx) g(\overline{\tilde{\chi}}, \psi_{\pi_F^{-d-n}})}{c^2 q^{-n}\mathrm{Vol}^2(\mathfrak{o}_F, dx)} \\
&= \frac{1}{c} q^{(-d-n)(s-1)} g(\overline{\tilde{\chi}}, \psi_{\pi_F^{-d-n}})
\end{aligned}$$

Consider the Gauss sum of $\tilde{\chi}$ and $\psi_{\pi_F^{-d-n}}$ defined by

$$g(\overline{\tilde{\chi}}, \psi_{\pi_F^{-d-n}}) = \int_{\mathfrak{o}_F^\times} \overline{\tilde{\chi}}(u)\psi_{\pi_F^{-d-n}}(u) d^*u.$$

Since ψ has conductor $\mathfrak{p}^{-d}$, then $\psi_{\pi_F^{-d-n}}$, defined by $\psi_{\pi_F^{-d-n}}(x) = \psi(\pi_F^{-d-n}x)$, has conductor $\mathfrak{p}^n$. For $a \in U/U_n$ and $1 + b\pi_F^n \in U_n$, we have

$$\psi_{\pi_F^{-d-n}}(a(1 + \pi_F^n b)) = \psi_{\pi_F^{-d-n}}(a)\psi_{\pi_F^{-d-n}}(a\pi_F^n b) = \psi_{\pi_F^{-d-n}}(a)$$

because $\mathfrak{p}^n = \pi_F^n \mathfrak{o}_F$ is the conductor of $\psi_{\pi_F^{-d-n}}$. As such,

$$\begin{aligned}
g(\overline{\tilde{\chi}}, \psi_{\pi_F^{-d-n}}) &= \sum_{x \in U/U_n} \psi_{\pi_F^{-d-n}}(x)\overline{\tilde{\chi}}(x) \int_{U_n} \overline{\tilde{\chi}}(u) d^*u = \mathrm{Vol}(U_n, d^*x) \sum_{x \in U/U_n} \overline{\tilde{\chi}}(x)\psi_{\pi_F^{-d-n}}(x) = \\
&= cq^{-n}\mathrm{Vol}(\mathfrak{o}_F, dx) \sum_{x \in U/U_n} \overline{\tilde{\chi}}(x)\psi_{\pi_F^{-d-n}}(x). \qquad (4.16)
\end{aligned}$$

Therefore,

$$\epsilon(\psi_{s,n}, \psi, dx) = \frac{1}{c} q^{(-d-n)(s-1)} g(\overline{\tilde{\chi}}, \psi_{\pi_F^{-d-n}}) = q^{-d(s-1)} q^{-ns}\mathrm{Vol}(\mathfrak{o}_F, dx) \sum_{x \in U/U_n} \overline{\tilde{\chi}}(x)\psi_{\pi_F^{-d-n}}(x). \qquad (4.17)$$

For F non-Archimedean case, we have

$$h(f_n, \chi_{s,n}, \psi, dx) = \begin{cases} q^{ds}\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) & \text{for } n = 0 \\ q^{(d+n)s} g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) & \text{for } n > 0 \end{cases}$$

and

$$h(\hat{f}_n, \check{\chi_{s,n}}, \psi, dx) = \begin{cases} q^d\mathrm{Vol}(\mathfrak{o}_F, dx)\mathrm{Vol}(\mathfrak{o}_F^\times, d^*x) & \text{for } n = 0 \\ cq^d\mathrm{Vol}^2(\mathfrak{o}_F, dx)\tilde{\chi}(-1) & \text{for } n > 0, \end{cases},$$

where: $\mathfrak{p}$ is the unique prime ideal of F, q is the order of the residue field $\mathfrak{o}_F/\mathfrak{p}\mathfrak{o}_F$, $\mathfrak{p}^{-d}$ is the conductor of the additive character ψ, $\mathfrak{p}^n$ is the conductor of $\chi_{s,n}$, and $\tilde{\chi}$ is the restriction of $\chi_{s,n}$ to $\mathfrak{o}_F$. Furthermore, note the dependence of h on dx and on d^*x.

Therefore, for all characteristic zero local fields, we have that

$$\gamma(\chi, \psi, dx) = \epsilon(\chi, \psi, dx)\frac{L(\check{\chi})}{L(\chi)},$$

where $\epsilon(\chi, \psi, dx)$ is an entire function of s whose image lies in $\mathbb{C}^\times$. Applying part (ii), it follows at once that

$$L(\chi)Z(\hat{f}, \check{\chi}) = \epsilon(\chi, \psi, dx)L(\check{\chi})Z(f, \chi). \tag{4.18}$$

Since $L(\chi)$, $L(\check{\chi})$, and $\epsilon(\chi, \psi, dx)$ do not have zeroes, then the poles of $Z(f, \chi)$ are no worse than those of $L(\chi)$, which is independent of f. □

4.6 Local Epsilon Factor and Root Number

The dependence of the epsilon factor $\epsilon(\chi, \psi, dx)$ on both the additive character ψ and Haar measure dx for any $\chi \in \mathrm{Hom}_{\mathrm{cont}}(F^\times, \mathbb{C}^\times)$ is stated in the following proposition. As mentioned in the remark before the local functional equation proof, it was not necessary for the pair (ψ, dx) to be self-dual. Three other important properties about the epsilon factor also will be proven.

Proposition 4.6.1. *(i) For every real number t,*

$$\epsilon(\chi, \psi, t \cdot dx) = t \cdot \epsilon(\chi, \psi, dx).$$

(ii) Let $a \in F^\times$, and let ψ_a denote the character defined by $\psi_a(x) = \psi(ax)$. Then

$$\epsilon(\chi, \psi_a, dx) = \chi(a)|a|_F^{-1}\epsilon(\chi, \psi, dx).$$

(iii) Let F be a non-Archimedean field with unique prime ideal $\mathfrak{p}$, and let $\mathfrak{p}^n$ and $\mathfrak{p}^{-d}$ be the conductors of χ and ψ, respectively. Then for every unramified character ν of $F^\times$ we have

$$\epsilon(\chi\nu, \psi, dx) = \nu(\pi^{d+n})\epsilon(\chi, \psi, dx),$$

where π is the uniformizing parameter for $\mathfrak{o}_F$. Note that since ν is unramified ($\nu(\mathfrak{o}_F^\times) = 1$), then $\nu(\pi^{m+n})$ is independent of the choice of uniformizing parameter.

(iv)

$$\epsilon(\check{\chi}, \psi, dx) = \frac{\chi(-1)}{\epsilon(\chi, \psi, dx)}.$$

(v)

$$\epsilon(\overline{\chi}, \psi, dx) = \chi(-1)\overline{\epsilon(\chi, \psi, dx)}.$$

Proof.

(i) Since the Fourier transform of a self-dual local field is dependent on the Haar measure, dx, chosen for F and the additive character, ψ, chosen, then we will denote the Fourier transform of a function $f \in S(F)$ by $(\hat{f}, \psi, dx)$. By definition, we have

$$(\hat{f}, \psi, tdx)(y) = \int_F f(x)\psi(xy)tdx = t\int_F f(x)\psi(xy)dx = t(\hat{f}, \psi, dx).$$

Although, $Z(f, \chi)$ is dependent on d^*x, and therefore on dx if we set $d^*x = dx/|x|_F$, the ratio $Z(\hat{f}\check{\chi})/Z(f, \chi)$ is independent of the measure chosen, whether we specify the multiplicative measure independent of dx or not. Therefore, we have

$$\frac{Z((\hat{f}, \psi, tdx), \check{\chi}, tdx)}{Z(f, \chi, tdx)} = t\frac{Z((\hat{f}, \psi, dx), \check{\chi}, dx)}{Z(f, \chi, dx)}$$

, and hence

$$\epsilon(\chi, \psi, t \cdot dx) = t \cdot \epsilon(\chi, \psi, dx).$$

(ii) With the same notation in part (i), it is clear that $(\hat{f}, \psi_a, dx)(x) = (\hat{f}, \psi, dx)(ax)$. Let $\chi = \mu|\cdot|_F^s$. So,

$$\begin{aligned} Z((\hat{f}, \psi_a, dx), \check{\chi}, dx) &= \int_{F^\times} (\hat{f}, \psi, dx)(ax)\check{\chi}(x)d^*x = \int_{F^\times} (\hat{f}, \psi, dx)(x)\check{\chi}(a^{-1}x)d^*x = \\ &= \check{\chi}(a^{-1}) \int_{F^\times} (\hat{f}, \psi, dx)(x)\check{\chi}(x)d^*x = \chi^{-1}(a^{-1})|a^{-1}|Z((\hat{f}, \psi, dx) = \\ &= \chi(a)|a|_F^{-1} Z(\hat{f}, \psi, dx). \end{aligned}$$

Therefore,

$$\frac{Z((\hat{f}, \psi_a, dx), \check{\chi}, tdx)}{Z(f, \chi, dx)} = \chi(a)|a|_F^{-1}\frac{Z((\hat{f}, \psi, dx), \check{\chi}, dx)}{Z(f, \chi, dx)}$$

, and hence

$$\epsilon(\chi,\psi,t\cdot dx)=\chi(a)|a|_F^{-1}\cdot\epsilon(\chi,\psi,dx).$$

(iii) Since ν is unramified, then there exists an $s'\in\mathbb{C}$ such that $\nu=|\cdot|_F^{s'}$ (Proposition 4.1.2). Also, $\chi=|\cdot|_F^{s}\tilde{\chi}$ for some $s\in\mathbb{C}$ and unitary $\tilde{\chi}$, the restriction of χ to $\mathfrak{o}_F$, with conductor $\mathfrak{p}^n$. The conductor of $\chi\nu$ is the same as the conductor of χ since ν is unramified. As in the local computations, we write $\chi\nu=\chi_{s+s',n}$. Note that this notation does not uniquely specify the character. If $n=0$, then we have by equation (4.15) that

$$\begin{aligned}\epsilon(\chi_{s+s',0},\psi,dx) &= =q^{-d(s+s'-1)}\mathrm{Vol}(\mathfrak{o}_F,dx)=q^{-ds'}q^{-d(s-1)}\mathrm{Vol}(\mathfrak{o}_F,dx)\\ &= |\pi_F^d|_F^{s'}q^{-ds}\mathrm{Vol}(\mathfrak{p}^{-d},dx)=\nu(\pi_F^d)\epsilon(\chi_{s,0},\psi,dx),\end{aligned}$$

where q is the order of the residue field $\mathfrak{o}_F/\mathfrak{p}\mathfrak{o}_F$. If $n>0$, then by equation (4.17) we have that

$$\begin{aligned}\epsilon(\chi_{s+s',n},\psi,dx) &= q^{-d(s+s'-1)}q^{-n(s+s')}\mathrm{Vol}(\mathfrak{o}_F,dx)\sum_{x\in U/U_n}\overline{\tilde{\chi}}(x)\psi_{\pi_F^{-d-n}}(x)\\ &= q^{-ds'}q^{-ns'}q^{-d(s-1)}q^{-ns}\mathrm{Vol}(\mathfrak{o}_F,dx)\sum_{x\in U/U_n}\overline{\tilde{\chi}}(x)\psi_{\pi_F^{-d-n}}(x)\\ &= \nu(\pi_F^{d+n})\epsilon(\chi_{s,n},\psi,dx).\end{aligned}$$

(iv) Applying part (ii) of the Theorem 4.5.3 twice, we obtain

$$Z(f,\chi)=\gamma(\chi,\psi,dx)Z(\hat{f},\check{\chi})=\gamma(\chi,\psi,dx)\gamma(\check{\chi},\psi,dx)Z(\hat{\hat{f}},\check{\check{\chi}}).$$

Using the translation invariance of the multiplicative Haar measure, self-duality of the pair (ψ,dx), and the fact that $\check{\check{\chi}}=\chi$, we obtain

$$\begin{aligned}Z(\hat{\hat{f}},\check{\check{\chi}}) &= \int_{F^\times}\hat{\hat{f}}(x)\chi(x)d^*x=\int_{F^\times}f(-x)\chi(x)d^*x=\int_{F^\times}f(x)\chi(-x)d^*x=\\ &= \chi(-1)\int_{F^\times}f(x)\chi(x)d^*x=\chi(-1)Z(f,\chi).\end{aligned}$$

Therefore, $\gamma(\check{\chi},\psi,dx)=\frac{\chi(-1)}{\gamma(\chi,\psi,dx)}$, which implies that

$$\epsilon(\check{\chi},\psi,dx)\frac{L(\check{\check{\chi}})}{L(\check{\chi})}=\frac{\chi(-1)}{\epsilon(\chi,\psi,dx)}\frac{L(\chi)}{L(\check{\chi})}$$

, and hence that

$$\epsilon(\check{\chi}, \psi, dx) = \frac{\chi(-1)}{\epsilon(\chi, \psi, dx)}.$$

(v) By inspection $\overline{\check{\chi}} = \check{\overline{\chi}}$. Since $\psi(x(-y)) = \overline{\psi(xy)}$, then

$$\hat{\overline{f}}(y) = \int_F \overline{f}(x)\psi(xy)dx = \int_F \overline{f(x)\psi(x(-y))}dx = \overline{\int_F f(x)\psi(x(-y))dx} = \overline{\hat{f}(-y)}.$$

Using this, we obtain

$$Z(\hat{\overline{f}}, \chi) = \int_{F^\times} \hat{\overline{f}}(x)\chi(x)d^*x = \int_{F^\times} \hat{\overline{f}}(-x)\chi(x)d^*x = \chi(-1)Z(\hat{\overline{f}}, \chi).$$

Applying part (ii) of the Theorem 4.5.3 and the above facts, we have that

$$\overline{Z(f,\chi)} = Z(\overline{f}, \overline{\chi}) = \gamma(\overline{\chi}, \psi, dx)Z(\hat{\overline{f}}, \check{\overline{\chi}}) = \gamma(\overline{\chi}, \psi, dx)\chi(-1)Z(\hat{\overline{f}}, \check{\overline{\chi}}) = \gamma(\overline{\chi}, \psi, dx)\chi(-1)\overline{Z(\hat{f}, \check{\chi})}.$$

On the other hand, we have

$$\overline{Z(f,\chi)} = \overline{\gamma(\chi, \psi, dx)Z(\hat{f}, \check{\chi})}.$$

Therefore, we have

$$\gamma(\overline{\chi}, \psi, dx) = \chi(-1)\overline{\gamma(\chi, \psi, dx)}.$$

Consequently,

$$\epsilon(\overline{\chi}, \psi, dx)\frac{L(\check{\overline{\chi}})}{L(\overline{\chi})} = \chi(-1)\overline{\epsilon(\chi, \psi, dx)\frac{L(\check{\chi})}{L(\chi)}}$$

, and hence that

$$\epsilon(\overline{\chi}, \psi, dx) = \chi(-1)\overline{\epsilon(\chi, \psi, dx)}.$$

$\square$

Suppose we wanted to preserve the self-duality of the pair (ψ, dx). If we wish to use a different additive character, say ψ_b, then what is the corresponding self-dual measure in terms of dx? This is really just a matter of being very careful with notation. Let $f \in \mathfrak{B}(F)$. For convenience, let $\hat{f} = (\hat{f}, \psi, dx)$. Then $(\hat{f}, \psi_b, dx)(y) = \hat{f}(by)$, and hence $(\hat{f}, \psi_b, dx) =$

$L_{b^{-1}}\hat{f}$. Let $g \in L^1(F)$. Then

$$\widehat{L_{b^{-1}}g}(y) = \int_F g(bx)\psi(xy)dx = |b|_F^{-1}\int_F g(z)\psi(zb^{-1}y)dz = |b|_F^{-1}\hat{g}(b^{-1}y).$$

That is, $\widehat{L_{b^{-1}}g} = |b|_F^{-1}L_b\hat{g}$. Letting $g = \hat{f}$ we obtain $\widehat{L_{b^{-1}}\hat{f}} = |b|_F^{-1}L_b\hat{\hat{f}}$. Then

$$(\hat{\hat{f}}, \psi_b, dx)(y) = L_{b^{-1}}\widehat{L_{b^{-1}}\hat{f}}(y) = L_{b^{-1}}|b|_F^{-1}L_b\hat{\hat{f}}(y) = |b|_F^{-1}f(-y).$$

As such, the measure $|b|_F^{1/2}dx$ is the self-dual measure with respect to ψ_b. By the proposition above, we obtain

$$\epsilon(\chi, \psi_b, |b|_F^{1/2}dx) = \chi(b)|b|_F^{-1/2}\epsilon(\chi, \psi, dx).$$

Definition 4.6.2. Let F be a local field with standard non-trivial character ψ and self-dual measure dx. For a unitary character $\tilde{\chi}$ of $F^\times$, one defines the *root number* $W(\tilde{\chi})$ by

$$W(\tilde{\chi}) = \epsilon(\tilde{\chi}|\cdot|_F^{1/2}, \psi, dx).$$

Proposition 4.6.3. $|W(\tilde{\chi})| = 1$.

