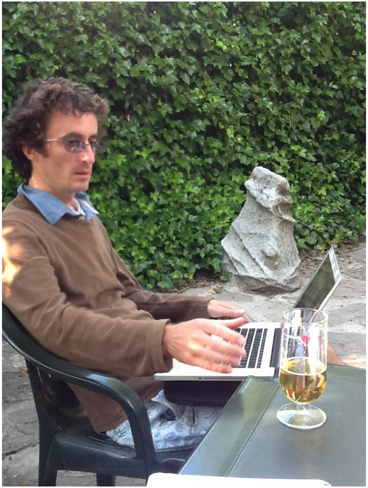
Stark's Conjecture and related topics

p-adic iterated integrals, modular forms of weight one, and Stark-Heegner points.

Henri Darmon

San Diego, September 20-22, 2013

(Joint with Alan Lauder and Victor Rotger)



Stark's conjectures

Stark's conjectures give complex analytic formulae for units in number fields (more precisely, for their *logarithms*) in terms of leading terms of Artin L-functions at $s = 0$.

Are there similar formulae for *algebraic points on elliptic curves*?

Heegner points, whose heights are related to L-series via the Gross-Zagier formula, are analogous to circular or elliptic units.

We refer to conjectural extensions of these as "Stark-Heegner points" because they would simultaneously generalise Stark units and Heegner points.

Modular forms of weight one

Let $g = \sum a_n(g)q^n$ be a cusp form of weight one, level N, and (odd) character χ.

Deligne-Serre. There is an odd two-dimensional Artin representation

$$\rho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

attached to g, satisfying

$$L(\rho_g, s) = L(g, s) = (2\pi)^2 \Gamma(s)^{-1} \int_0^{i\infty} y^s g(iy) \frac{dy}{y}.$$

From Artin representations to weight one forms

Buzzard-Taylor, Khare-Wintenberger. Conversely, if ρ is an odd, irreducible two-dimensional Artin representation, there is a weight one newform g satisfying

$$L(\rho, s) = L(g, s).$$

Odd two-dimensional Artin representations are therefore an ideal testing ground for Stark's conjectures.

Stark units attached to forms of weight one

Let $H_g :=$ the field cut out by the Artin representation ρ_g;

$L \subset \mathbb{Q}(\zeta_n) :=$ field generated by the fourier coefficients of g;

$V_g :=$ the L-vector space underlying ρ_g.

Conjecture (Stark). *Let g be a cuspidal newform of weight one, with Fourier coefficients in L. Then there is a unit $u_g \in (\mathcal{O}_{H_g}^\times \otimes L)^{\sigma_\infty = 1}$ satisfying*

$$L'(g, 0) \left(= \int_0^\infty g(iy) \frac{dy}{y} \right) = \log u_g.$$

A real quadratic example

$K = \mathbb{Q}(\sqrt{5})$.

The prime $29 = \lambda\bar{\lambda} = \left(\frac{11-\sqrt{5}}{2}\right)\left(\frac{11+\sqrt{5}}{2}\right)$ splits in K.

$\psi_g =$ character of K of order 4 and conductor $\lambda\infty_1$.

Inducing ψ_g from K to $\mathbb{Q}$ yields an odd, irreducible representation ρ_g which *cannot be obtained* as the induced representation from an imaginary quadratic field.

It corresponds to a cusp form

$$g \in S_1(5 \cdot 29, \chi), \quad \chi^4 = 1, \qquad \mathrm{cond}(\chi) = 5 \cdot 29.$$

Stark's calculation

$$\begin{aligned} \frac{1}{2}\int_0^\infty (g+\bar{g})(iy)\frac{dy}{y} &= 1.65074962913147\cdots \\ \log(u) &= 1.65074962913158\cdots, \end{aligned}$$

where

$$u = \frac{(3+2\sqrt{5}) + \sqrt{7+2\sqrt{5}} + \sqrt{(20+14\sqrt{5}) + (6+4\sqrt{5})\sqrt{7+2\sqrt{5}}}}{4}.$$

Classification of odd two-dimensional Artin representations

By projective image, in order of increasing arithmetic complexity:

A. Reducible representations (sums of Dirichlet characters).

B. Dihedral, induced from an imaginary quadratic field.

C. Dihedral, induced from a real quadratic field.

D. Tetrahedral case: projective image A_4.

E. Octahedral case: projective image S_4.

F. Icosahedral case: projective image A_5.

The status of Stark's conjecture

A. In the reducible case, it follows from the theory of circular units and Dirichlet's class number formula.

B. In the imaginary dihedral case, it follows from the theory of elliptic units and from *Kronecker's limit formula* (as Stark observes).