Proof. For $\chi = \tilde{\chi}|\cdot|_F^{1/2}$, we have

$$\chi(x)\overline{\chi}(x) = \tilde{\chi}(\tilde{x})|x|_F^{1/2}\overline{\tilde{\chi}(\tilde{x})}|x|_F^{1/2} = |x|\tilde{\chi}(\tilde{x})\overline{\tilde{\chi}(\tilde{x})} = |x| = (\tilde{\chi}(\tilde{x})|x|_F^{1/2})(\tilde{\chi}(\tilde{x})|x|_F^{1/2})^{-1}|x| = \chi(x)\check{\chi}(x)$$

since $\tilde{\chi}$ is a unitary character. Thus $\overline{\chi} = \check{\chi}$. By part (iv) and (v) of the above proposition we have that

$$\epsilon(\chi, \psi, dx)^{-1} = \overline{\epsilon(\chi, \psi, dx)},$$

which proves the theorem. $\square$

Let F be a characteristic zero non-Archimedean field with unique prime $\mathfrak{p}$ and uniformizing parameter π_F. Let ψ be the standard non-trivial character of F with conductor $\mathfrak{D}_F^{-1} = \mathfrak{p}^{-d}$ and let dx be the associated self-dual measure. Note that dx has the property $\mathrm{Vol}(\mathfrak{o}_F, dx) = q^{-d/2}$. Let $\tilde{\chi}$ be a unitary multiplicative character with conductor $\mathfrak{p}^n$. If $n > 0$, then by 4.17 we have that

$$\begin{aligned}
W(\tilde{\chi}) = \epsilon(\tilde{\chi}|\cdot|_F^{1/2}, \psi, dx) &= q^{-d(1/2-1)} q^{-n/2} \mathrm{Vol}(\mathfrak{o}_F, dx) \sum_{x \in U/U_n} \overline{\tilde{\chi}}(x) \psi_{\pi_F^{-d-n}}(x) \\
&= q^{d/2} q^{-n/2} q^{-d/2} \sum_{x \in U/U_n} \overline{\tilde{\chi}}(x) \psi_{\pi_F^{-d-n}}(x) \\
&= q^{-n/2} \sum_{x \in U/U_n} \overline{\tilde{\chi}}(x) \psi(\pi_F^{-d-n} x). \qquad (4.19)
\end{aligned}$$

If $n = 0$, then by 4.15 we have that

$$W(\tilde{\chi}) = \epsilon(\tilde{\chi}|\cdot|_F^{1/2}, \psi, dx) = q^{-d(1/2-1)} \mathrm{Vol}(\mathfrak{o}_F, dx) = q^{d/2} q^{-d/2} = 1.$$

Put

$$G = \sum_{x \in U/U_n} \tilde{\chi}(x) \psi(\pi_F^{-d-n} x).$$

If $n = 0$, then $G = 1$ because $\tilde{\chi}(1)\psi(\pi_F^{-d} 1) = 1$. Furthermore, we have that

$$g(\tilde{\chi}, \psi_{\pi_F^{-d-n}}) = c q^{-n-d/2} G.$$

See equation 4.16. According to Lemma 4.5.7, for $n > 0$, we have that

$$|g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})|^2 = c \mathrm{Vol}(\mathfrak{o}_F, dx) \mathrm{Vol}(U_n, d^* x) = c^2 q^{-d/2} \mathrm{Vol}(\mathfrak{p}^n, d^* x) = c^2 q^{-n-d}.$$

Therefore, for $n > 0$ we have

$$|G|^2 = \frac{1}{c^2} q^{2n+d} |g(\tilde{\chi}, \psi_{\pi_F^{-d-n}})|^2 = q^n,$$

and for $n = 0$ we have

$$|G|^2 = 1.$$

By 4.19, we see that $W(\tilde{\chi}) = q^{-n/2} G$ for $n > 0$. As such, we obtain the relation

$$q^{-n} |G|^2 = |W(\tilde{\chi})|,$$

, and hence that $|W(\tilde{\chi})| = 1$ by direct calculation.

For $F = \mathbb{R}$, there are two unitary multiplicative characters, 1 and sgn. In the local computations, specifically equations 4.7 and 4.9, we found that

$$W(\tilde{\chi}) = \begin{cases} 1 & \text{for } \tilde{\chi} = 1 \\ i & \text{for } \tilde{\chi} = \text{sgn}\,. \end{cases}$$

Consequently, $|W(\tilde{\chi})| = 1$.

For $F = \mathbb{C}$, the unitary characters are the maps $\chi_{0,n} : re^{-i\theta} \mapsto e^{-in\theta}$. According to equation 4.11, we have

$$W(\tilde{\chi_{s,n}}) = W(\chi_{0,n}) = i^{|n|}.$$

Again, we have that $|W(\chi_{0,n})| = 1$.

4.7 Adelic Schwartz-Bruhat Functions and the Riemann-Roch Theorem

One of the most important and useful results of abelian harmonic analysis is the Poisson summation formula, which relates the averages of a function over a lattice to its Fourier transform. The Poisson summation formula will help us to establish the global functional equation.

Definition 4.7.1. Let K be a global field. Let ν be a place of K and K_ν be the completion of K with respect to ν. Define

$$S(\mathbb{A}_K) = \otimes_\nu^{'} S(K_\nu) = \{f = \otimes f_\nu : f_\nu \in S(K_\nu) \forall \nu \text{ and } f_\nu = \mathbf{1}_{\mathfrak{o}_\nu} \text{ for almost all } \nu\},$$

where $\mathbf{1}_{\mathfrak{o}_\nu}$ is a characteristic function of $\mathfrak{o}_\nu$. A function $f \in S(\mathbb{A}_K)$ is called an *adelic Schwartz-Bruhat function.*

According to Proposition 3.1.9, it makes sense to write

$$f(x) = \prod_\nu f_\nu(x_\nu)$$

for all $x = (x_\nu) \in \mathbb{A}_K$. Also, let dx denote the Haar measure on $\mathbb{A}_K$, and define $L^2(\mathbb{A}_K)$ using this measure. It can be shown that $S(\mathbb{A}_K)$ is dense in $L^2(\mathbb{A}_K)$.

Proposition 4.7.2. *For each place ν of K, let ψ_ν be the standard unitary character on K_ν. Then the restriction of ψ_ν to $\mathfrak{o}_\nu$ is trivial for almost all ν. Hence,*

$$\psi_K\left(\prod_\nu x_\nu\right) = \prod_\nu \psi_\nu(x_\nu) \text{ for } x = (x_\nu) \in \mathbb{A}_K$$

is a well-defined non-trivial character on the adeles. Furthermore, $\psi(\alpha) = 1$ for $\alpha \in K$.

Proof. Recall that the conductor of ψ_ν is the inverse different of K_ν. Since the inverse different is trivial for all but finitely many places ν, then $\psi_\nu|_{\mathfrak{o}_\nu} = 1$ for all but finitely many places ν, and hence $\prod_\nu \psi_\nu$ is a well-defined character on $\mathbb{A}_K$. See Proposition 3.1.6. First, let us first restrict ourselves to $K = \mathbb{Q}$ in order to show that ψ is trivial on the embedding of $K = \mathbb{Q}$ into $\mathbb{A}_\mathbb{Q}$. In this case, $\psi_\mathbb{Q} = \psi_\infty \times \prod_p \psi_p$, where ψ_p is the standard non-trivial additive character on $\mathbb{Q}_p$, and where $\psi_\infty = e^{-2\pi i \cdot}$ because the infinite prime of $\mathbb{Q}$ is a real prime (the completion at the usual absolute value is $\mathbb{R}$). Recall that if $\alpha \in \mathbb{Q}$, then there is a unique expansion of the form

$$\alpha = \sum_p \frac{a_p}{p^{\nu_p}} + b,$$

where $a_p, \nu_p, b \in \mathbb{Z}$ and $a_p = 0$ for all but finitely many primes. This was proved explicitly when we constructed the non-trivial additive character on $\mathbb{Q}_p$. Applying this unique representation of α, we obtain

$$\psi_\mathbb{Q}(\alpha) = \prod_p \psi_p(\alpha) = \psi_\infty(\alpha) \prod_{p \not| \infty} \psi_p(\frac{a_p}{p^{\nu_p}}) = e^{-2\pi i a} \prod_p e^{\frac{2\pi a_p}{p^{\nu_p}}} = e^{-2\pi b} = 1.$$

Note the convenience of the negative sign in the infinite character $e^{-2\pi i a}$.

One can show that

$$\sum_{\nu|p} \mathrm{tr}_{K_\nu/\mathbb{Q}_p}(\cdot) = \mathrm{tr}_{K/\mathbb{Q}}(\cdot).$$

See Neukrich's text, *Algebraic Number Theory*, page 164, for a proof of this fact.

Then, for a finite extension K of $\mathbb{Q}$, we have

$$\psi_K(\alpha) = \prod_p \prod_{\nu|p} \psi_p(\mathrm{tr}_{K_\nu/\mathbb{Q}_p}(\alpha)) = \prod_p \psi_p\left(\sum_{\nu|p} \mathrm{tr}_{K_\nu/\mathbb{Q}_p}(\alpha)\right) = \prod_\nu \psi_p(\mathrm{tr}_{K/\mathbb{Q}}(\alpha) = 1.$$

This completes the proof. □

Remark 4.7.3. The standard character ψ_K factors through the trace map from $\mathbb{A}_K$ to $\mathbb{A}_{\mathbb{Q}}$, defined by

$$\begin{array}{rccc} \mathrm{tr}: & \mathbb{A}_K & \longrightarrow & \mathbb{A}_{\mathbb{Q}} \\ & (x_\nu)_\nu & \longmapsto & \left(\sum_{\nu | p} \mathrm{tr}_{K_\nu/\mathbb{Q}_p}(x_\nu) \right), \end{array}$$

where p ranges over all rational primes of $\mathbb{Q}_P$. That is,

$$\psi_K(x) = \psi_{\mathbb{Q}}(\mathrm{tr}(x)) \quad \text{for all } x \in \mathbb{A}_K.$$

Proposition 4.7.4. *Let K be a number field with the standard character ψ_K, as defined above. Then the following assertions hold:*

(i) The map $\alpha_{\psi_K} : \mathbb{A}_K \to \hat{\mathbb{A}_K}$, defined by $y \mapsto \psi_{K,y}$, where $\psi_{K,y}(x) = \psi_K(xy)$, is an isomorphism.

(ii) The map $\beta_{\psi_K} : K \to \widehat{\mathbb{A}_K/K}$, defined by $x \mapsto \psi_{K,x}$, where x is identified with its embedding in $\mathbb{A}_K$, is an isomorphism. Hence, by part (i), the translation $\psi_{K,y}$ of ψ_K is trivial on K if and only if $y \in K$.

Proof.

(i) In Proposition 3.1.7, we proved that the dual group of the restricted direct product of G_ν with respect to H_ν (a compact-open subgroup of G_ν) is the restricted direct product of $\hat{G}_\nu$ with respect to $K(G_\nu, H_\nu)$, the subgroup of characters on G_ν that restrict to the trival map on H_ν. Therefore, $\widehat{\mathbb{A}_K} \cong \prod' \hat{K}_v$, where the restricted direct product is taken with respect to the characters of K_ν that restrict to the identity on $\mathfrak{o}_\nu$. We have proved above that $\psi_K \in \hat{\mathbb{A}_K}$. In Proposition 4.2.1, we proved that $\alpha_{\psi_\nu} : K_\nu \to \hat{K_\nu}$, defined by $y \mapsto \psi_{\nu,y}$, is topological group isomorphism for all local fields K_ν. Therefore, the map $\alpha_{\psi_K} : \mathbb{A}_K \to \hat{\mathbb{A}_K}$, defined by $a = (a_\nu) \mapsto \psi_{K,a} = \prod_\nu \psi_{\nu,a_\nu}$, is an isomorphism.

(ii) Since ψ_K is trivial on K, then ψ_K induces a character on $\mathbb{A}_K/K$. That is, for any coset $a + K \in \mathbb{A}_K/K$, we have that $\psi_K(a + K) = \psi_K(a)$. The map $\beta_{\psi_K} : K \to \widehat{\mathbb{A}_K/K}$, defined by $x \mapsto \psi_{K,x}$, is therefore a well-defined map from K into $\widehat{\mathbb{A}_K/K}$. Indeed, $\psi_{K,\alpha}(x + K) = \psi_{K,\alpha}(x) = \psi_K(\alpha x) = 1$ for all $x \in K$ because $\alpha x \in K$. Since $\mathbb{A}_K/K$ is compact, then by Proposition 1.3.4, $\widehat{\mathbb{A}_K/K}$ is discrete. Recall that from the functorial nature of Pontryagin duality, we have $K^\perp \cong \mathbb{A}_K/K$, where $K^\perp$ is the closed set of characters of $\mathbb{A}_K$ that are trivial on K. Since $\mathbb{A}_K$ is self-dual, then $K^\perp/K$ is a closed subgroup of the compact quotient group $\mathbb{A}_K/K$. Therefore, $K^\perp/K$ is both discrete and compact, and hence finite. Since $K^\perp/K$ is a K-vector space, then $K^\perp/K = 0$, which proves that $\widehat{\mathbb{A}_K/K} \cong K$.

□

Remark 4.7.5. Let $K = \mathbb{Q}$. Then part (ii) yields $\hat{\mathbb{Q}} \cong \mathbb{A}_\mathbb{Q}/\mathbb{Q}$. Let us reinvestigate this case. For any additive character ψ of $\mathbb{A}_\mathbb{Q}$, we can define a character on $\mathbb{Q}$ via the diagonal embedding of $\mathbb{Q}$ into $\mathbb{A}_\mathbb{Q}$. By part (i) of the above theorem, we have that every character on $\mathbb{A}_\mathbb{Q}$ is of the form $\psi_{\mathbb{Q},a}$ for some $a = (a_p) \in \mathbb{A}_\mathbb{Q}$. Let $q \in \mathbb{Q}$. Then

$$\psi_{\mathbb{Q},a}(q) = \psi_\mathbb{Q}((a_p)\cdot(q,q,\cdots)) = \psi_\infty(a_\infty \cdot q)\cdot \prod_{p \text{ finite}} \psi_p(a_p\cdot q) = e^{-2\pi i a_\infty \cdot q}\cdot \prod_{p \text{ finite}} e^{2\pi i\{a_p\cdot q\}_p},$$

where $\{a_p \cdot q\}_p$ are the fractional parts of $a_p \cdot q$ in $\mathbb{Q}_p$. Also, $\psi_{\mathbb{Q},a}$ is a homomorphism of $\mathbb{Q}$ into S^1 because $\psi_\mathbb{Q}$ is a homomorphism of $\mathbb{A}_\mathbb{Q}$ into S^1. However, if $a \in \mathbb{A}_\mathbb{Q} \cap \mathbb{Q}$, that is, $a = (q,q,\cdots)$ for some $q \in \mathbb{Q}$, then the Proposition 4.7.2 tells us that $\psi_{\mathbb{Q},a}$ is the trivial character on $\mathbb{Q}$. If $a, b \in \mathbb{A}_\mathbb{Q}$ and $a - b \in \mathbb{A}_\mathbb{Q} \cap \mathbb{Q}$, then $\psi_{\mathbb{Q},a} = \psi_{\mathbb{Q},b}$. To show that every character of $\mathbb{Q}$ takes this form, and that $\psi_{\mathbb{Q},a} = \psi_{\mathbb{Q},b}$ if and only if $a - b \in \mathbb{A}_\mathbb{Q} \cap \mathbb{Q}$, is precisely the content of part (ii) of the above proposition. It can, however, be shown more directly. See Keith Conrad's [6] notes on the Character Group of $\mathbb{Q}$.

The topological group isomorphism $\hat{\mathbb{Q}} \cong \mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is analogous to $\hat{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z} \cong S^1$, where that hat over $\mathbb{Z}$ is signifying the dual group and not the projective limit. In Proposition 3.2.6, we saw that $\mathbb{A}_\mathbb{Q}/\mathbb{Q} \cong \lim\limits_{\leftarrow} \mathbb{R}/n\mathbb{Z}$. As such, we now obtain the topological group isomorphism $\hat{\mathbb{Q}} \cong \lim\limits_{\leftarrow} \mathbb{R}/n\mathbb{Z}$, which helps solidify the aforementioned analogy. Furthermore, from basic algebraic topology, we know that the universal cover of S^1 is $\mathbb{R}$. Recall that a *covering space* of S^1 is a space C with a continuous surjective map $p : C \to S^1$ such that for every $z \in S^1$, there exists an open neighborhood U of z, such that $p^{-1}(U)$ is a disjoint union of open sets in C, each of which is mapped homeomorphically onto U by p. The map $\rho : \mathbb{R} \to S^1$, defined by $p(t) = (\cos(t), \sin(t))$, is an infinite cover (infinitely many open sets in the pre-image). In addition, $\mathbb{R}$ is simply connected and satisfies the following property: if the mapping $q : C \to S^1$ is any cover of S^1, where C is connected, then there exists a covering map $f : \mathbb{R} \to C$ such that $\rho \circ f = q$. We call $\mathbb{R}$ the universal cover of S^1. See

Munkres [22], Chapter 13, Section 80 for an introduction on universal covers. There is a strong connection between coverings and the fundamental group in nice enough spaces. Since S^1 is path-connected, locally path-connected and semi-locally simply connected, then there is a bijection between equivalence classes of path-connected covers of S^1 and the conjugacy classes of subgroups of the fundamental group of S^1, namely $\mathbb{Z}$. See Munkres [22], Chapter 13, for an introduction to this correspondence. This connection is very similar to the connection between algebraic field extensions and the Galois group. Covering spaces play the role of algebraic field extensions and the fundamental group plays the role of the Galois group. The map $\rho_n : \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a covering of degree n. By the correspondence between conjugacy classes of subgroups of $\mathbb{Z}$ and equivalence classes of path-connected covers of $\mathbb{R}/\mathbb{Z}$, we know that all the finite degree coverings of $\mathbb{R}/\mathbb{Z}$ come from such maps. Therefore, every finite cover of S^1 comes from $\hat{\mathbb{Q}} \cong \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} = \varprojlim \mathbb{R}/n\mathbb{Z}$. Such an object is called an algebraic universal covering. Continuing with the analogy of field extensions, the algebraic universal cover plays the role of the algebraic closure. The Galois group of the covering $\hat{\mathbb{Q}} \cong \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \cong \varprojlim \mathbb{R}/n\mathbb{Z} \to S^1$ is $\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$. This whole construction is a simple example of Grothendieck's general construction of the algebraic fundamental group for abelian varieties.