C. Stark has numerically verified many real dihedral cases.

The "exotic" (tetrahedral, octahedral and icosahedral) cases appear to have been relatively less studied, even numerically.

Stark-Heegner points

Let E be an elliptic curve attached to $f \in S_2(\Gamma_0(N))$.

To extend Stark's conjecture to elliptic curves, it is natural to replace Artin L-series by Hasse-Weil-Artin L-series

$$L(E, \rho_g, s) = L(f \otimes g, s).$$

Remark. The BSD conjecture leads us to expect that the leading terms of $L(E, \rho_g, s)$ ought to encode the *Néron-Tate heights* of global points on E, and not their logarithms, which in any case are not numbers at all but elements of $\mathbb{C}/\Lambda$.

p-adic methods

Motivation. This issue does not arise in a p-adic setting: the p-adic logarithms of global points are well-defined p-adic numbers.

In fact, p-adic logarithms of global points *do* arise as leading terms of p-adic L-series attached to elliptic curves:

a) The Katz p-adic L-function (Rubin, 1992);

b) The Mazur-Swinnerton Dyer p-adic L-function (Perrin-Riou, 1993);

c) Various types of p-adic Rankin L-functions attached to $f \otimes \theta_\psi$ (Bertolini-D, 1995; Bertolini-D-Prasanna, 2008);

d) p-adic Garrett-Rankin L-functions attached to $f \otimes g \otimes h$ (D-Rotger, 2012).

p-adic iterated integrals

D, Rotger. The leading terms of p-adic Garrett-Rankin L-functions can be expressed in terms of certain explicit analytic expressions, referred to as *"p-adic iterated integrals"*.

These iterated integrals are attached to a triple (f, g, h) of newforms of weights $(2, k, k)$, $k \geq 1$.

Their definition is based on the theory of p-adic and overconvergent modular forms.

p-adic and overconvergent modular forms

Let χ be a Dirichlet character of modulus N prime to p.

$M_k(Np, \chi)$ the space of *classical* modular forms of weight k, level Np and character χ;

$M_k^{(p)}(N, \chi)$ the corresponding space of p-adic modular forms;

$M_k^{\mathrm{oc}}(N, \chi)$ the subspace of *overconvergent* modular forms.

The latter is a p-adic Banach space, on which the Atkin U_p operator acts *completely continuously*.

$$M_k(Np, \chi) \subset M_k^{\mathrm{oc}}(N, \chi) \subset M_k^{(p)}(N, \chi).$$

Coleman's classicality theorem. If h is overconvergent and ordinary (slope zero) of weight ≥ 2, then h is classical.

The d operator

Let $d = q\frac{d}{dq}$ be the Atkin-Serre d operator on p-adic modular forms.

$$d^j(\sum_n a_n q^n) = \begin{cases} \sum_n n^j a_n q^n & \text{if } j \geq 0; \\ \sum_{p \nmid n} n^j a_n q^n & \text{if } j < 0. \end{cases}$$

- If $f \in M_2^{\mathrm{oc}}(N)$, then

$$F := d^{-1} f \in M_0^{\mathrm{oc}}(N).$$

- If h belongs to $M_k(Np, \chi)$, then

$$F \times h \in M_k^{\mathrm{oc}}(N, \chi), \qquad e_{\mathrm{ord}}(F \times h) \in M_k(Np, \chi) \otimes \mathbb{C}_p,$$

where $e_{\mathrm{ord}} := \lim_n U_p^{n!}$ is Hida's *ordinary projector*.

p-adic iterated integrals: definition

Suppose

$$f \in S_2(N), \qquad \gamma \in M_k(Np, \chi)^\vee, \quad h \in M_k(N, \chi).$$

Definition

The *p-adic iterated integral* of f and h along γ is

$$\int_\gamma f \cdot h := \gamma(e_{\mathrm{ord}}(F \times h)) \in \mathbb{C}_p.$$

The terminology is motivated from the case $k = 2$, where f and h correspond to differentials on a modular curve.

Remark: They differ from those that arise in Chen's theory and Coleman's p-adic extension, where one focusses on integrands that are "path independent".

Lauder's "fast ordinary projection" algorithm

- Given an overconvergent form, represented as a truncated q-series $g = \sum_{n=1}^{N} a_n q^n$, the calculation of

$$e_{\mathrm{ord}}(g) \pmod{p^M}$$

typically requires (in favorable circumstances) applying U_p to g roughly M times.

- But the first N fourier coefficients of $U_p^M g$ depend on knowing the first Np^M fourier coefficients of g: so this naive algorithm runs in "exponential time" in the desired p-adic accuracy.

- Alan Lauder's fast "ordinary projection" algorithm calculates the ordinary projection in "polynomial time".

- Our experiments rely crucially on this powerful tool.

The set-up

$f \in S_2(N)$ corresponds to an elliptic curve E;

$g \in M_1(N, \chi^{-1})$, $\quad h \in M_1(N, \chi)$ classical weight one eigenforms;

$V_{gh} := V_g \otimes V_h$, a 4-dimensional self-dual Artin representation,

H_{gh} the field cut out by it.

Let $g_\alpha \in M_1(Np, \chi^{-1})$ be an ordinary p-stabilisation of g attached to a root α_g of the Hecke polynomial

$$x^2 - a_p(g)x + \chi^{-1}(p) = (x - \alpha_g)(x - \beta_g).$$

Assume that $\gamma = \gamma_{g_\alpha}$ has the same system of Hecke eigenvalues as g_α,

$$\gamma_{g_\alpha} \in M_k(Np, \chi)^\vee[g_\alpha]$$

The question

Give an arithmetic interpretation for

$$\int_{\gamma_{g_\alpha}} f \cdot h, \qquad \text{as } \gamma_{g_\alpha} \in M_1(Np, \chi)^\vee[g_\alpha],$$

in terms of the arithmetic of E over the field H_{gh}.

Some assumptions

I. Certain local signs in the functional equation for $L(E, V_{gh}, s)$ are all 1. In particular, $L(E, V_{gh}, s)$ vanishes to *even order* at $s = 1$.

II. The self-dual representation V_{gh} breaks up as

$$V_{gh} = V_1 \oplus V_2 \oplus W, \qquad \text{and}$$

$$\mathrm{ord}_{s=1} L(E, V_1, s) = \mathrm{ord}_{s=1} L(E, V_2, s) = 1, \qquad L(E, W, 1) \neq 0.$$

The BSD conjecture then predicts that V_1 and V_2 occur in $E(H_{gh}) \otimes L$ with *multiplicity one*.

III. The frobenius σ_p at p acting on V_1 (resp V_2) has the eigenvalue $\alpha_g \alpha_h$ (resp. $\alpha_g \beta_h$).

IV. (Not essential) The eigenvalues $(\alpha_g \alpha_h, \alpha_g \beta_h)$ do not arise in (V_2, V_1) at the same time, when $V_1 \neq V_2$.

The conjecture

Stark-Heegner Conjecture (D-Lauder-Rotger)

Under the above assumptions,

$$\int_{\gamma_{g_\alpha}} f \cdot h = \frac{\log_{E,p}(P_1)\log_{E,p}(P_2)}{\log_p u_{g_\alpha}}, \qquad \textit{where}$$

- $P_j \in V_j$-isotypic component of $E(H_{gh}) \otimes L$, and

$$\sigma_p P_1 = \alpha_g \alpha_h \cdot P_1, \qquad \sigma_p P_2 = \alpha_g \beta_h \cdot P_2;$$