Let K be a number field. Let K_ν be the completion of K at the νth place. Let us fix the Haar measures dx_ν on local fields as in Definition 3.2.10 and let dx on $\mathbb{A}_K$ be defined as in Definition 3.2.12. That is, dx is the the unique Haar measure such that for each finite set S of places of K, necessarily containing the infinite places, the restriction dx_S of dx to

$$\mathbb{A}_{K,S} = \prod_{\nu \in S} K_\nu \times \prod_{\nu \notin S} \mathfrak{o}_\nu$$

is precisely the product measure $dx_S = \prod_{\nu \in S} dx_\nu$. We write $dx = \prod_\nu dx_\nu$ for the Haar measure on $\mathbb{A}_K$. In Proposition 4.2.4 we proved that the dx_ν are self-dual with respect to the standard non-trivial additive characters, ψ_ν. Since dx_ν is self-dual with respect to ψ_ν for all ν, then dx on $\mathbb{A}_K$ is self-dual with respect to ψ_K by Proposition 3.1.10. Let $|\alpha|_{\mathbb{A}_K} = \prod_\nu |\cdot|_\nu$, where $|\cdot|_\nu$ is the normalized absolute value of the completion K_ν. In Chapter 3, Proposition 3.2.13, we showed that $\mu(\alpha M) = |\alpha|_{\mathbb{A}_K}\mu(M)$ for any Haar measure μ on $\mathbb{A}_K$ and for any measurable set M with $0 < \mu(M) < \infty$. As such, this property holds for our choice of dx on $\mathbb{A}_K$.

Theorem 4.7.6. *The mapping $f \mapsto \hat{f}$ defines an automorphism of $S(\mathbb{A}_K)$ that, moreover, extends to an isometry of $L^2(\mathbb{A}_K)$.*

Proof. Let $f \in S(\mathbb{A}_K)$. Since $S(\mathbb{A}_K)$ is generated by functions of the form $f = \otimes_\nu f_\nu$ where $f_\nu \in S(K_\nu)$, then $\hat{f} = \otimes_\nu \hat{f}_\nu$. We have already seen that for all local fields K_ν, $\hat{f}_\nu \in S(K_\nu)$ for all ν. Furthermore, the conductor of the standard character ψ_ν is the inverse different of K_ν, and the inverse different is $\mathfrak{o}_\nu$ for all but finitely many ν. Also, $f_\nu = \mathbf{1}_{\mathfrak{o}_\nu}$ for all but finitely many ν. Let ν be place such that the conductor of ψ_ν is $\mathfrak{D}_F^{-1} = \mathfrak{o}_\nu$ and such that f_ν is a characteristic function of $\mathfrak{o}_\nu$. There are infinitely many ν of that type. Then

$$\hat{f}_\nu(y_\nu) = \int\limits_{K_\nu} f_\nu(x_\nu)\psi_\nu(x_\nu y_\nu)dx_\nu = \int\limits_{\mathfrak{o}_\nu} \psi_K(x_\nu y_\nu)dx = \begin{cases} \mathrm{Vol}(\mathfrak{o}_\nu, dx_\nu) = 1 & \text{if } y_\nu \in \mathfrak{o}_\nu \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we picked dx such that $\mathrm{Vol}(\mathfrak{o}_\nu, dx_\nu) = N(\mathfrak{D}_{K_\nu})^{-1/2}$. Since the inverse different in this case is $\mathfrak{o}_\nu$, then $N(\mathfrak{D}_{K_\nu})^{-1/2} = 1$. Therefore, $\hat{f} = \prod_\nu \hat{f}_\nu \in S(\mathbb{A}_K)$ because $\hat{f}_\nu \in S(K_\nu)$ for all ν and because $\hat{f}_\nu = \mathbf{1}_{\mathfrak{o}_\nu}$ for all but finitely many ν.

Let $h = \overline{\hat{f}}$. Then

$$\hat{h}(y) = \int\limits_{\mathbb{A}_K} \overline{\hat{f}(x)}\psi_K(xy)dx = \int\limits_{\mathbb{A}_K} \overline{\hat{f}(x)\psi_K(-xy)}dx = \overline{\int\limits_{\mathbb{A}_K} \hat{f}(x)\psi_K(x(-y))dx} = \overline{f(-(-y))} = \overline{f(y)}$$

by the Fourier inversion theorem and self-duality

$$\hat{\hat{g}}(y) = g(-y).$$

Applying the above result and Fubini's theorem we obtain

$$\begin{aligned} \int\limits_{\mathbb{A}_K} |f(x)|^2 dx &= \int\limits_{\mathbb{A}_K} f(x)\overline{f(x)}dx = \int\limits_{\mathbb{A}_K} f(x)\hat{h}(x)dx = \\ &= \int\limits_{\mathbb{A}_K} f(x)\left(\int\limits_{\mathbb{A}_K} h(y)\psi_K(yx)dy\right)dx = \int\limits_{\mathbb{A}_K} h(y)\left(\int\limits_{\mathbb{A}_K} f(x)\psi_K(xy)dx\right)dy = \\ &= \int\limits_{\mathbb{A}_K} \hat{f}(x)h(x)dx = \int\limits_{\mathbb{A}_K} |\hat{f}(x)|^2 dx \end{aligned}$$

Since the Schwartz-Bruhat functions are dense in $L^2(\mathbb{A}_K)$, a Hilbert space, then Fourier transform may be extended by continuity to an isometry on $L^2(\mathbb{A}_K)$. This follows from the fact that $L^2(\mathbb{A}_K)$ is a Hilbert space. See Rudin's *Real and Complex Analysis* Chapter 4 Lemma 4.16 [26]. □

We want to consider the set of functions on $\mathbb{A}_K$ that are invariant with respect to additive translations by elements of K. For example, ψ_K, as chosen above, is translation invariant because ψ_K is an additive character that is trivial on K. For $\phi \in S(\mathbb{A}_K)$, set

$$\tilde{\phi}(x) = \sum_{\gamma \in K} \phi(\gamma + x).$$

If the function $\tilde{\phi}$ is convergent for all $x \in \mathbb{A}_K$, then it is invariant under translation by K because additive translation is an automorphism of K.

Definition 4.7.7. Let f be a complex-valued function on $\mathbb{A}_K$ such that both $\tilde{f}$ and $\tilde{\hat{f}}$ are *normally convergent*; both are absolutely and uniformly convergent on compact sets. Then we say that f is *admissible.*

Lemma 4.7.8. *All $f \in S(\mathbb{A}_K)$ are admissible.*

Proof. Let $f \in S(\mathbb{A}_K)$. Let C be a compact set of $\mathbb{A}_K$. We know that compact sets of the local fields K_ν are of the form $\mathfrak{p}_\nu^{n_\nu}$, where $\mathfrak{p}$ is the unique prime ideal of K_ν and where $n_\nu \in \mathbb{Z}$. Therefore, without loss of generality, by enlarging K, we may take it to be of the form

$$\prod_{\nu \in S_\omega} C_\nu \times \prod_{\nu \in S} \mathfrak{p}_\nu^{n_\nu} \times \prod_{\nu \notin S \cup S_\omega} \mathfrak{o}_\nu,$$

where S is the finite set of finite places such that $f|_{\mathfrak{o}_\nu} \neq 1$, and S_ω is the set of infinite places. Since the characteristic functions of $\mathfrak{p}_\nu^{m_\nu}$ generate $S(K_\nu)$, then we may assume that f_ν for all $\nu \in S$ are characteristic functions. Define the fractional ideal I in $\mathfrak{o}_K$ by

$$I = \prod_{\nu \in S} \mathfrak{p}_\nu^{k_\nu},$$

where $k_\nu = \inf\{n_\nu, m_\nu\}$ and where $\mathfrak{p}_\nu$ really should be thought of as $\mathfrak{p}_\nu \cap \mathfrak{o}_K$. Suppose that $f(\gamma + z) \neq 0$ for some $z \in C$ and for some $\gamma \in K$. Since we assumed that for all $\nu \in S$, f_ν is a characteristic function of $\mathfrak{p}_\nu^{m_\nu}$, then the νth components of $\gamma + z \in \mathbb{A}_K$, $\nu \in S$ are necessarily in $\mathfrak{p}^{m_\nu}$ for all $\nu \in S$. Consequently, $\gamma \in \mathfrak{p}^{m_\nu} - z_\nu \subset \mathfrak{p}_\nu^{k_\nu}$ for all $\nu \in S$. In addition, $\gamma \in \mathfrak{o}_\nu$ for $\nu \notin S \cup S_\omega$. Therefore,

$$|\tilde{f}(z)| = |\sum_{\gamma \in K} f(\gamma + z)| \leq \sum_{\gamma \in K} \prod_{\nu} |f_\nu(\gamma + z_\nu)| = \sum_{\gamma \in I} \prod_{\nu \in S_\omega} |f_\nu(\gamma + z_\nu)| = \sum_{\gamma \in I} |f_\omega(\gamma + z_\omega|,$$

where

$$f_\omega = \prod_{\nu \in S_\omega} f_\nu \in S(\prod_{\nu \in S_\omega} K_\nu) \text{ and } z_\omega = (z_\nu)_{\nu \in S_\omega}.$$

In the previous chapter, we showed that K is a discrete subgroup of $\mathbb{A}_K$. Hence, I is a discrete subgroup of $\prod_{\nu \in S_\omega} K_\nu$. Since for $\nu \in S_\omega$, $f_\nu \in S(K_\nu)$ has a uniform bound on the compact set C_ν of K_ν and decreases rapidly with z_ν, then the analogous statement for f_ω is true. We now will want to show that for all but finitely many $\gamma \in I$, $f_\omega(\gamma + z_\omega) = 0$. Since I is a discrete subgroup of $\prod_{\nu \in S_\omega} K_\nu$, where K_ν is either isomorphic to $\mathbb{R}$ or $\mathbb{C}$, the number of $\gamma \in I$ that occur in any shell of radius B and thickness ΔB can only grow at most by powers of the radius B. However, since $|f_\omega|$ goes to zero more rapidly than any polynomial, then we know that the number of terms in the summation above is finite. We have shown above that the Fourier transform maps $S(\mathbb{A}_K)$ to $S(\mathbb{A}_K)$; thus, $\hat{f}$ is normally convergent by the above. $\square$

We are now ready to prove the *Poisson Summation Formula.*

Theorem 4.7.9. *Let $f \in S(\mathbb{A}_K)$. Then $\tilde{f} = \tilde{\hat{f}}$; that is,*

$$\sum_{\gamma \in K} f(\gamma + x) = \sum_{\gamma \in K} \hat{f}(\gamma + x)$$

for all $x \in \mathbb{A}_L$.

Proof. If $\phi \in \mathbb{A}_K$ is a K-invariant function on $\mathbb{A}_K$, then ϕ induces a function on $\mathbb{A}_K/K$. Then the Fourier transform of $\phi : \mathbb{A}_K/K \to \mathbb{C}$ can be realized as a function on K since K is the dual group of $\mathbb{A}_K/K$. That is, for all $z \in K$,

$$\hat{\phi}(z) = \int_{\mathbb{A}_K/K} \phi(t)\psi_K(tz)\overline{dt},$$

where $\overline{dt}$ is the quotient Haar measure on the compact group $\mathbb{A}_K/K$ induced by dt on $\mathbb{A}_K$. The quotient measure $\overline{dt}$ is characterized by the relation

$$\int\limits_{\mathbb{A}_K/K} \tilde{f}(t)\overline{dt} = \int\limits_{\mathbb{A}_K/K} \left(\sum_{\gamma\in K} f(\gamma+t)\right)\overline{dt} = \int\limits_{\mathbb{A}_K} f(t)dt$$

for all continuous and admissible functions f on $\mathbb{A}_K$. In order to proceed, we will establish two lemmas.

Lemma 4.7.10. *For every $f \in S(\mathbb{A}_K)$, we have*

$$\hat{f}|_K = \hat{\tilde{f}}|_K.$$

Proof. Fix $z \in K$. By definition, and the fact that $\psi_K|_K = 1$, we obtain

$$\begin{aligned}
\hat{\tilde{f}}(z) &= \int\limits_{\mathbb{A}_K/K} \tilde{f}(t)\psi_K(tz)\overline{dt} = \int\limits_{\mathbb{A}_K/K} \left(\sum_{\gamma\in K} f(\gamma+t)\right)\psi_K(tz)\overline{dt} = \\
&= \int\limits_{\mathbb{A}_K/K} \left(\sum_{\gamma\in K} f(\gamma+t)\psi_K((\gamma+t)z)\right)\overline{dt} = \int\limits_{\mathbb{A}_K} f(t)\psi_K(tz)dt = \hat{f}(z).
\end{aligned}$$

$\square$

Lemma 4.7.11. *For every $f \in S(\mathbb{A}_K)$ and $x \in K$, we have*

$$\tilde{f}(x) = \sum_{\gamma\in K} \hat{\tilde{f}}(\gamma)\overline{\psi_K}(\gamma x).$$

Proof. Since $\hat{f}|_K = \hat{\tilde{f}}|K$, then

$$\left|\sum_{\gamma\in K} \hat{\tilde{f}}(\gamma)\overline{\psi_K}(\gamma x)\right| = \left|\sum_{\gamma\in K} \hat{f}(\gamma)\overline{\psi_K}(\gamma x)\right| \leq \sum_{\gamma\in K} |\hat{f}(\gamma)|$$

because ψ_K is unitary. Therefore, the expression on the right-hand side of the lemma is normally convergent, since $f \in S(\mathbb{A}_K)$ is admissible. Also,

$$\sum_{\gamma\in K} \hat{\tilde{f}}(\gamma)$$

is convergent for the same reason. Recall that $\widehat{\mathbb{A}_K/K} \cong K$. In this way, $\hat{\hat{f}} \in L^1(K)$. Since K is discrete, then

$$\sum_{\gamma \in K} \hat{\hat{f}}(\gamma)\overline{\psi_K}(\gamma x)$$

is the Fourier transform of $\hat{\tilde{f}}$ evaluated at $-x$. Since $\hat{\hat{\tilde{f}}}(-x) = \tilde{f}(x)$ by the Fourier inversion formula, then the lemma is proved. **Note that it is here where self-duality is needed**

$\square$

Let us return to the proof of the Poisson summation formula. Applying the second lemma with $x = 0$ and then the first lemma, we obtain

$$\tilde{f}(0) = \sum_{\gamma \in K} \hat{\tilde{f}}(\gamma)\overline{\psi_K}(0) = \sum_{\gamma \in K} \hat{\tilde{f}}(\gamma) = \sum_{\gamma \in K} \hat{f}(\gamma).$$

Since $\tilde{f}(0) = \sum_{\gamma \in K} f(\gamma)$, then

$$\sum_{\gamma \in K} f(\gamma) = \sum_{\gamma \in K} \hat{f}(\gamma).$$

$\square$

We now will proceed with the number field analogue of the geometric Riemann-Roch Theorem.