- u_{g_α} = *Stark unit* in $Ad^0(V_g)$-isotypic part of $(\mathcal{O}_{H_g}^\times) \otimes L$;

$$\sigma_p u_{g_\alpha} = \frac{\alpha_g}{\beta_g} \cdot u_{g_\alpha}.$$

Remarks about the Stark-Heegner conjecture

• The RHS of this conjecture belongs to $L \otimes \mathbb{Q}_p$, because

$$(\alpha_g \alpha_h)(\alpha_g \beta_h) = \alpha_g^2 \chi(p) = \alpha_g / \beta_g.$$

• The term $\log_p(u_{g_\alpha})$ that appears in the denominator can be viewed as a p-adic avatar of $\langle g_\alpha, g_\alpha \rangle$, and is defined over the field cut out by the *adjoint* of V_g. In particular it depends only on the projective representation attached to g.

• The unit u_{g_α} is closely related to the Stark units that will come up in Bill Duke's lecture tomorrow.

Theoretical evidence

Theorem (D, Lauder, Rotger)

If g and h are theta series attached to the same *imaginary quadratic field K, and the prime p splits in K, then the Stark-Heegner conjecture holds.*

- The points P_1 and P_2 are expressed in terms of Heegner points;
- the unit u_{g_α} in terms of *elliptic units*.

The ingredients in the proof

1. The relation described in D, Rotger between $\int_{\gamma_{g_\alpha}} f \cdot h$ and the Garrett-Rankin L-function $L_p(f \otimes g \otimes h)$.

2. When $g = \theta_{\psi_g}$ and $h = \theta_{\psi_h}$ are theta series, a factorisation

$$L_p(f \otimes \theta_{\psi_g} \otimes \theta_{\psi_h}) = L_p(f \otimes \theta_{\psi_1}) L_p(f \otimes \theta_{\psi_2}) \times \eta^{-1},$$

$\psi_1 = \psi_g \psi_h, \quad \psi_2 = \psi_g \psi_h', \quad \eta =$ *ratio of periods.*

3. The p-adic Gross-Zagier formula of Bertolini, D, Prasanna, relating the appropriate values of $L_p(f \otimes \theta_{\psi_j})$ to Heegner points over ring class fields of K.

4. The period ratio η can be interpreted as a value of the Katz p-adic L-function for K; the Stark unit u_{g_α} of the denominator arises from Katz's p-adic variant of the Kronecker limit formula.

Remarks about the proof

The assumption that p splits in K is used crucially;

- in Katz's p-adic Kronecker limit formula;
- in the p-adic Gross-Zagier formula of Bertolini-D-Prasanna for $L_p(f \otimes \theta_\psi)$, which is based on a similar circle of ideas.

("The CM points need to lie on the p-ordinary locus of the modular curve".)

However, the conjecture on p-adic iterated integrals still makes sense when p is inert in K.

Although many of many of our tools of our proof break down, Heegner points are still available, making this setting specially tantalising.

Examples, examples!

B. Gross, in a letter to B. Birch, 1982:

> *"The fun of the subject seems to me to be in the examples."*

An imaginary dihedral example, p inert

$K = \mathbb{Q}(\sqrt{-83})$ a quadratic imaginary field of class number3.

$H =$ be the Hilbert class field of K.

$g =$ cusp form attached to the cubic character of $\mathrm{cl}(K)$:

$$g \in S_1(83, \chi_K), \qquad h = E(1, \chi_K) \in M_1(83, \chi_K).$$

We considered the p-adic iterated integrals attached to:

$$\begin{aligned} (f, g, g): \quad V_{gg} &= \mathbb{Q} \oplus \mathbb{Q}(\chi_K) \oplus V_g, \\ (f, g, h): \quad V_{gh} &= V_g \oplus V_g. \end{aligned}$$

Heegner points

Let $f \in S_2(83)$ be the newform attached to

$$E = 83A : y^2 + xy + y = x^3 + x^2 + x.$$

The curve E has rank 1 over $\mathbb{Q}$, and rank 3 over H.

$E(H) \otimes \mathbb{Q}$ is generated by three Heegner points P_1, P_2, P_3 which are permuted by $\mathrm{Gal}(H/K)$ and whose x coordinates satisfy $x^3 - x^2 + x - 2 = 0$.

Embed $H \longrightarrow \mathbb{Q}_{5^2}$ so that $P_1 \in E(\mathbb{Q}_5), \quad P_2, P_3 \in E(\mathbb{Q}_{5^2})$.

$$\begin{aligned} P &:= P_1 + P_2 + P_3 = (1, -3) \in E(\mathbb{Q}), \\ Q^+ &:= 2P_1 - (P_2 + P_3) \in E(H)_0^{\sigma_5 = 1}, \\ Q^- &:= P_2 - P_3 \in E(H)_0^{\sigma_5 = -1}. \end{aligned}$$

The experiment

The prime $p = 5$ is inert in K.

We calculated that

$$\begin{aligned}
\int_{\gamma_{g_+}} f \cdot g = \int_{\gamma_{g_-}} f \cdot g &= \frac{16 \log_{E,5}(P) \log_{E,5}(Q^-)}{5 \log_5(u)} \\
\int_{\gamma_{g_+}} f \cdot h = \int_{\gamma_{g_-}} f \cdot h &= \frac{16 \log_{E,5}(Q^+) \log_{E,5}(Q^-)}{5 \log_5(u)},
\end{aligned}$$

modulo 5^{70}, in agreement with the conjectures.

A real dihedral example

Stark's calculation, from an earlier slide: $K = \mathbb{Q}(\sqrt{5})$, $29 = \lambda\lambda'$

$\psi_g = \psi_h^{-1} =$ quartic character of conductor $\lambda\infty_1$.

$$g \in S_1(145, \chi^{-1})_L, \qquad h \in S_1(145, \chi)_L, \qquad L = \mathbb{Q}(i).$$

$$V_{gh} = L \oplus L[\chi_5] \oplus V_{\psi_-}, \quad \psi_- := \psi_g/\psi_g'.$$

The character ψ_- cuts out the quartic subfield H of the narrow ring class field of K of conductor 29.

$$H = \mathbb{Q}(\sqrt{5}, \sqrt{29}, \sqrt{\delta}), \quad \delta = \frac{-29 + 3\sqrt{29}}{2}.$$

The conjecture in this case involves points on elliptic curves defined over H, and in the minus part for $\mathrm{Gal}(H/\mathbb{Q}(\sqrt{5}, \sqrt{29}))$.

A real dihedral example, cont'd

$$E = 17A : y^2 + xy + y = x^3 - x^2 - x - 14.$$

$$E(\mathbb{Q})_L = 0, \quad E(K)_L = L \cdot P, \quad P = \left(\frac{392}{20}, \frac{-1995 + 7218\sqrt{5}}{200} \right).$$

$E(H)$ is generated by the Galois conjugates of

$$\begin{aligned} Q &= \left(\frac{-220777 - 17703\sqrt{145}}{5800}, \frac{214977 + 17703\sqrt{145}}{11600} \right. \\ &\quad \left. + \frac{28584525 + 3803103\sqrt{5} + 1645605\sqrt{29} + 2364771\sqrt{145}}{290000}\sqrt{\delta} \right). \end{aligned}$$

To calculate the point Q, we used the theory of Stark-Heegner points arising from "integration on $\mathcal{H}_{17} \times \mathcal{H}$".

A real quadratic example, cont'd

A calculation reveals that

$$\int_{\gamma_\pm} f \cdot h = \frac{\log_{E,17}(P)\log_{E,17}(Q)}{3 \cdot 17 \cdot \log_{17}((1+\sqrt{5})/2)}$$

to an accuracy of 16 significant 17-adic digits.

Remark. Unlike what often happens with numerical verifications of Stark's conjectures, both the left and right-hand sides of this expression are *explicit* 17*-adic analytic expressions*. It might therefore not be out of reach to prove the resulting equality, and it would be of great interest to do so.

An abelian example

$\chi_1 =$ cubic Dirichlet character of conductor 19,

$\chi_2 =$ quadratic Dirichlet character of conductor 3;

$\chi = \chi_1\chi_2$. $\qquad L = \mathbb{Q}(\zeta_3)$.