Theorem 4.7.12 (Riemann-Roch). *Let $x \in \mathbb{I}_K$. Let $f \in S(\mathbb{A}_K)$. Then*

$$\sum_{\gamma \in K} f(\gamma x) = \frac{1}{|x|_{\mathbb{A}_K}} \sum_{\gamma \in K} \hat{f}(\gamma x^{-1}).$$

Proof. Fix an $x \in \mathbb{I}_K$. Since f is admissible, then f_x, defined by $f_x(y) = f(xy)$, is in $S(\mathbb{A}_K)$, and hence admissible. So, the sum on the left is normally convergent. The Poisson summation formula applied to f_x yields

$$\sum_{\gamma \in K} f_x(\gamma) = \sum_{\gamma \in K} \hat{f}_x(\gamma).$$

Computing the Fourier transform of f_x we obtain

$$\hat{f}_x(\gamma) = \int_{\mathbb{A}_K} f(yx)\psi_K(y\gamma)dy = \frac{1}{|x|_{\mathbb{A}_K}} \int_{\mathbb{A}_K} f(y)\psi_K(yx^{-1}\gamma)dy = \frac{1}{|x|_{\mathbb{A}_K}} \hat{f}(\gamma x^{-1}).$$

This completes the proof. $\square$

4.8 Idele-Class Characters

Proposition 4.8.1. *Every idele-class character χ has the factorization $\chi{=}\tilde{\chi}|\cdot|^s$ where $\tilde{\chi}$ is a unitary character.*

Proof. Let $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}_K/K^*, \mathbb{C}^\times)$. Let ν_∞ be an infinite place of K. Consider the subgroup $V(\mathbb{I}_K) = \{(t_{\nu_\infty}, 1, 1, \ldots) : t_{\nu_\infty} \in \mathbb{R}_+^\times\}$ of $\mathbb{I}_K$. Then $|\,(t_{\nu_\infty}, 1, 1, \ldots)\,|_{\mathbb{A}_K} = t_{\nu_\infty}$ if ν_∞ is a real place and $= t_{\nu_\infty}^2$ if ν_∞ is a complex place. We have at once that the map $|\cdot|_{\mathbb{A}_K} : V(\mathbb{I}_K) \to \mathbb{R}_+^\times$ is an isomorphism. Since we uniquely can write any idele in the form $x = |x|_{\mathbb{A}_K} \cdot y$ where $y \in \mathbb{I}_K^1$, then the map $\phi : V(\mathbb{I}_K) \times \mathbb{I}_K^1 \to \mathbb{I}_K$, defined by $(\alpha, \beta) \mapsto \alpha\beta$, is an isomorphism. Moreover, we have the short exact sequence

$$1 \to C_K^1 = \mathbb{I}_K^1/K^* \to C_K = \mathbb{I}_K/K^* \to V(\mathbb{I}_K) = \mathbb{R}_+^\times \to 1.$$

Recall that $C_K^1 = \mathbb{I}_K^1/K^*$ is compact, where $\mathbb{I}_K^1 = \mathrm{Ker}(|\cdot|_{\mathbb{A}_K})$. Since a quasi-character is continuous, then $\chi(\mathbb{I}_K^1/K^*)$ is a compact subgroup of $\mathbb{C}^\times$, and hence is contained in S^1. Therefore, $\chi|_{\mathbb{I}_K^1/K^*} = \tilde{\chi}$ is a unitary character on $\mathbb{I}_K^1/K^*$. Now, consider $\tilde{\chi}^{-1}\chi$, which, by definition, is trivial on $\mathbb{I}_K^1/K^*$. Since $\mathbb{I}_K/\mathbb{I}_K^1 \cong \mathbb{R}_+^\times$, then $\tilde{\chi}^{-1}\chi = |\cdot|^s$ for some $s \in \mathbb{C}$. See Prop 4.1.3 for a proof of this fact. Therefore, an arbitrary quasi-character on C_K is of the form $\alpha \mapsto \tilde{\chi}(\tilde{\alpha})|\alpha|^s$ where $\tilde{\alpha}$ is characterized by the relation $\alpha = \tilde{\alpha}\beta$, for some unique $\beta \in C_K^1$. $\square$

Remark 4.8.2. An idele-class character, χ, is called unramified if $\chi|_{\mathbb{I}_1} = 1$. We say that two idele-class characters are equivalent if their quotient is unramified. Each equivalence class is of the form

$$\{\tilde{\chi}|\cdot|^s : s \in \mathbb{C}\}$$

for some fixed unitary character $\tilde{\chi}$. Let us investigate the idele-class characters of $\mathbb{Q}$.

Example 4.8.3. Let $K = \mathbb{Q}$. In the adele and idele chapter, we showed that $\mathbb{I}_\mathbb{Q} \cong \mathbb{Q}^* \times \mathbb{R}_+^\times \times \hat{\mathbb{Z}}^\times$, where $\hat{\mathbb{Z}}^\times = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^\times = \prod_p \mathbb{Z}_p^\times$. Recall from the chapter concerning topological groups that $\hat{\mathbb{Z}}^\times = \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^\times$ is a projective limit of discrete groups, and hence is a profinite group. Since $\hat{\mathbb{Z}}^\times$ is a profinite group, then, by Theorem 1.1.40, it is compact and totally disconnected.

We see that $C_\mathbb{Q} = \mathbb{I}_\mathbb{Q}/\mathbb{Q}^* \cong \mathbb{R}_+^\times \times \hat{\mathbb{Z}}^\times$. The quotient topology on $C_\mathbb{Q}$ is equivalent to the product topology on $\mathbb{R}_+^\times \times \hat{\mathbb{Z}}^\times$. Let χ be a quasi-character of $\mathbb{I}_\mathbb{Q}$ that is trivial on $\mathbb{Q}^*$, or, equivalently, an idele-class character. More specifically, let χ be a finite order character. If χ is a finite-order quasi-character, then χ must be trivial on $\mathbb{R}_+^\times$ because $\mathbb{R}_+^\times$ is a divisible group. As such, χ is a character on $\hat{\mathbb{Z}}^\times$, a totally disconnected compact group. Furthermore, $\chi = \prod_p \chi_p$, where χ_p is a continuous unitary character on $\mathbb{Z}_p^\times$ such that $\chi_p|_{\mathbb{Z}_p^\times} = 1$ for all but finitely many primes p. See Proposition 3.1.5 for a proof of this fact. It relies on the fact that χ is continuous and that there are no small subgroups of S^1. For the remaining χ_{p_i}, $i = 1, \dots, r$ that are not trivial on $\mathbb{Z}_{p_i}$, there exists a largest subgroup $U_{n_i} = 1 + p_i^{n_i}\mathbb{Z}_{p_i}$ such that $\chi_{p_i}|_{U_{n_i}} = 1$. Again, we use a "no small subgroups" argument. Consequently, the kernel of χ contains

$$\mathbb{R}_+^\times \times \prod_{i=1}^r U_{n_i} \times \prod_{p \neq p_i} \mathbb{Z}_p^\times,$$

and χ factors through the quotient group

$$\prod_{i=1}^r \mathbb{Z}_{p_i}^\times / \prod_{i=1}^r U_{n_i} \cong \prod_{i=1}^r (\mathbb{Z}/p_i^{n_i}\mathbb{Z})^\times \cong (\mathbb{Z}/n\mathbb{Z})^\times,$$

where $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ by the Chinese remainder theorem. Therefore, χ induces a multiplicative character $\chi_n : (\mathbb{Z}/n\mathbb{Z})^\times \to S^1$. Such a character is called a *Dirichlet character*. In sum, any idele-class character χ of $\mathbb{Q}$ takes the form $\chi = \tilde{\chi}| \cdot |_{\mathbb{A}_K}^s$, where $\tilde{\chi}$ is a finite order character induced from a Dirichlet character mod n.

In general, let χ be an idele-class character of a number field K. Let S be the finite set of finite places ν of K such that $\chi_\nu|_{\mathfrak{o}_\nu^\times} \neq 1$. Let S_∞ be the finite set of infinite primes. Let $\mathfrak{p}_\nu$ be the unique prime associated to a finite place ν of K. Then there exists a largest subgroup $U_{n_\nu} = 1 + \mathfrak{p}_\nu^{n_\nu}$ of $\mathfrak{o}_\nu^\times$ such that $\chi_\nu | U_{n_\nu} = 1$. As such

$$\prod_{\nu \in S_\infty} \{1\} \times \prod_{\nu \in S} U_{n_\nu} \times \prod_{\nu \notin S \cup S_\infty} \mathfrak{o}_\nu^\times \subset \mathrm{Ker}(\chi).$$

We call $\mathfrak{m} = \prod_{\nu \in S} \mathfrak{p}_\nu^{n_\nu}$ the *modulus of* χ.

We will now show how a a Dirchlet character on $(\mathbb{Z}/n\mathbb{Z})^\times$ induces a finite order idele-class character. A Dirichlet character first can be extended to $\mathbb{Z}/n\mathbb{Z}$ by extending by 0, and then extended again to a map $\underline{\chi_N} : \mathbb{Z} \to S^1 \cup \{0\}$ by pulling back under the projection $\mathrm{pr}_N : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. We call such a character $\underline{\chi_N}$ a *classical Dirichlet character.* Suppose $N|M$. Then χ_N determines a character $\chi_M = \chi_N \circ \mathrm{pr}_{M,N}$ of $\mathbb{Z}/m\mathbb{Z}$ by pulling back under the projection

$$\mathrm{pr}_{M,N} : (\mathbb{Z}/M\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times.$$

Furthermore, we can extend this character χ_M to a character $\underline{\chi_M}$ on $\mathbb{Z}$. The projection maps

$$\mathrm{pr}_N : \hat{\mathbb{Z}}^\times \to (\mathbb{Z}/n\mathbb{Z})^\times$$

are surjective and continuous. As such, every Dirichlet character χ_N can be viewed as a continuous character χ of $\hat{\mathbb{Z}}^\times$. The *conductor* of χ is the smallest N_0 such that χ is trivial on the kernel of the projection pr_{N_0}. Note that the smallest N_0 corresponds to the largest subgroup (kernel) of $\hat{\mathbb{Z}}^\times$. It follows that $N_0|N$ and that χ is the pullback of a unique character χ_{N_0} of $\mathbb{Z}/N_0\mathbb{Z}$. As such, there is a collection of Dirichlet characters that pullback to χ; we call $\underline{\chi_{N_0}}$ a *primitive* classical Dirichlet character and all others *imprimitive* classical Dirichlet characters. Therefore, Dirichlet characters are in bijective correspondence with continuous characters of $\hat{\mathbb{Z}}^\times$ and, moreover, of continuous finite order characters of the idele-class group $C_{\mathbb{Q}}$.

The equivalence classes of quasi-characters of $C_{\mathbb{Q}} = \mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^*$ are of the form

$$\{\tilde{\chi}|\cdot|^s : s \in \mathbb{C}\},$$

where $\tilde{\chi}$ is a character on $\hat{\mathbb{Z}}^\times$ induced from a Dirichlet Character. Since there are countably many conductors of $\tilde{\chi}$, and only finitely many Dirchlet Characters for a given conductor, then domain of idele-class characters is isomorphic to countably many copies of the complex plane.

4.9 The Meromorphic Continuation and Functional Equation of the Global Zeta Function

Let K be a number field and let ψ_K the standard adelic character. Let dx_ν be the self-dual additive measure with respect to ψ_ν. We set

$$d^*x_\nu = \frac{q_\nu}{q_\nu - 1} \cdot \frac{dx_\nu}{|x_\nu|_\nu}$$

to be the Haar measure of the multiplicative group of the completion of K with respect to finite places, ν, of K. We have seen that $\mathrm{Vol}(\mathfrak{o}_\nu^\times, d^*x_\nu) = \mathrm{Vol}(\mathfrak{o}_\nu, dx) = N(\mathfrak{D}_{K_\nu}^{-1}) = q_\nu^{-d_\nu/2}$. Also, we set $d^*x_\nu = dx_\nu/|x_\nu|_\nu$ to be the Haar measure for the multiplicative group of the completion of K with respect to the infinite places. Since $\mathfrak{D}_{K_\nu}^{-1} = \mathfrak{o}_\nu$ for all but finitely many primes, then $\mathrm{Vol}(\mathfrak{o}_\nu^\times, d^*x_\nu) = 1$ for all but finitely many places ν of K. Therefore, $\prod_\nu d^*x_\nu$ is a Haar measure on $\mathbb{I}_K$ by Proposition 3.1.8.

Definition 4.9.1. Let $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}_K/K^*, \mathbb{C}^\times)$. For $f \in S(\mathbb{A}_K)$, define the *global zeta function* by

$$Z(f,\chi) = \int_{\mathbb{I}_K} f(x)\chi(x)d^*x.$$

Note that since the restricted direct product topology of $\mathbb{I}_K$ is stronger than the subspace topology induced by $\mathbb{A}_K$, then f is necessarily continuous on $\mathbb{I}_K$.

Just as the local zeta function was a function on the domain of quasi-characters of a local field F, $Z(f,\chi)$ is a function on the domain of idele-class characters of a given number field K. In the following theorem, we first will prove that $Z(f,\chi)$ is absolutely and uniformly convergent on the domain of idele-class characters of exponent greater than 1. Then we will prove that in the equivalence class of unramified characters, $Z(f,\chi)$ can be meromorphically continued to the whole s-plane with two simple-poles at $s = 0$ and $s = 1$; on all other equivalence classes, $Z(f,\chi)$ can be analytically continued to the whole s-plane.

Theorem 4.9.2. *For all idele-class characters $\chi = \tilde{\chi}|\cdot|^s$ and $f \in S(\mathbb{A}_K)$, the global zeta function $Z(f,\chi)$ is normally convergent in $\sigma = \Re(s) > 1$. Furthermore, $Z(f,\chi)$ extends to a meromorphic function of s and satisfies the functional equation*

$$Z(f,\chi) = Z(\hat{f},\check{\chi}).$$

The continuation is entire in all classes of idele-class characters except for the class of unramified characters, which is given by the set

$$\{\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}_K/K^*, \mathbb{C}^\times) : \tilde{\chi} = |\cdot|^{-i\tau} \textit{ with } \tau \in \mathbb{R}\}.$$

For a given class representative $\chi = |\cdot|^{s-i\tau}$, $Z(f, |\cdot|^{s-i\tau})$ *has simple poles at* $s = i\tau$ *and* $s = 1 + i\tau$, *with corresponding residues given by*

$$-\mathrm{Vol}(C_K^1)f(0) \quad \textit{and} \quad \mathrm{Vol}(C_K^1)\hat{f}(0),$$

respectively. The volume of C_K^1 *is taken with respect to the quotient measure on* C_K *defined by both* d^*x *and the counting measure on* K^*. *We will compute* $\mathrm{Vol}(C_K^1)$ *in the last section of this chapter.*

Proof. Since $f \in S(\mathbb{A}_K)$, then f_ν is a characteristic function of $\mathfrak{o}_\nu$ for all but finitely many finite places ν of K. Let S be the finite set of finite places for which $f_\nu \in S(K_\nu)$ is not a characteristic function of $\mathfrak{o}_\nu$. For all finite places ν of K, let $\mathfrak{p}_\nu$ be the unique prime of K_ν and let π_ν be a uniformizing parameter of $\mathfrak{p}_\nu$. We may take f_ν for $\nu \in S$ to be a characteristic function of $\mathfrak{p}_\nu^{m_\nu} = \pi_\nu^{m_\nu}\mathfrak{o}_\nu$ by linearity and translation invariance of the Haar measure. Let S_ω be the set of infinite places of K. As such, the product

$$\prod_\nu c_\nu \int\limits_{K_\nu - \{0\}} |f_\nu(x_\nu)||x_\nu|_\nu^{\sigma-1} dx_\nu,$$

where $c_\nu = q_\nu/(q_\nu - 1)$ for finite places and $c_\nu = 1$ for infinite places, is equal to

$$\prod_{\nu \in S} c_\nu \int\limits_{\pi_\nu^{m_\nu}\mathfrak{o}_\nu - \{0\}} |x_\nu|_\nu^{\sigma-1} dx_\nu \times \prod_{\nu \in S_\omega} \int\limits_{K_\nu - \{0\}} |f_\nu(x_\nu)||x_\nu|_\nu^{\sigma-1} dx_\nu \times \prod_{\nu \notin S \cup S_\omega} c_\nu \int\limits_{\mathfrak{o}_\nu - \{0\}} |x_\nu|_\nu^{\sigma-1} dx_\nu.$$

In part (i) of Theorem 4.5.3, for Archimedean fields ($\nu \in S_\omega$) we showed that the integral

$$\int\limits_{K_\nu - \{0\}} |f_\nu(x_\nu)||x_\nu|_\nu^{\sigma-1} dx_\nu$$

is finite for $\sigma > 0$. Since the number of infinite places is finite, then the product of the Archimedean integrals is equal to some positive real M. Furthermore, for non-Archimedean fields, and hence ν finite, we showed that

$$c_\nu \int\limits_{\pi_\nu^{m_\nu}\mathfrak{o}_\nu - \{0\}} |x_\nu|_\nu^{\sigma-1} dx_\nu = \mathrm{Vol}(\mathfrak{o}_\nu^\times, d^*x_\nu)\frac{q_\nu^{-m_\nu\sigma}}{1 - q_\nu^{-\sigma}}$$

for $\sigma > 0$. Also, $c_\nu = \frac{q_\nu}{(q_\nu - 1)}$ was chosen so that $\mathrm{Vol}(\mathfrak{o}_\nu^\times, d^*x_\nu) = N(\mathfrak{D}_{K_\nu}^{-1})$. We obtain at once that the product of integrals is equal to

$$M\left(\prod_{\nu\in S}\frac{q_\nu^{-m_\nu\sigma}}{1-q_\nu^{-\sigma}}\cdot N(\mathfrak{D}_{K_\nu}^{-1})\right)\left(\prod_{\nu\notin S\cup S_\omega}\frac{1}{1-q_\nu^{-\sigma}}\cdot N(\mathfrak{D}_{K_\nu}^{-1})\right).$$

The convergence of the global zeta function is determined completely by the convergence of the infinite product

$$\prod_{\nu\notin S\cup S_\omega}\frac{1}{1-q_\nu^{-\sigma}},$$

since S and S_ω is finite and $\mathfrak{D}_{K_\nu}^{-1} = \mathfrak{o}_\nu$ for all but finitely many places ν of K. An infinite product of complex numbers $\prod_{n=1}^{\infty} a_n$ of complex numbers a_n is said to converge if the sequence of partial products $P_n = a_1 \cdots a_n$ has a nonzero limit. If we fix the principal branch of logarithm, then $\prod_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1} \log a_n$ converges, where log denotes the principal branch of the logarithm. See Alfhors, *Complex Analysis* Chapter V 2.2 ([1]). A product is called absolutely convergent if the series converges absolutely. Therefore, in order to determine the region of convergence of the product, we examine the logarithm

$$\sum_{\nu\notin S\cup S_\omega}\log\left(\frac{1}{1-q_\nu^{-\sigma}}\right) = \sum_{\nu\notin S\cup S_\omega}\sum_{m=1}^{\infty}\frac{q_\nu^{-m\sigma}}{m}.$$

The number of ν lying over a given rational prime p is bounded by $n = [K:\mathbb{Q}]$. Also, $q_\nu = p^f$ where f is the residue degree of $K_\nu/\mathbb{Q}_p$. That is q_ν is a positive integer power of p So, if $\mathfrak{p}_\nu$ lies above p, then $q_\nu^{-m\sigma} \le p^{-m\sigma}$ for m positive. Letting p run over the set of positive rational primes, we obtain

$$\sum_{\nu\notin S\cup S_\omega}\log\left(\frac{1}{1-q_\nu^{-\sigma}}\right) = \sum_p \sum_{\substack{\nu|p,\\ \nu\notin S\cup S_\omega}}\sum_{m=1}^{\infty}\frac{q_\nu^{-m\sigma}}{m} \le \sum_p\sum_{m=1}^{\infty}\frac{p^{-m\sigma}}{m}.$$

The product on the right converges absolutely for $\sigma \ge 1+\delta$ for every $\delta > 0$. Indeed, $mp^{m\sigma} \ge p^{m(1+\delta)}$ for $m > 0$ so the series has the convergent majorant

$$\sum_p\sum_{m=1}^{\infty}p^{-m(1+\delta)} = \sum p(p^{1+\delta}-1)^{-1} \le 2\sum_p\frac{1}{p^{1+\delta}},$$

which converges by the p-test. Since

$$\prod_{\nu} c_\nu \int\limits_{K_\nu - \{0\}} |f_\nu(x_\nu)||x_\nu|_\nu^{\sigma-1} dx_\nu$$

is convergent in $\sigma > 1$, then

$$\prod_{\nu} \int\limits_{K_\nu^*} f_\nu(x_\nu)\chi_\nu(x_\nu) dx_\nu^*$$

is normally convergent in $\sigma > 1$. We have that $\chi = \tilde{\chi}|\cdot|_{\mathbb{A}_K}^s = \prod_\nu \tilde{\chi}_\nu|\cdot|_\nu^s$ from a proposition in the adele and idele chapter. By the analysis from the adele and idele chapter, we have that

$$|Z(f,\chi)| = \left|\int\limits_{\mathbb{I}_K} f(x)\chi(x)d^*x\right| = \prod_{\nu}\left|\int\limits_{K_\nu^*} f_\nu(x_\nu)\chi_\nu(x_\nu)d^*x_\nu\right| \le \prod_{\nu} c_\nu \int\limits_{K_\nu - \{0\}} |f_\nu(x_\nu)||x_\nu|_\nu^{\sigma-1} dx_\nu.$$

Therefore,, $Z(f,\chi) = Z(f,\tilde{\chi},s)$ is normally convergent in $\sigma = \Re(s) > 1$. In order to show that $Z(f,\chi)$ is holomorphic for $\sigma > 1$, we need to justify exchanging the order of the derivative d/ds and the integral. We see that

$$\frac{d}{ds} f(x)\tilde{\chi}(x)|x|_{\mathbb{A}_K}^s = f(x)\tilde{\chi}(x)\frac{d}{ds}e^{s\log(|x|_{\mathbb{A}_K})} = f(x)\tilde{\chi}(x)\log(|x|_{\mathbb{A}_K})|x|_{\mathbb{A}_K}^s,$$

which is continuous and absolutely integrable for $\sigma > 1$. Therefore,

$$\frac{d}{ds}Z(f,\tilde{\chi},s) = \frac{d}{ds}\int\limits_{\mathbb{I}_K} f(x)\tilde{\chi}(x)|x|_{\mathbb{A}_K}^s d^*x = \int\limits_{\mathbb{I}_K} f(x)\tilde{\chi}(x)\log(|x|_{\mathbb{A}_K})|x|_{\mathbb{A}_K}^s d^*x,$$

which proves that $Z(f,\chi)$ is holomorphic in $\sigma > 1$.

If we fix an infinite place of K, then $\mathbb{I}_K \cong \mathbb{R}_+^\times \times \mathbb{I}_K^1$. We see by our choice of measures above that d^*x on $\mathbb{I}_K$ is equivalent to $\frac{dt}{t} \cdot d^*x$ on $\mathbb{R}_+^\times \times \mathbb{I}_K^1$. Therefore, if $\sigma > 1$, then applying Fubini's theorem with both $f \in S(\mathbb{A}_K)$ and $\sigma \ge 1$, we obtain

$$Z(f,\chi) = \int\limits_{\mathbb{I}_K} f(x)\chi(x)d^*x = \iint\limits_{\mathbb{R}_+^\times \times \mathbb{I}_K^1} f(tx)\chi(tx)\frac{dt}{t}d^*x = \int\limits_0^\infty \int\limits_{\mathbb{I}_K^1} f(tx)\chi(tx)d^*x\frac{dt}{t},$$

where the product tx takes place at the fixed infinite component of x. Define

$$Z_t(f,\chi) = \int\limits_{\mathbb{I}_K^1} f(tx)\chi(tx)d^*x.$$

We will now apply Riemann-Roch to establish a functional equation for $Z_t(f,\chi)$.

Proposition 4.9.3. *The function $Z_t(f,\chi)$ satisfies the relation*

$$Z_t(f,\chi) = Z_{t^{-1}}(\hat{f},\check{\chi}) + \hat{f}(0)\int\limits_{C_K^1} \check{\chi}(x/t)d^*x - f(0)\int\limits_{C_K^1} \chi(tx)d^*x.$$

Proof. By definition, $C_K^1 = \mathbb{I}_K^1/K^*$. Since K^* is discrete in $\mathbb{I}_K^1$, then the Haar measure on K^* is the counting measure. Then

$$Z_t(f,\chi) = \int\limits_{C_K^1} \left(\sum_{a\in K^*} f(atx)\chi(atx)\right) d^*x = \int\limits_{C_K^1} \left(\sum_{a\in K^*} f(atx)\right) \chi(tx)d^*x$$

since $\chi|_{K^*} = 1$, by hypothesis. To apply the Riemann-Roch theorem, we need to sum over K, not K^*. In order to do this, we add $f(0)\int\limits_{C_K^1} \chi(tx)d^*x$ to $Z_t(f,\chi)$. That is,

$$Z_t(f,\chi) + f(0)\int\limits_{C_K^1} \chi(tx)d^*x = \int\limits_{C_K^1} \left(\sum_{a\in K} f(atx)\right) \chi(tx)d^*x.$$

Applying the Riemann-Roch theorem to the sum on the right-hand side and then using the change of variable $x \mapsto x^{-1}$, we obtain

$$\begin{aligned}
\int\limits_{C_K^1} \left(\sum_{a\in K} f(atx)\right) \chi(tx)d^*x &= \int\limits_{C_K^1} \left(\sum_{a\in K} \hat{f}(at^{-1}x^{-1})\right) \frac{\chi(tx)}{|tx|_{\mathbb{A}_K}}d^*x \\
&= \int\limits_{C_K^1} \left(\sum_{a\in K} \hat{f}(at^{-1}x)\right) |t^{-1}x|_{\mathbb{A}_K}\chi(tx^{-1})d^*x \\
&= \int\limits_{C_K^1} \left(\sum_{a\in K^*} \hat{f}(at^{-1}x)\right) \check{\chi}(x/t)d^*x + \hat{f}(0)\int\limits_{C_K^1} \check{\chi}(x/t)d^*x \\
&= Z_{t^{-1}}(\hat{f},\check{\chi}) + \hat{f}(0)\int\limits_{C_K^1} \check{\chi}(x/t)d^*x
\end{aligned}$$

since $\check{\chi} = \chi^{-1}|\cdot|$. This completes the proof of the functional equation of $Z_t(f,\chi)$. □

We may break up $Z(f,\chi)$ as follows:

$$Z(f,\chi) = \int\limits_0^1 Z_t(f,\chi)\frac{1}{t}dt + \int\limits_1^\infty Z_t(f,\chi)\frac{1}{t}dt.$$

We see that

$$\int_1^\infty Z_t(f,\chi)\frac{1}{t}dt = \int_{\{x\in\mathbb{I}_K : |x|_{\mathbb{A}_K}\geq 1\}} f(x)\chi(x)d^*x.$$

The integral on the right-hand side is normally convergent for $\sigma > 1$. However, for smaller σ and $|x|_{\mathbb{A}_K} > 1$, the convergence is better. Therefore, the integral is normally convergent for all $s \in \mathbb{C}$. We now will use the functional equation for $Z_t(f,\chi)$ to investigate the integral from 0 to 1:

$$\int_0^1 Z_t(f,\chi)\frac{1}{t}dt = \int_0^1 \left(Z_{t^{-1}}(\hat{f},\check{\chi}) + \hat{f}(0)\check{\chi}(t^{-1}) \int_{C_K^1} \check{\chi}(x)d^*x - f(0)\chi(t)\int_{C_K^1} \chi(x)d^*x \right)\frac{1}{t}dt.$$

Applying the change of variable $t \mapsto t^{-1}$ to the first integral in the sum, we obtain

$$\int_0^1 Z_{t^{-1}}(\hat{f},\check{\chi})\frac{1}{t}dt = \int_1^\infty Z_t(\hat{f},\check{\chi})\frac{1}{t}dt,$$

which is convergent for all σ by the argument above (recall that $\hat{f} \in S(\mathbb{A}_K)$). Now we are left to analyze

$$R(f,\chi) := \int_0^1 \hat{f}(0)\check{\chi}(t^{-1}) \int_{C_K^1} \check{\chi}(x)d^*x\frac{1}{t}dt - \int_0^1 f(0)\chi(t)\int_{C_K^1}\chi(x)d^*x\frac{1}{t}dt.$$

There are two cases to consider.

(i) If χ is nontrivial on $\mathbb{I}_K^1$, then

$$\int_{C_K^1} \check{\chi}(x)d^*x \quad \text{and} \quad \int_{C_K^1} \chi(x)d^*x$$

are both zero by orthogonality of characters ($R(f,\chi)=0$). Therefore,

$$\int_0^1 Z_t(f,\chi)\frac{1}{t}dt = \int_1^\infty Z_t(\hat{f},\check{\chi})\frac{1}{t}dt$$

, and hence

$$Z(f,\chi) = \int_1^\infty Z_t(f,\chi)\frac{1}{t}dt + \int_1^\infty Z_t(\hat{f},\check{\chi})\frac{1}{t}dt.$$

So, when χ is nontrivial on $\mathbb{I}_K^1$, then $Z(f,\chi)$ extends to an entire function.

(ii) If $\chi = \tilde{\chi}|\cdot|^s$ is trivial on $\mathbb{I}_K^1$, then $\chi = |\cdot|^{s'}$, where $s' = s - i\tau$, and

$$\begin{aligned} R(f,\chi) &= \hat{f}(0)\mathrm{Vol}(C_K^1)\int_0^1 t^{s'-2}dt - f(0)\mathrm{Vol}(C_K^1)\int_0^1 t^{s'-1}dt \\ &= \frac{\hat{f}(0)\mathrm{Vol}(C_K^1)}{s'-1} - \frac{f(0)\mathrm{Vol}(C_K^1)}{s'}. \end{aligned}$$

Consequently,

$$\int_0^1 Z_t(f,\chi)\frac{1}{t}dt = \int_1^\infty Z_t(\hat{f},\check{\chi})\frac{1}{t}dt + \frac{\hat{f}(0)\mathrm{Vol}(C_K^1)}{s'-1} - \frac{f(0)\mathrm{Vol}(C_K^1)}{s'}$$

, and hence

$$Z(f,\chi) = \int_1^\infty Z_t(f,\chi)\frac{1}{t}dt + \int_1^\infty Z_t(\hat{f},\check{\chi})\frac{1}{t}dt + \frac{\hat{f}(0)\mathrm{Vol}(C_K^1)}{s'-1} - \frac{f(0)\mathrm{Vol}(C_K^1)}{s'}.$$

If $\chi = \tilde{\chi}|\cdot|^s$ is trivial on $\mathbb{I}_K^1$ and $\tilde{\chi} \neq |\cdot|^{i\tau}$, then the global zeta function $Z(f,\chi)$ is holomorphic everywhere However, if $\tilde{\chi} = |\cdot|^{i\tau}$, then $Z(f,\chi)$ has simple poles at $s = i\tau$ and $s = 1 + i\tau$, with respective residues $-\mathrm{Vol}(C_K^1)$ and $\mathrm{Vol}(C_K^1)$.

In either case, we have

$$\begin{aligned} Z(f,\chi) &= \int_1^\infty Z_t(f,\chi)\frac{1}{t}dt + \int_1^\infty Z_t(\hat{f},\check{\chi})\frac{1}{t}dt + R(f,\chi) \\ &= \int_1^\infty \int_{\mathbb{I}_K^1} f(tx)\chi(tx)d^*x\frac{1}{t}dt + \int_1^\infty \int_{\mathbb{I}_K^1} \hat{f}(tx)\check{\chi}(tx)d^*x\frac{1}{t}dt + R(f,\chi). \end{aligned}$$

We have that $\hat{\hat{f}}(x) = f(-x)$, since dx is self-dual relative to ψ_K on $\mathbb{A}_K$. In addition, $\check{\check{\chi}} = \chi$ by definition. Applying these two facts, we obtain

$$\begin{aligned} Z(\hat{f},\check{\chi}) &= \int_1^\infty Z_t(\hat{f},\check{\chi})\frac{1}{t}dt + \int_1^\infty Z_t(\hat{\hat{f}},\check{\check{\chi}})\frac{1}{t}dt + R(\hat{f},\check{\chi}) \\ &= \int_1^\infty \int_{\mathbb{I}_K^1} \hat{f}(tx)\check{\chi}(tx)d^*x\frac{1}{t}dt + \int_1^\infty \int_{\mathbb{I}_K^1} f(-tx)\chi(tx)d^*x\frac{1}{t}dt + R(f,\chi). \end{aligned}$$

By inspection, $R(\hat{f}, \check{\chi}) = R(f, \chi)$. Furthermore, since χ is an idele-class character, and hence trivial on K^*, then $\chi(-tx) = \chi(tx)$. Finally, we have that

$$Z(f, \chi) = Z(\hat{f}, \check{\chi}).$$

$\square$

4.10 **Hecke L-Functions**

Let $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}_K/K^*, \mathbb{C}^\times)$ (an idele-class character), for a number field K. We saw in Proposition 4.8.1 that every χ can be written as $\tilde{\chi}|\cdot|^s_{\mathbb{A}_K}$, where $\tilde{\chi}$ is a unitary character and where $s \in \mathbb{C}$. Denote by σ the real part of s and call it the exponent of χ. Furthermore, by Proposition 3.1.5, we may, at each place ν of K, define a local character

$$\begin{aligned} \chi_\nu : K_\nu^* &\rightarrow \mathbb{C}^\times \\ t &\mapsto \chi(1, \ldots, 1, t, 1, \ldots, 1), \end{aligned}$$

where t is in the νth component. Then $\chi(y) = \prod_\nu \chi_\nu(y)$. Since the restriction of $\chi_\nu|_{\mathfrak{o}_\nu} = 1$ for all but finitely many places, then this product makes sense.

Definition 4.10.1. Let $L(\chi_\nu)$ be defined as in 4.1, 4.2, 4.3, We define the *global L-function* of χ in terms of its local versions by the product expansion

$$L(\chi) = \prod_\nu L(\chi_\nu),$$

whenever this is convergent.

Lemma 4.10.2. *$L(\chi)$ is absolutely convergent, nonzero, and holomorphic whenever the exponent $\sigma = \Re(s)$ of χ is greater than 1.*

Proof. $L(\chi)$ is nonzero because $L(\chi_\nu)$ is nonzero for all quasi-characters χ_ν. See Remark 4.1.6. Write $\chi = \tilde{\chi}|\cdot|^s$ with $\sigma = \Re s$. By definition, $L(\chi_\nu) = 1$ if ν is a finite place and χ_ν is ramified ($\chi_\nu|_{\mathfrak{o}_\nu \neq 1}$). Since $\chi_\nu|_{\mathfrak{o}_\nu} = 1$ for all but finitely many finite places, then χ_ν is unramified for almost all ν. In addition, there are only a finite number of non-Archimedean places, ν; $L(\chi_\nu)$ is holomorphic for all $\Re(s) > 0$ since they come from gamma functions.

Hence, we may ignore infinite places and those finite places for which χ_ν is ramified.

$$\prod_\nu |L(\chi_\nu)| = \prod_\nu \frac{1}{|1-\tilde{\chi}_\nu(\pi_\nu)|\pi_\nu|_\nu^s|} = \prod_\nu \frac{1}{|1-\tilde{\chi}_\nu(\pi_\nu)q_\nu^{-s}.|}$$

In order to show that the product is convergent for $\sigma > 1$, then we must show that the logarithm of the product converges for $\sigma > 1$. See [20], page 373, for an explanation of complex products. Taking the logarithm, we obtain

$$\log\left(\prod_\nu |L(\chi_\nu)|\right) = \sum_\nu \log\left(\frac{1}{|1-\tilde{\chi}_\nu(\pi_\nu)q_\nu^{-s}|}\right) = \Re\left(\sum_\nu \sum_{m>0} \frac{\tilde{\chi}_\nu(\pi_\nu)^m q_\nu^{-ms}}{m}\right).$$

Since $\tilde{\chi}$ is unitary (image in S^1) and since $|q_\nu^{-ms}| = |e^{-ms\log q_\nu}| = |e^{-m\sigma\log q_\nu}||e^{-im\Im(s)\log q_\nu}| = |e^{-m\sigma\log q_\nu}| = q_\nu^{-m\sigma}$, then it suffices to show the convergence of

$$\sum_\nu \sum_{m>0} \frac{q_\nu^{-m\sigma}}{m}.$$

Replicating the argument from the beginning of Theorem 4.9.2, we can establish the convergence of this infinite sum for $\sigma > 1$. $\square$

We will adopt the notation of [24] and define the Hecke L-function as follows.