Eisenstein series: $g = E(1, \chi^{-1})$, $\qquad h = E(\chi_1, \chi_2)$,

$$V_{gh} = L[\chi_1] \oplus L[\chi_1^{-1}] \oplus L[\chi_2] \oplus L[\chi_2^{-1}].$$

Remark: The Artin representation $L(\chi_1)$ is not self-dual.

Goal: Use the conjecture to construct points defined over the cubic subfield H of $\mathbb{Q}(\zeta_{19})$.

An abelian example, cont'd

$$E = 42a : y^2 + xy + y = x^3 + x^2 - 4x + 5$$

$$E(\mathbb{Q})_L = 0, \qquad E(K_{\chi_2})_L = 0, \qquad E(H)_L = L \cdot P_{\chi_1} \oplus L \cdot P_{\bar{\chi}_1},$$

$$H = \mathbb{Q}(\alpha), \quad \text{where} \quad \alpha^3 - \alpha^2 - 6\alpha + 7 = 0.$$

$$P = (64\alpha^2 + 80\alpha - 195, 1104\alpha^2 + 1424\alpha - 3391).$$

An abelian example, cont'd

We take $p = 7$, and find:

$$\int_\gamma f.h = \frac{64}{7 \cdot 9} \frac{\log_{E,7}(P_{\chi_1}) \log_{E,7}(P_{\bar{\chi}_1})}{\log_7(u_\chi) + \log_7(u_{\bar{\chi}})} \pmod{7^{35}},$$

where u_χ is a *Gross-Stark* unit (a 7 unit) attached to the *odd* sextic character χ,for which $\chi(7) = 1$.

The presence of p-units (Gross-Stark units) rather than genuine units in the denominator of the formula is specific to settings where g is an Eisenstein series.

Stark's retirement gift

Stark's "Class fields and modular forms of weight one" concludes: *"A meaningful numerical verification for $N = 133$ would be of some interest."*

This level is one of the smallest where an exotic form (with projective image A_4) arises.

Stark is alluding to his conjectures on $L'(g, 0)$, but the comment applies equally well to the Stark-Heegner conjectures!

Stark's retirement gift: A Stark-Heegner point attached to the exotic A_4 form of level 133.

The last week of August, in Benasque, Spain

A tetrahedral example

χ= sextic Dirichlet character of conductor 133;

$g \in S_1(133, \chi) :=$ the unique A_4 form.

$h = \bar{g}$. (Note that $L = \mathbb{Q}(\zeta_{12})$).

$$V_{gh} = L \oplus W_g, \qquad W_g := \mathrm{Ad}^0(V_g).$$

The A_4 extension cut out by W_g is the normal closure of $\mathbb{Q}(a)$,

$$a^4 + 3a^2 - 7a + 4 = 0.$$

The Stark-Heegner conjecture involves points of elliptic curves defined over this field.

A tetrahedral example, cont'd

Let $f \in S_2(91)$ be the eight two cusp form attached to

$$E = 91B : y^2 + y = x^3 + x^2 - 7x + 5.$$

$$E(\mathbb{Q})_L = L \cdot P, \qquad P = (-1, 3).$$

$$E(\mathbb{Q}(a)))_L = L \cdot P \oplus L \cdot Q,$$

$$Q = (9a^3 + 5a^2 + 31a - 45, -a^3 + 16a^2 + 16a + 83).$$

The point $Q = Q_1$ and its Galois translates Q_2, Q_3 and Q_4 generate a copy of W_g in $E(H)$.

A tetrahedral example, cont'd

Let $p = 13$.

Embed $H \longrightarrow \bar{\mathbb{Q}}_{13}$ so that σ_{13} fixes Q_1 and permutes Q_2, Q_3 and Q_4 cyclically, and set:

$$Q' = Q_2 + \zeta_3 Q_3 + \zeta_3^2 Q_4.$$

Alan checked (just one week before this conference!) that

$$\int_\gamma f \cdot h = \frac{4 \log_{E,13}(P) \log_{E,13}(Q')}{13\sqrt{3} \log_{13}(u_g)},$$

to 20 digits of 13-adic accuracy.

Happy retirement!