Definition 4.10.3. Let $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{I}_K/K^*, \mathbb{C}^\times)$ (an idele-class character). For complex s, define the *Hecke L-function* $L(s,\chi)$ by

$$L(s,\chi) = L(\chi|\cdot|^s). \tag{4.20}$$

Let $\chi_f = \prod_{\nu \text{ finite}} \chi_\nu$ and $\chi_\infty = \prod_{\nu|\infty} \chi_\nu$, where χ_ν is the restriction of χ to the νth place. We define the finite and infinite versions of $L(\chi)$ by

$$L(s,\chi_f) = \prod_{\nu \text{ finite}} L(s,\chi_\nu)$$

and

$$L(s,\chi_\infty) = \prod_{\nu|\infty} L(s,\chi_\nu),$$

respectively.

Traditionally, $L(s,\chi_f)$ is denoted $L(s,\chi)$, and $L(s,\chi)$ is denoted $\Lambda(s,\chi)$. Let us consider the trivial idele-class character $\chi = 1$. Note that $\chi = 1$ belongs to the class of unramified idele-class characters $\{|\cdot|_{\mathbb{A}_K}^{-i\tau} : \tau \in \mathbb{R}\}$. Then

$$L(s, 1_f) = \prod_{\nu \text{ finite}} \frac{1}{1 - |\pi_\nu|^s} = \prod_{\nu \text{ finite}} \frac{1}{1 - N(\mathfrak{p}_\nu)^{-s}},$$

where $\mathfrak{p}_\nu$ is the unique prime associated to the completion of K at ν and N is the absolute norm. That is $N(\mathfrak{p}_\nu) = [\mathfrak{o}_K : \mathfrak{p}_\nu \mathfrak{o}_K] = [\mathfrak{o}_\nu : \mathfrak{p}_\nu \mathfrak{o}_\nu] = q_\nu$.

For an arbitrary number field K, $L(s, 1_f)$ is called the *Dedekind zeta function of* K and is denoted $\zeta_K(s)$. Let $x \in \mathbb{R}$ and $x > 0$. Applying the unique factorization of integral ideals of K into prime ideals (finite places of K) and multiplicity of the norm, we have

$$\prod_{N(\mathfrak{p}) \leq x} \left(1 - N(\mathfrak{p})^{-\sigma}\right)^{-1} = \prod_{N(\mathfrak{p}) \leq x} \left(\sum_{m=0}^{\infty} N(\mathfrak{p})^{-m\sigma}\right) \geq \sum_{N(\mathfrak{a}) \leq x} N(\mathfrak{a})^{-\sigma},$$

where the last sum is over nonzero integral ideals $\mathfrak{a}$ of K. Since the product on the left converges uniformly and absolutely for $\Re s = \sigma \geq 1 + \delta$ for any $\delta > 0$ by the above lemma, then $\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$ converges uniformly and absolutely for $\Re s = \sigma \geq 1 + \delta$ for any $\delta > 0$. In addition, we have

$$\left| \prod_{N(\mathfrak{p}) \leq x} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1} - \sum_{N(\mathfrak{a}) \leq x} N(\mathfrak{a})^{-s} \right| \leq \sum_{N(\mathfrak{a}) \geq x} N(\mathfrak{a})^{-\sigma}.$$

Since $\sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}$ is absolutely and normally convergent for $\sigma > 1$, then the right-hand side converges to zero, proving that

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}} = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s} \tag{4.21}$$

is absolutely and uniformly convergent for $\sigma > 1$. We will now prove the main theorem. If $K = \mathbb{Q}$, then, for $\Re(s) > 1$, we have that

$$L(s, 1_f) = \prod_p \frac{1}{1 - p^{-s}} = \sum_{n \geq 1} \frac{1}{n^s}$$

is the *Riemann zeta function*.

Theorem 4.10.4. *Let $\tilde{\chi}$ be a unitary idele-class character with factorization $\tilde{\chi} = \prod_\nu \tilde{\chi}_\nu$ Let $\psi = \prod_\nu \psi_\nu$ be a non-trivial adelic character that is trivial on K. Then $L(s, \tilde{\chi})$, which is a* *priori defined and holomorphic in $\{s \in \mathbb{C} : \Re(s) > 1\}$, admits a meromorphic continuation to the whole s-plane, and satisfies the functional equation*

$$L(1-s, \overline{\tilde{\chi}}) = \epsilon(s, \tilde{\chi}) L(s, \tilde{\chi})$$

where

$$\epsilon(s, \tilde{\chi}) = \prod_{\nu} \epsilon(\tilde{\chi_\nu} |\cdot|^s, \psi_\nu, dx_\nu) \in \mathbb{C}^\times,$$

for some choice of self-dual pair ($\psi = \prod_\nu \psi_\nu$ and $dx = \prod_\nu dx_\nu$.) The global epsilon factor is, in fact, independent of the this pair. Furthermore, $L(s, \tilde{\chi}_f)$, as defined above, admits a meromorphic continuation to the whole s-plane and satisfies the functional equation

$$L(1-s, \overline{\tilde{\chi}_f}) \prod_{\nu \text{ real}} \Gamma_{\mathbb{R}}(1-s+n_\nu) \prod_{\nu \text{ cplx}} \Gamma_{\mathbb{C}}(1-s+\frac{|n_\nu|}{2}) = \epsilon(s, \tilde{\chi}) \prod_{\nu \text{ real}} \Gamma_{\mathbb{R}}(s+n_\nu) \prod_{\nu \text{ cplx}} \Gamma_{\mathbb{C}}(s+\frac{|n_\nu|}{2}) L(s, \tilde{\chi}_f),$$

where n_ν is defined as follows:

(i) If ν is a finite place, then let $n_\nu > 0$ be the integer such that $\mathfrak{p}_\nu^{n_\nu}$ is the conductor of $\tilde{\chi}_\nu$.

(ii) If ν is a complex place, then let $n_\nu \in \mathbb{Z}$ be the integer such that $\tilde{\chi}_\nu : re^{-i\theta} \mapsto e^{-in\theta}$.

(iii) If ν is a real place, then let $n_\nu = 0$ if $\tilde{\chi}_\nu = 1$, and $n_\nu = 1$ if $\tilde{\chi}_\nu = \mathrm{sgn}$.

Let r_2 be the number of complex places of K (i.e. the number of non-conjugate complex embeddings of K). Let d_K be the global discriminant. The meromorphic continuation of $L(s, \chi)$ is entire unless χ is unramified – that is, $\chi = |\cdot|_{\mathbb{A}_K}^{-i\tau}$ for some $\tau \in \mathbb{R}$ – in which case there exists simple poles at $s = i\tau$ and $s = i + i\tau$, with residues $-(2\pi)^{-r_2}\mathrm{Vol}(C_K^1)$ and $(2\pi)^{-r_2}|d_K|^{1/2}\mathrm{Vol}(C_K^1)$, respectively.

Also, the meromorphic continuation of $L(s, \chi_f)$ is entire unless χ is unramified, in which case there exists a simple pole at $s = i + i\tau$ with residue $\mathrm{Vol}(C_K^1)$. Since $L(s, \chi_f) = L(s - i\tau, 1_f) = \zeta_K(s - i\tau)$, then we see that $\zeta_K(s)$ admits a meromorphic continuation to the entire s-plane with a simple pole at $s = 0$ and corresponding residue $\mathrm{Vol}(C_K^1)$, and satisfies the functional equation

$$\zeta_K(1-s)\Gamma_{\mathbb{R}}^{r_1}(1-s)\Gamma_{\mathbb{C}}^{r_2}(1-s) = |d_K|^{1/2-s}\, \Gamma_{\mathbb{R}}^{r_1}(s)\Gamma_{\mathbb{C}}^{r_2}(s)\zeta_K(s).$$

Proof. Let us fix the standard non-trivial adelic character ψ_K and the unique Haar measure dx that is self-dual relative to ψ_K. If we can show that $L(s, \tilde{\chi})$ is meromorphic everywhere, then the functional equation will follow. Indeed, let us choose a factorizable $f = \otimes_\nu f_\nu \in$

$S(\mathbb{A}_K)$ such that each $f_\nu \in S(K_\nu)$ and $f_\nu = \chi_{\mathfrak{o}_\nu}$ for all but finitely many places ν of K; these functions generate $S(\mathbb{A}_K)$. In Theorem 4.7.6, we proved that $\hat{f} = \otimes_\nu \hat{f}_\nu$ for such a function. Let d^*x be the multiplicative measure on $\mathbb{I}_K$ used in the proof of the global functional equation. Thus, $d^*x = \prod_\nu d^*x_\nu$, where $\mathrm{Vol}(\mathfrak{o}_\nu, d^*x_\nu) = N(\mathfrak{D}_\nu)^{-1/2} = 1$ for all but finitely many places ν of K. For every idele-class character

$$\chi := \tilde{\chi}|\cdot|^s_{\mathbb{A}_K},$$

we have $\chi = \prod_\nu \chi_\nu$, where

$$\chi_\nu = \tilde{\chi_\nu}|\cdot|^s_\nu$$

and $\chi_\nu|_{\mathfrak{o}_\nu} = 1$, hence $\tilde{\chi_\nu}|_{\mathfrak{o}_\nu} = 1$ for all but finitely many places of K (Proposition 3.1.5). As such, $f_\nu\chi_\nu$ and $\hat{f}_\nu\check{\chi}_\nu$ are characteristic functions of $\mathfrak{o}_\nu$ for all but finitely many finite places ν of K. Note that we also have $\hat{f}_\nu \in S(K_\nu)$ for all places ν of K. Therefore, by Proposition 3.1.9, we have that

$$Z(\hat{f}, \check{\chi}) = \int\limits_{\mathbb{I}_K} \hat{f}(x)\check{\chi}(x)d^*x = \prod_\nu \int\limits_{K^*} \hat{f}_\nu(x_\nu)\check{\chi}_\nu(x_\nu)d^*x_\nu = \prod_\nu Z(\hat{f}_\nu, \check{\chi}_\nu)$$

and

$$Z(f, \chi) = \prod_\nu Z(f_\nu, \chi_\nu).$$

The global functional equation (Theorem 4.9.2), in combination with the above product decomposition, yields

$$\prod_\nu Z(f_\nu, \chi_\nu) = \prod_\nu Z(\hat{f}_\nu, \check{\chi}_\nu) \quad \Leftrightarrow \quad 1 = \prod_\nu \frac{Z(\hat{f}_\nu, \check{\chi}_\nu)}{Z(f_\nu, \chi_\nu)}.$$

In order to apply the global functional equation, we needed that dx be the unique measure which is self-dual with respect to ψ_K. Explicitly, this is used in the proof of the global functional equation when applying the Riemann-Roch theorem. Applying the local functional equation (Theorem 4.5.3) we have that

$$1 = \prod_\nu \frac{\epsilon(\tilde{\chi_\nu}|\cdot|^s_\nu, \psi_\nu, dx_\nu)L(1-s, \tilde{\chi_\nu}^{-1})}{L(s, \tilde{\chi_\nu})} = \frac{\prod_\nu \epsilon(\tilde{\chi_\nu}|\cdot|^s_\nu, \psi_\nu, dx_\nu)L(1-s, \tilde{\chi}^{-1})}{L(s, \tilde{\chi})}$$

by the definitions of $L(s, \tilde{\chi})$ and $L(1-s, \tilde{\chi}^{-1})$. Note that $\tilde{\chi_\nu}^{-1} = \overline{\tilde{\chi_\nu}}$, since $\tilde{\chi_\nu}$ is unitary. Now, suppose we pick another non-trivial $\phi \in \widehat{\mathbb{A}_K/K}$. Since $\widehat{\mathbb{A}_K/K} \cong K$, then $\phi = \psi_{K,b}$ for some

$b \in K^*$. As such, $\phi = \prod_\nu \psi_{\nu,b}$, where ψ_ν is the standard non-trivial additive character on K_ν. Unlike the local functional equation, where we freely could choose an additive character and not worry about adjusting the measure to be self-dual with respect to that character, here we cannot do so. The measure dx' that is self-dual with respect to ϕ is precisely $\prod_\nu dx'_\nu$, where dx'_ν is the self-dual measure with respect to ψ_ν. The measure $|b|_\nu^{1/2} dx_\nu$ is the corresponding self-dual measure with respect to $\psi_{\nu,b}$. Indeed,

By Proposition 4.6.1, we have that

$$\begin{aligned}
\prod_\nu \epsilon(\tilde{\chi_\nu}|\cdot|_\nu^s, \psi_{\nu,b}, |b|_\nu^{1/2} dx_\nu) &= \prod_\nu |b|_\nu^{s-1/2} \tilde{\chi_\nu}(b) \epsilon(\tilde{\chi_\nu}|\cdot|_\nu^s, \psi_\nu, dx_\nu) \\
&= |b|_{\mathbb{A}_K}^{s-1/2} \tilde{\chi}(b) \prod_\nu \epsilon(\tilde{\chi_\nu}|\cdot|_\nu^s, \psi_\nu, dx_\nu) \\
&= \prod_\nu \epsilon(\tilde{\chi_\nu}|\cdot|_\nu^s, \psi_\nu, dx_\nu).
\end{aligned}$$

Set

$$\epsilon(s, \tilde{\chi}) = \prod_\nu \epsilon(\tilde{\chi_\nu}|\cdot|^s, \psi_\nu, dx_\nu).$$

Therefore, if $L(s, \tilde{\chi})$ is meromorphic as a function of s, then the requisite functional equation and meromorphic continuation of $L(s, \chi)$ holds. Because $Z(f, \chi) = Z(f, \tilde{\chi}|\cdot|_{\mathbb{A}_K}^s)$ is a meromorphic function in the whole s-plane (Theorem 4.9.2), if we find a function $f = \otimes_\nu f_\nu \in S(\mathbb{A}_K)$ with the property that

$$Z(f, \tilde{\chi}|\cdot|_{\mathbb{A}_K}^s) = h(s, \tilde{\chi}) L(s, \tilde{\chi}), \tag{4.22}$$

where h is a nonzero meromorphic function, then we will have that $L(s, \tilde{\chi})$ is a meromorphic function of s. Let ψ_ν be the additive characters induced by the adelic character ψ_K. Since we chose the standard non-trivial adelic character ψ_K, then $\psi_\nu = \psi_p(\mathrm{tr}_{K_\nu/\mathbb{Q}_p}(\cdot))$ for all finite ν with $\nu | p$. The conductor of ψ_ν is $\mathfrak{D}_\nu^{-1} = \pi_\nu^{-d_\nu} \mathfrak{o}_\nu$. If ν is a finite place, then let $n_\nu > 0$ be the integer such that $\mathfrak{p}_\nu^{n_\nu}$ is the conductor of $\tilde{\chi_\nu}$. If ν is a complex place, then let $n_\nu \in \mathbb{Z}$ be the integer such that $\tilde{\chi_\nu} : re^{-i\theta} \mapsto e^{-in\theta}$. If ν is a real place, then let $n_\nu = 0$ if $\tilde{\chi_\nu} = 1$ and $n_\nu = 1$ if $\tilde{\chi_\nu} = \mathrm{sgn}$. In Theorem 4.5.3, we picked a local function $f_\nu \in S(K_\nu)$ such that

$$Z_\nu(f_\nu, \tilde{\chi_\nu}|\cdot|_\nu^s) = h_\nu(f_\nu, \tilde{\chi_\nu}|\cdot|_\nu^s, \psi_\nu, dx_\nu) L(s, \tilde{\chi_\nu})$$

and

$$Z_\nu(\hat{f}_\nu, \overline{\tilde{\chi_\nu}}|\cdot|_\nu^{1-s}) = h_\nu(\hat{f}_\nu, \overline{\tilde{\chi_\nu}}|\cdot|_\nu^{1-s}, \psi_\nu, dx_\nu) L(1-s, \overline{\tilde{\chi_\nu}}),$$

where h_ν is an entire and everywhere nonzero function for all places ν of K . Note that f_ν and h_ν are dependent on χ_ν, ψ_ν, dx_ν, and, technically, d^*x_ν if one does not specify the multiplicative measure $d^*x_\nu = dx_\nu/|\cdot|_\nu$. In this case, the measures dx_ν and d^*x_ν correspond to the product decomposition of the chosen dx and d^*x on the adeles and ideles, respectively. That is, dx_ν is the unique Haar measure that is self-dual with respect to ψ_ν (i.e. $\hat{\hat{f}}_\nu(x) = f(-x)$) and satisfies $\mathrm{Vol}(\mathfrak{o}_\nu, dx) = N(\mathfrak{p}_\nu)^{-1/2} = q_\nu^{-d_\nu/2}$, where $\mathfrak{p}_\nu$ is the unique prime of K_ν and q_ν is the order of the residue field $\mathfrak{o}_\nu/\mathfrak{p}_\nu\mathfrak{o}_\nu$. Also, d^*x_ν is the unique measure such that $\mathrm{Vol}(\mathfrak{o}_\nu, d^*x) = N(\mathfrak{D}_K)^{-1/2} = q_\nu^{-d_\nu/2}$.

In the table below, we will list f_ν, $h(s, \tilde{\chi_\nu})$, and $h'(1-s, \tilde{\chi_\nu}^{-1})$ by local field (completion type), as computed in Theorem 4.5.3, relative to n_ν, ψ_ν, dx_ν, and d^*x_ν, induced from $\tilde{\chi_\nu}$, ψ_K, dx and d^*x, respectively:

K_ν	$f_\nu(x)$	$h_\nu(f_\nu, \tilde{\chi_\nu}|\cdot|_\nu^s, \psi_\nu, dx_\nu)$	$h_\nu(\hat{f}_\nu, \overline{\tilde{\chi_\nu}}|\cdot|_\nu^{1-s}, \psi_\nu, dx_\nu)$
$\mathbb{R}$	$\begin{cases} e^{-\pi x^2} & \text{if } n_\nu = 0 \\ xe^{-\pi x^2} & \text{if } n_\nu = 1 \end{cases}$	1	$\begin{cases} 1 & \text{if } n_\nu = 0 \\ i & \text{if } n_\nu = 1 \end{cases}$
$\mathbb{C}$	$\begin{cases} \frac{\overline{x}^{n_\nu} e^{-2\pi x\overline{x}}}{2\pi} & \text{for } n_\nu \geq 0 \\ \frac{x^{-n_\nu} e^{-2\pi x\overline{x}}}{2\pi} & \text{for } n_\nu < 0 \end{cases}$	1	$i^{\lvert n_\nu \rvert}$
n-A	$\begin{cases} \psi_\nu(x) & \text{if } x \in (\mathfrak{p}_\nu^{-d_\nu - n_\nu}) \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} q_\nu^{d_\nu(s-1/2)} & \text{if } n_\nu = 0 \\ q_\nu^{d_\nu(s-1/2)+n_\nu(s-1)} G'_\nu & \text{if } n_\nu > 0 \end{cases}$	$\begin{cases} 1 & \text{if } n_\nu = 0 \\ \tilde{\chi_\nu}(-1) & \text{if } n_\nu > 0, \end{cases}$

where

$$G'_\nu = \frac{q_\nu}{q_\nu - 1} \sum_{x \in U_\nu/U_{\nu,n}} \tilde{\chi_\nu}(x) \psi_\nu(\pi_\nu^{-d_\nu - n_\nu} x).$$

As we can see, specific test functions, $f_\nu \in S(K_\nu)$, were chosen for certain equivalence classes of quasi-characters, hence the dependence of f_ν on n_ν. Recall that $Z(f_\nu, \chi_\nu)$ is a function on the equivalence class of local quasi-characters. The test functions also were dependent on the additive character and measure chosen. For the Archimedean cases, the dependence of f_ν on the additive characters and measure is characterized by the property $\hat{f} = f$. Whereas, in the non-Archimedean case, the dependence is more apparent with a d_ν (the valuation of the conductor of the additive character ψ_ν) appearing in the definition of the function. While the function h_ν is dependent on f_ν, the gamma and epsilon factors are not (Lemma 4.5.4).

For the idele-class character χ defined by $\chi = \tilde{\chi}|\cdot|_{\mathbb{A}_K}^s$, we have $\chi_\nu|_{\mathfrak{o}_\nu} = 1$ for all but finitely many places of K, hence χ_ν is unramified ($n_\nu = 0$) for all but finitely many finite places of K. Since we have fixed ψ_K, the standard non-trivial adelic character, then $\psi_K = \otimes_\nu \psi_\nu$, where ψ_ν is the standard non-trivial additive character of K_ν. The conductor of ψ_ν is $\mathfrak{o}_\nu$ for all but finitely many finite places ν of K because the inverse different is trivial for all but finitely many finite places ν of K. Hence, $d_\nu = 0$ for all but finitely many finite places of K. Let S_0 denote the finite set of finite places of K for which χ_ν is ramified ($n > 0$). Let T denote the finite set of finite places of K for which $\mathfrak{D}_\nu^{-1} \neq \mathfrak{o}_\nu$ ($d_\nu \neq 0$). As such, we have $f_\nu = \mathbf{1}_{\mathfrak{o}_\nu}$ for all but finitely many finite places $\nu \in S_0 \cup T$ and $f_\nu \in S(K_\nu)$ for all places of K. Also, $h_\nu = 1$ for all places $\nu \notin S_0 \cup T$. Therefore, the function $f = \otimes_\nu f_\nu \in S(K)$ satisfies 4.22 with

$$h(s, \tilde{\chi}) = \prod_{\nu \in S_0 \cap T} q_\nu^{d_\nu(s-1/2)+n_\nu(s-1)} G'_\nu \cdot \prod_{\nu \in S_0 \cap T^c} q_\nu^{d_\nu(s-1/2)} G'_\nu \cdot \prod_{\nu \in S_0^c \cap T} q^{d_\nu(s-1/2)}. \tag{4.23}$$

Since h is a nonzero meromorphic function, then $L(s, \tilde{\chi})$ is meromorphic as a function of s.

In the table below, we will list $\epsilon(\tilde{\chi_\nu}|\cdot|_\nu^s, \psi_\nu, dx)$ by local field (completion type), as computed in Theorem 4.5.3, relative to n_ν, ψ_ν, dx_ν, and d^*x_ν induced from $\tilde{\chi_\nu}$, ψ_K, dx and d^*x, respectively:

K_ν	$\epsilon(\tilde{\chi_\nu}\|\cdot\|_\nu^s, \psi_\nu, dx)$
$\mathbb{R}$	$\begin{cases} 1 & \text{if } n_\nu = 0 \\ i & \text{if } n_\nu = 1 \end{cases}$
$\mathbb{C}$	$i^{\|n_\nu\|}$
n-A	$\begin{cases} q_\nu^{d_\nu(1/2-s)} = N(\mathfrak{D}_\nu)^{1/2-s} & \text{for } n_\nu = 0 \\ q_\nu^{d_\nu(1/2-s)} q_\nu^{-n_\nu s} G_\nu & \text{for } n_\nu > 0, \end{cases}$

where

$$G_\nu = \sum_{x \in U_\nu / U_{\nu,n}} \overline{\tilde{\chi}_\nu}(x) \psi_\nu(\pi_\nu^{-d_\nu - n_\nu} x).$$

Therefore,

$$\epsilon(s, \tilde{\chi}) = i_\infty \cdot \prod_{\nu \in S_0 \cap T} q_\nu^{d_\nu(1/2-s)} q_\nu^{-n_\nu s} G_\nu \cdot \prod_{\nu \in S_0 \cap T^c} q_\nu^{d_\nu(1/2-s)} G_\nu \cdot \prod_{\nu \in S_0^c \cap T} N(\mathfrak{D}_\nu)^{1/2-s}, \tag{4.24}$$

where

$$i_\infty = \prod_{\nu \text{ complex}} i^{|n_\nu|} \cdot \prod_{\nu \text{ real}} i^{4(1-n_\nu)+n_\nu}.$$

Hence,

$$L(1-s, \overline{\tilde{\chi}}) = \epsilon(s, \tilde{\chi}) L(s, \tilde{\chi}).$$

Separating the finite and infinite components of the L-series, we obtain

$$L(1-s, \overline{\tilde{\chi}_f}) L_(1-s, \overline{\tilde{\chi}_\infty}) = \epsilon(s, \tilde{\chi}) L_(s, \tilde{\chi}_f) L_(s, \tilde{\chi}_\infty),$$

where

$$L(s, \tilde{\chi}_\infty) = \prod_{\nu \text{ real}} \Gamma_{\mathbb{R}}(s + n_\nu) \cdot \prod_{\nu \text{ cplx}} \Gamma_{\mathbb{C}}(s + \frac{|n_\nu|}{2}) \tag{4.25}$$

and

$$L(1-s, \overline{\tilde{\chi}_\infty}) = \prod_{\nu \text{ real}} \Gamma_{\mathbb{R}}(1-s+n_\nu) \cdot \prod_{\nu \text{ cplx}} \Gamma_{\mathbb{C}}(1-s+\frac{|n_\nu|}{2}).$$

Consequently,

$$L(1-s, \overline{\tilde{\chi}_f}) \prod_{\nu \text{ real}} \Gamma_{\mathbb{R}}(1-s+n_\nu) \prod_{\nu \text{ cplx}} \Gamma_{\mathbb{C}}(1-s+\frac{|n_\nu|}{2}) = \epsilon(s, \tilde{\chi}) \prod_{\nu \text{ real}} \Gamma_{\mathbb{R}}(s+n_\nu) \prod_{\nu \text{ cplx}} \Gamma_{\mathbb{C}}(s+\frac{|n_\nu|}{2}) L(s, \tilde{\chi}_f).$$

The simple poles and residues of $L(s, \tilde{\chi})$ are determined completely by the simple poles and residues of $Z(f, \tilde{\chi}|\cdot|_{\mathbb{A}_K}^s)$ since h is nonzero. In Theorem 4.9.2, we determined that the poles of $Z(f, \tilde{\chi}|\cdot|_{\mathbb{A}_K}^s)$ are in the equivalence class of unramified characters, $\tilde{\chi} = |\cdot|_{\mathbb{A}_K}^{-i\tau}$, at $s = i\tau$ and $s = 1 + i\tau$. Recall that unramified means that $\tilde{\chi}|_{\mathbb{I}_K^1} = 1$. Since $\tilde{\chi} = |\cdot|_{\mathbb{A}_K}^{-i\tau}$, then $n_\nu = 0$ for all places ν of K. Consequently, we choose test functions:

$$f_\nu(x) = \psi_\nu(x) \cdot \mathbf{1}_{\mathfrak{p}_\nu^{-d_\nu}}(x) = \mathbf{1}_{\mathfrak{p}_\nu^{-d_\nu}}(x) \qquad \text{for } \nu \text{ finite,}$$

$$f_\nu(x) = e^{-\pi x^2} \qquad \text{for } \nu \text{ real,}$$

and

$$f_\nu(x) = (2\pi)^{-1} e^{-2\pi x\overline{x}} \qquad \text{for } \nu \text{ complex.}$$

Again, note that the local different is trivial for all but finitely many places ν, so $\otimes_\nu f_\nu \in S(\mathbb{A}_K)$. For all real and non-Archimedean places ν, we see that $f_\nu(0) = 1$. However, for complex places ν, we have $f_\nu(0) = (2\pi)^{-1}$. Let r_2 be the number of complex places (number of non-conjugate embeddings). Then $f(0) = \prod_\nu f_\nu(0) = (2\pi)^{-r_2}$ and the residue at $s = i\tau$ is $-(2\pi)^{-r_2}\mathrm{Vol}(C_K^1)$.

Let us compute $\hat{f}(0)$. By Lemma 4.5.8, we have that

$$\widehat{f_\nu} = \widehat{\mathbf{1}_{\mathfrak{p}_\nu^{-d_\nu}}} = \mathrm{Vol}(\mathfrak{p}^{-d_\nu}, dx)\mathbf{1}_{\mathfrak{o}_\nu} = |\mathfrak{p}^{-d_\nu}|_\nu \mathrm{Vol}(\mathfrak{o}_\nu, dx)\mathbf{1}_{\mathfrak{o}_\nu} = N(\mathfrak{p}^{-d_\nu})^{-1} N(\mathfrak{D}_\nu)^{-1/2}\mathbf{1}_{\mathfrak{o}_\nu} = N(\mathfrak{D}_\nu)^{1/2}\mathbf{1}_{\mathfrak{o}_\nu},$$

for finite places ν. Since $\hat{f}_\nu(0) = N(\mathfrak{D}_\nu)^{1/2}$ for all finite places ν and $\hat{f}_\nu(0) = (2\pi)^{-1}$ for all complex places ν, then

$$\hat{f}(0) = (2\pi)^{-r_2} \prod_{\nu \text{ finite}} N(\mathfrak{D}_\nu)^{1/2} = (2\pi)^{-r_2}|d_K|^{1/2}.$$

Therefore, the residue at $s = 1 + i\tau$ is equal to $(2\pi)^{-r_2}|d_K|^{1/2}\mathrm{Vol}(C_K^1)$.

Let T denote the finite set of finite places of K for which $\mathfrak{D}_\nu^{-1} \neq \mathfrak{o}_\nu$ ($d_\nu \neq 0$). Then

$$\epsilon(s, |\cdot|_{\mathbb{A}_K}^{-i\tau}) = \prod_{\nu\in T} \epsilon(s, |\cdot|_{\mathbb{A}_K}) = \prod_{\nu\in T} N(\mathfrak{D}_\nu)^{1/2-s+i\tau} = |d_K|^{1/2-s+i\tau}. \tag{4.26}$$

Therefore,

$$L(1-s, |\cdot|_{\mathbb{A}_K}^{i\tau}) = |d_K|^{1/2-s-i\tau} L(s, |\cdot|_{\mathbb{A}_K}^{-i\tau}). \tag{4.27}$$

We have that $L\left(s, |\cdot|_{\mathbb{A}_K}^{-i\tau}\right) = L(s - i\tau, 1)$, and the same relation holds for the finite and infinite parts of the L-series. Then by separating the L-function into finite and infinite parts, we obtain

$$L(1-s+i\tau, 1_f)\Gamma_{\mathbb{R}}^{r_1}(1-s+i\tau)\Gamma_{\mathbb{C}}^{r_2}(1-s+i\tau) = |d_K|^{1/2-s-i\tau}\Gamma_{\mathbb{R}}^{r_1}(s-i\tau)\Gamma_{\mathbb{C}}^{r_2}(s-i\tau)L(s-i\tau, 1_f).$$

Let $\tau = 0$ and replace $L(1-s+i\tau, 1_f)$ with $\zeta_K(s)$ in order to obtain the functional equation for the Dedekind zeta function. Since by equation 4.23, $h(s, |\cdot|_{\mathbb{A}_K}^{-i\tau}) = h(s - i\tau, 1) =$

$\prod_\nu q^{d_\nu(s-i\tau-1/2)} = |d_K|^{s-i\tau-1/2}$, then we have that

$$\begin{aligned} L(s-i\tau, 1_f) &= |d_K|^{1/2-s+i\tau} Z(f, |\cdot|_{\mathbb{A}_K}^{s-i\tau})(L(s-i\tau, 1_\infty))^{-1} \\ &= |d_K|^{1/2-s+i\tau} Z(f, |\cdot|_{\mathbb{A}_K}^{s-i\tau}) \Gamma_{\mathbb{R}}^{-r_1}(s-i\tau)\Gamma_{\mathbb{C}}^{-r_2}(s-i\tau) L(s-i\tau, 1_f). \end{aligned}$$

The gamma function has a simple pole at $s = 0$ and no zeroes, so $L(s-i\tau, 1_\infty)$ has no zeroes and a pole of order $r_1 + r_2$ at $s = i\tau$ (see 4.25). Being that $Z(f, |\cdot|_{\mathbb{A}_K}^{s-i\tau})$ has a simple pole at $i\tau$, then $L(s, \tilde{\chi}_f)$ is actually holomorphic at $s = 0$; the order of the poles of $L(s-i\tau, 1_\infty)$, which is necessarily greater than 1, always out number the simple pole of the global zeta function . Therefore, $L(s-i\tau, 1_f)$ is meromorphic with a only simple pole at $s = 1 + i\tau$. Evaluating the gamma functions at 1, we obtain $\Gamma_{\mathbb{R}}(1) = \pi^{-1/2}\Gamma(1/2) = 1$ and $\Gamma_{\mathbb{C}}(1) = (2\pi)^{-1}\Gamma(1) = (2\pi)^{-1}$, and hence $L(1, 1_\infty) = (2\pi)^{-r_2}$. Therefore,

$$\mathrm{Res}_{s=1+i\tau} L(s-i\tau, 1_f) = |d_K|^{-1/2}(2\pi)^{-r_2} \mathrm{Res}_{s=1+i\tau} Z(f, |\cdot|_{\mathbb{A}_K}^{s-i\tau}) \mathrm{Res}_{s=1+i\tau} = \mathrm{Vol}(C_K^1).$$

□

Proposition 4.10.5. *Let K be a global field. For any unitary idele-class character $\tilde{\chi} = (\tilde{\chi_\nu})$, put*

$$W(\chi) = \prod_\nu W(\chi_\nu).$$

Then $|W(\chi)| = 1$.

Proof. Let $\chi = \tilde{\chi}|\cdot|_{\mathbb{A}_K}^s$ (Proposition 4.8.1). Then $\chi_\nu = \tilde{\chi_\nu}|\cdot|_\nu^s$. Recall that $W(\chi_\nu) = \epsilon(\chi_\nu, \psi, dx)|_{s=1/2} = \epsilon(\tilde{\chi_\nu}|\cdot|_\nu^{1/2}, \psi, dx)$ and $|W(\tilde{\chi_\nu})| = 1$ by Proposition 4.6.3. Then

$$W(\chi)\overline{W(\chi)} = \prod_\nu W(\chi_\nu)\overline{W(\chi_\nu)} = 1.$$

□

4.11 The Volume of C_K^1 and the Regulator

Let K be a number field. As the section title suggests, our main goal is to compute $\mathrm{Vol}(C_K^1)$. Recall that the volume of C_K^1 is taken with respect to the quotient measure on C_K, defined by d^*x and the counting measure on K^*. In doing so we will obtain the residue of

$\zeta_K(s) = L(s, 1_f)$ at $s = 1$. Let us define $|\cdot|_\nu$ for all completions of K at ν, as in the beginning of the chapter. For a finite set S of places of K, let us define the set of S*-ideles of* K by

$$\mathbb{I}_{K,S} = \{x = (x_\nu) \in \mathbb{I}_K : |x_\nu|_\nu = 1, \ \forall \nu \notin S\}.$$

If $S = \emptyset$, then $\mathbb{I}_{K,S} \subseteq \mathbb{I}_K^1$. However, even if S is not the empty-set, then we define the norm-one version of the S-ideles by

$$\mathbb{I}_{K,S}^1 := \mathbb{I}_K^1 \cap I_{K,S}.$$

Note that $\mathbb{I}_{K,S}^1$ is a subgroup of $\mathbb{I}_K^1$. We have not made the requirement that S include the set of infinite places. We will follow Ramakrishnan and Valenza [24], Chapter 7, Section 4, and will find the volume of $C_K^1 = \mathbb{I}_K^1/K^*$ in three steps.

Step One. Let us assume that S is nonempty. We know that K^* is a subgroup of $\mathbb{I}_K^1$, but not necessarily $\mathbb{I}_{K,S}^1$. That is, for $k \in K^*$ we have $|(k, k, \ldots)|_{\mathbb{A}_K} = \prod_\nu |k|_\nu = 1$, but this doesn't imply that $|k|_\nu = 1$ for $\nu \in S$. However, we can consider $K^* \cap \mathbb{I}_{K,S}^1$, which is necessarily a subgroup of $\mathbb{I}_{K,S}^1$ and consists precisely the elements of K^* with $|k|_\nu = 1$ for all $\nu \notin S$. In this way, it makes sense to consider the quotient group $\mathbb{I}_{K,S}^1/K^* \cap \mathbb{I}_{K,S}^1$, which, by the second isomorphism theorem, is isomorphic to $\mathbb{I}_{K,S}^1 \cdot K^*/K^*$. Furthermore, $\mathbb{I}_{K,S}^1/K^* \cap \mathbb{I}_{K,S}^1$ is a subgroup of the norm-one idele-class group $C_K^1 = \mathbb{I}_K^1/K^*$. Consider the following projection map

$$\rho : C_K^1 = \mathbb{I}_K^1/K^* \longrightarrow \left(\mathbb{I}_K^1/K^*\right) / \left(\mathbb{I}_{K,S}^1 \cdot K^*/K^*\right).$$

Clearly, $\mathrm{Ker}\rho = \mathbb{I}_{K,S}^1 \cdot K^*/K^*$. By the third isomorphism theorem, we have

$$\left(\mathbb{I}_K^1/K^*\right) / \left(K^* \cdot \mathbb{I}_{K,S}^1/K^*\right) \cong \mathbb{I}_K^1/K^* \cdot \mathbb{I}_{K,S}^1,$$

which we denote by $C_{K,S}$. Summarizing, we obtain the short exact sequence of abelian groups:

$$1 \longrightarrow \mathbb{I}_{K,S}^1/K^* \cap \mathbb{I}_{K,S}^1 \xrightarrow{inc} C_K^1 \xrightarrow{\rho} C_{K,S} \longrightarrow 1.$$

Note that

$$I_{K,S} = \{x = (x_\nu) \in \mathbb{I}_K : x_\nu \in \mathfrak{o}_\nu^\times,\ \forall \nu \notin S\} = \prod_{\nu \in S} K_\nu^* \times \prod_{\nu \notin S} \mathfrak{o}_\nu^\times,$$

where $\mathfrak{o}_\nu^\times = \{\pm 1\}$ if ν is a real place, $\mathfrak{o}_\nu^\times = S^1$ if ν is a complex place, and $\mathfrak{o}_\nu^\times$ are the elements of norm 1 in the ring of integers if ν is a finite place. Hence, $I_{K,S}$ is open in $\mathbb{I}_K$ because the restricted direct product topology on $\mathbb{I}_{K,S}$, induced by $\mathbb{I}_K$, is the same as the product topology on $\prod_\nu K_\nu^*$. We proved this in Proposition 3.1.2. As such, $I^1_{K,S} = I_{K,S} \cap \mathbb{I}^1_K$ is an open subgroup of $\mathbb{I}^1_K$. In the adeles and ideles chapter, we proved that $\mathbb{I}^1_K/K^*$ is compact. Since $\mathbb{I}^1_K/K^*$ is compact and $I^1_{K,S}$ is open, then by Proposition 1.1.13, we have that $C_{K,S}$ is a finite group.

Let h_S denote the order of $C_{K,S}$. Therefore,

$$\mathrm{Vol}(C^1_K) = h_S \cdot \mathrm{Vol}(\mathbb{I}^1_{K,S}/K^* \cap \mathbb{I}^1_{K,S}). \tag{4.28}$$

We now are reduced to computing the volume of the second factor.

Step Two. Take $S = S_\infty$, the set of Archimedean places of K. Let r_1 be the number of real places. Let r_2 be the number of complex places (one half of the number of conjugate embeddings). Let $|\cdot|$ denote the usual complex absolute value, which restricts to the usual real absolute value. Define

$$\begin{aligned} \lambda \ \ : \ \ & \mathbb{I}_{K,S_\infty} \to \mathbb{R}^{r_1+r_2} \\ & (x_\nu) \mapsto (\log|x_\nu|)_{\nu \in S_\infty}. \end{aligned}$$

Then we have

$$\lambda\left((x_\nu \cdot y_\nu)_\nu\right) = \left(\log\left(|x_\nu \cdot y_\nu|\right)\right)_{\nu \in S_\infty} = \left(\log\left(|x_\nu|\right) + \log\left(|y_\nu|\right)\right)_{\nu \in S_\infty} = \lambda\left((x_\nu)\right) + \lambda\left((y_\nu)\right).$$

Therefore, λ is a homomorphism of groups. Since $\log(|\cdot|) : K_\nu^* \to \mathbb{R}$ is continuous for all $\nu \in S_\infty$, then λ is continuous. Let H denote a hyperplane in $\mathbb{R}^{r_1+r_2}$ given by

$$H := \{t = (t_\nu) \in \mathbb{R}^{r_1+r_2} : \sum_{\nu \text{real}} t_\nu + 2 \sum_{\nu \text{complex}} t_\nu = 0\}.$$

This construction is analogous to the Minkowski lattice theory used to prove Dirichlet's unit theorem. See Neukirch [23], Chapter 1, Sections 4, 5, and 7, for a proof of the Dirichlet's unit theorem.

Lemma 4.11.1. *The logarithm map has the following properties:*

(i) $\mathrm{Im}(\lambda) = H$

(ii) $\mathrm{Ker}(\lambda) = \mathbb{I}^1_{K,\emptyset} (= \mathbb{I}_{K,\emptyset})$.

Proof. (i) Let $x = (x_\nu) \in \mathbb{I}^1_{K,S_\infty}$. Since $\prod_\nu |x_\nu|_\nu = |x|_{\mathbb{A}_K} = 1$ and $|x_\nu|_\nu = 1$ for all $\nu \notin S_\infty$, then

$$\prod_{\nu \in S_\infty} |x_\nu|_\nu = 1 \quad \forall\, x = (x_\nu) \in \mathbb{I}^1_{K,S_\infty}.$$

Note that $|\cdot|_\nu$ is the square of the usual absolute value for ν complex and is equal to the usual absolute value for ν real. Let $x = (x_\nu) \in \mathbb{I}^1_{K,S_\infty}$. Then

$$\sum_{\nu \text{ real}} \log|x_\nu| + 2\sum_{\nu \text{ complex}} \log|x_\nu| = \sum_{\nu \text{ real}} \log|x_\nu| + \sum_{\nu \text{ complex}} \log|x_\nu|^2 = \log\left(\prod_{\nu \in S_\infty} |x_\nu|_\nu\right) = \log(1) = 0.$$

Therefore, $\mathrm{Im}(\lambda) \subseteq H$. Let $t = (t_\nu) \in H$. Then pick the idele x such that $x_\nu = 1$ for all ν finite and $x_\nu \in K_\nu$ such that $|x_\nu| = t_\nu$ for all $\nu \in S_\infty$. By construction, $\lambda(x) = t$. Consequently, $\mathrm{Im}(\lambda) = H$. This proves part (i).

(ii) Let $x = (x_\nu) \in \mathbb{I}^1_{K,\emptyset}$. Then $|x_\nu|_\nu = 1$ for all ν of K, which implies that $|x_\nu| = 1$ for all $\nu \in S_\infty$. Consequently, $\lambda(x) = (0, \ldots, 0)$, which implies that $\mathbb{I}^1_{K,\emptyset} \subseteq \mathrm{Ker}(\lambda)$. Let $x \in \mathrm{Ker}(\lambda)$. Then $\log|x_\nu| = 0$ for all ν, which implies that $|x_\nu|_\nu = 1$ for all $\nu \in S_\infty$. Since $x \in \mathbb{I}^1_{K,S_\infty}$, then $|x_\nu|_\nu = 1$ for all finite places ν. Therefore, $x \in \mathbb{I}^1_{K,\emptyset}$, which implies that $\mathrm{Ker}(\lambda) = \mathbb{I}^1_{K,\emptyset} (= \mathbb{I}_{K,\emptyset})$.

□

Let us define $R_S := K \cap \mathbb{A}_{K,S}$, the *ring of S-integers of K*, where

$$\mathbb{A}_{K,S} = \{x \in \mathbb{A}_K : x_\nu \in \mathfrak{o}_\nu,\ \forall\, \nu \notin S\}.$$

Then $R_\infty = K \cap \mathbb{A}_{K,S_\infty}$ consists of the elements that are in $\mathfrak{o}_\nu$ for all finite places ν. Since $\mathfrak{o}_K = \cap_{\nu \text{ finite}}\, \mathfrak{o}_\nu$, then $R_{S_\infty} = \mathfrak{o}_K$. The group of invertible elements of $A_{K,S}$ is clearly $\mathbb{I}_{K,S}$. In addition, by Proposition **??**, we know from that the elements of K^* have adelic norm 1.

Therefore,

$$R_S^\times = K^* \cap \mathbb{I}_{K,S} = K^* \cap \mathbb{I}^1_{K,S},$$

which implies that $\mathfrak{o}_K^\times = R_{S_\infty}^\times = K^* \cap \mathbb{I}^1_{K,S}$.

Definition 4.11.2. We will call the restriction of λ to $K^* \cap \mathbb{I}^1_{K,S_\infty} = \mathfrak{o}_K^\times$ the *regulator map,* and will denote it as $\mathrm{reg}(x)$.

The above lemma tells us that

$$\mathrm{Ker}(\mathrm{reg}) = K^* \cap \mathbb{I}^1_{K,\emptyset}.$$

Let $k \in K^* \cap \mathbb{I}^1_{K,\emptyset}$. Then $|k|_\nu = 1$ for all places ν of K. Therefore $K^* \cap \mathbb{I}^1_{K,\emptyset}$ is a subgroup of K^*, consisting of elements whose absolute values are bounded. It must therefore be a finite subgroup of K^*, and hence consists precisely of the group of roots of unity in K, denoted μ_K. That is,

$$\mathrm{Ker}(\mathrm{reg}) = \mu_K.$$

Let

$$w_K = \mathrm{Card}(\mu_K) \quad \text{and } L = \mathrm{reg}(\mathfrak{o}_K^\times).$$

As such, L is a discrete subgroup of H, which is isomorphic to $\mathbb{R}^r$, where $r = r_1 + r_2 - 1$. Since $\mathbb{I}_{K,S_\infty}$ is open in $\mathbb{I}^1_K$, then its image in $\mathbb{I}_{K,S_\infty}/K^* \cap \mathbb{I}^1_{K,S}$ is an open subgroup of the compact group $\mathbb{I}^1_K/K^*$. Since every open subgroup of a topological group is closed (Proposition 1.1.9), then $\mathbb{I}_{K,S_\infty}/K^* \cap \mathbb{I}^1_{K,S}$ is a closed subgroup of the compact group $\mathbb{I}^1_K/K^*$, which implies that $\mathbb{I}_{K,S_\infty}/K^* \cap \mathbb{I}^1_{K,S}$ is compact. Therefore, H/L is compact. In sum, L is a discrete and compact subgroup of H. This makes L a full lattice in H. See Neukirch [23], Chapter 1, Proposition 4.2 and Lemma 4.3, for proofs that a discrete subgroup is a lattice and that a lattice is full if and only if the quotient is compact .

Step Three. By definition, $\mathbb{I}_{K,\emptyset}$ admits the product decomposition

$$\prod_{\nu \text{ real}} \mathfrak{o}_\nu \times \prod_{\nu \text{ complex}} \mathfrak{o}_\nu \times \prod_{\nu \text{ finite}} \mathfrak{o}_\nu.$$

Let us construct the **product** Haar measure $d^\times x$ on $\mathbb{I}^1_{K,\emptyset}$ as follows:

(i) For ν real, we let $d^\times x_\nu$ be a clounting measure on $\mathfrak{o}_\nu^\times = \{\pm 1\}$.

(ii) For ν complex, we let $d^\times x_\nu$ be the Lebesgue measure on $\mathfrak{o}_\nu^\times = S^1$.

(iii) For ν finite, we let $d^\times x_\nu = d^* x_\nu$, the normalized measure on K_ν^*, such that $\mathrm{Vol}(\mathfrak{o}_\nu^\times, d^* x_\nu) = N(\mathfrak{D}_\nu)^{-1/2}$, where $\mathfrak{D}_\nu$ is the different of K_ν.

Then

$$\mathrm{Vol}(\mathfrak{o}_\nu) = \begin{cases} 2 & \text{for } \nu \text{ real} \\ 2\pi & \text{for } \nu \text{ complex} \\ N(\mathfrak{D})^{-1/2} & \text{for } \nu \text{ finite.} \end{cases}$$

The discriminant of K, denoted $\Delta_{K/\mathbb{Q}}$, is equal to $N_{K/\mathbb{Q}}(D_K)$. Let d_K be the integer defined up to a sign by $\Delta_K = d_K\mathbb{Z}$. We also have that $\mathfrak{D}_K = \prod_\nu \mathfrak{D}_\nu$ (See Neukirch [23], Chapter 3 Section 2). Therefore,

$$|d_K| = \prod_{\nu \text{ finite}} N(\mathfrak{D}_\nu).$$

As such, we get, relative to this measure,

$$\mathrm{Vol}(\mathbb{I}_{K,\emptyset}, d^\times x) = 2^{r_1}(2\pi)^{r_2}|d_K|^{1/2}. \tag{4.29}$$

Theorem 4.11.3. *Let K be a number field. Then*

$$\mathrm{Res}_{s=i\tau} Z(\hat{f},\chi) = -\frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|d_K|}} \quad \text{and} \quad \mathrm{Res}_{s=1+i\tau} Z(\hat{f},\chi) = \frac{2^{r_1} h_K R_K}{w_K}.$$

and

$$\mathrm{Res}_{s=1}\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|d_K|}},$$

where h_k is the class number of K and where R_K is the regulator of K, which is the volume of H/L relative quotient measure induced by the map λ_* defined below.

Proof. From the analysis done in step two, we have the following commutative diagram, all of whose columns and rows are exact:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 0 \arrow[d] & \\
1 \arrow[r] & \mu_K \arrow[r] \arrow[d] & \mathfrak{o}_K^\times \arrow[r, "\mathrm{reg}"] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathbb{I}^1_{K,\emptyset} \arrow[r] \arrow[d] & \mathbb{I}^1_{K,s_\infty} \arrow[r, "\lambda"] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \mathbb{I}^1_{K,\emptyset}/\mu_K \arrow[r] \arrow[d] & \mathbb{I}^1_{K,s_\infty}/\mathfrak{o}_K^\times \arrow[r, "\lambda_*"] \arrow[d] & H/L \arrow[r] \arrow[d] & 0 \\
 & 1 & 1 & 0 &
\end{tikzcd}$$

The regulator R_K of K is computed with respect to the quotient measure induced by both the measure on $\mathbb{I}^1_{K,\emptyset}$ established in step three and the standard measure on the idele group. From equations 4.28 and 4.29, we obtain $\mathrm{Vol}(C_K^1) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|d_K|}}$. The residues then follow from Theorem 4.9.2 and Theorem 4.10.4. $\square$

Conclusion

Recall that in section 8 of Chapter 4, we constructed idele-class characters

$$\chi : \mathbb{I}_K/K^* \to \mathbb{C}^\times.$$

Later in that chapter, in section 10, we saw that χ can be factored into a product of local quasi-characters $\chi_\nu : K_\nu^* \to \mathbb{C}^\times$. Let us now change our notation to

$$K_\nu^* = \mathrm{GL}_1(K_\nu), \qquad \mathbb{I}_K = \mathrm{GL}_1(\mathbb{A}_K), \quad \text{and} \quad K^* = \mathrm{GL}_1(K).$$

With this difference in notation, we can see that the idele-class character

$$\chi : \mathrm{GL}_1(K)\backslash\mathrm{GL}_1(\mathbb{A}_K) \to \mathbb{C}^\times$$

factors into a product of local quasi-characters $\otimes_\nu \chi_\nu$, where $\chi_\nu : \mathrm{GL}_1(K_\nu) \to \mathbb{C}^\times$. For $K = \mathbb{Q}$, we know there are many classical Dirichlet characters that are associated to a single idele-class character χ, but only one of which is primitive. In Kudla's article [17], "From Modular Forms to Automorphic Representations", the author sketches the passage of holomorphic modular forms f of weight k and level N to automorphic representations $\pi = \pi(f)$ of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Like the idele-class characters, the automorphic representations have the factorization $\pi = \otimes_\nu \pi_\nu$ into representations of the groups $\mathrm{GL}_2(\mathbb{Q}_p)$. Additionally, there will be many modular forms associated to π , but there will be only one primitive form. The major difference between the two approaches is that the representation π and local components π_ν are infinite dimensional and involve nonabelian harmonic analysis. Kudla's article provides a comprehensive, but brief, introduction to the Langland correspondence and the Langland L-function; it is highly recommended in concert with this work.

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