# Mock plectic points

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Let $K/\mathbb{Q}$ be a quadratic imaginary field and $E_{/\mathbb{Q}}$ an elliptic curve. Suppose that the conductor $N$ of $E_{/\mathbb{Q}}$ is coprime to the discriminant of $K$, then we can write

$$N = N^+ \cdot N^-$$

where a prime divisor of $N$ divides $N^-$ if and only if it is inert in $K$. Suppose that $N^-$ is square-free and denote by $\omega(N^-)$ the number of its prime factors, then the sign of the functional equation of $E_{/K}$ is given by

$$\varepsilon(E/K) = (-1)^{\omega(N^-)+1}. \tag{1}$$

On the one hand, when $\omega(N^-)+1$ is even, Waldspurger's formula expresses the special $L$-value $L(E/K,1)$ in terms of CM points on a Shimura set associated to the quaternion algebra ramified at *all places* $v$ such that $\varepsilon_v(E/K) = -1$, i.e, the definite quaternion algebra of discriminant $N^-$. On the other hand, when $\omega(N^-)+1$ is odd, the Gross–Zagier–Zhang formula describes the central value of the first derivative $L'(E/K,1)$ using Heegner points on a Shimura curve associated to the quaternion algebra ramified at *all places* $v \neq \infty$ such that $\varepsilon_v(E/K) = -1$, i.e., the indefinite quaternion algebra of discriminant $N^-$.

Suppose momentarily that $E_{/F}$ is a modular elliptic curve over a totally real number field $F$. For quadratic CM extensions $K/F$, Nekovar and Scholl's plectic conjectures predict that the pattern outlined above continues: it should be possible to express $L^{(r)}(E/K,1)$ in terms of CM points on an $r$-dimensional quaternionic Shimura variety associated to the quaternion algebra ramified at *all places* $v \neq \infty_1, \dots, \infty_r$ such that $\varepsilon_v(E/K) = -1$, where $\infty_1, \dots, \infty_r$ is a choice of $r$ Archimedean places of $F$. Some $p$-adic evidence for these expectations where given in the form of *plectic Stark–Heegner (SH) points* ([FGM22], [FG23b], [FG23a]) under additional assumptions regarding the existence of $r$ $p$-adic places $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ such that $\varepsilon_{\mathfrak{p}_j}(E/K) = -1$. The construction of plectic SH points was inspired by the $p$-adic uniformization of quaternionic Shimura varieties attached to quaternion algebras ramified at several $p$-adic places, but it applies to arbitrary quadratic extensions $K/F$ without restrictions on their signatures.

The aim of this article is to construct invariants for *rational* elliptic curves $E_{/\mathbb{Q}}$ that conjecturally control the central value of the second derivative $L^{(2)}(E/K,1)$. For parity reasons we require that $\varepsilon(E/K) = +1$, hence Equation (1) implies the existence of $p \parallel N^-$. The idea is to use CM points on a "mock quaternionic Shimura surface" attached to the quaternion algebra ramified at all places $v \neq p, \infty$ such that $\varepsilon_v(E/K) = -1$. The main difference from the construction of plectic SH points resides in utilizing an Archimedean and a $p$-adic place at the same time. Even though Archimedean and $p$-adic topologies do not interact well, we overcome this hurdle using algebraicity results for Heegner points.

Remark 0.1. The construction of "rank two" mock plectic points extends without problem to totally real number fields, but it is not clear how to define higher rank analogues, i.e., for ranks greater than two.

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Acknowledgements

To do..

## 1. Construction

We are interested in studying cases where $\varepsilon(E/K) = +1$, so we assume that $\omega(N^-)$ is odd, and we fix a rational prime $p$ dividing $N^-$. For notational convenience we set $(N^+_*, N^-_*) = (N^+p, N^-/p)$. Let $B/\mathbb{Q}$ be the indefinite quaternion algebra of discriminant $N^-_*$, in particular it is split at $p$, and let $R$ be an Eichler order of level $N^+_*$. By fixing an isomorphism $\iota_\infty\colon B\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{\sim} \mathrm{M}_2(\mathbb{R})$ we can let the group $\Gamma$ of norm one elements of $R^\times$ modulo $\mathbb{Z}^\times$ act on the Poincaré upper-half plane $\mathcal{H}$ by Möbius transformation. The Riemann surface $\Gamma\backslash\mathcal{H}$ has a canonical model over $\mathbb{Q}$ which we denote by $X_B(N^+_*)$. By modularity there is a non-trivial $\mathbb{Q}$-rational morphism

$$\varphi\colon X_B(N^+_*) \longrightarrow E.$$

For a group $G$, a $\mathbb{Z}[G]$-module $M$, and a $\mathbb{Z}$-module $N$ we are going to use the notation

$$\mathrm{H}^i\big(G, M, N\big) := \mathrm{H}^i\big(G, \mathrm{Hom}(M,N)\big).$$

Consider the following element determined by the modular parametrization

$$\varphi_{\mathbb{C}} \in \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}\big[X_B(N^+_*)(\mathbb{C})\big], E(\mathbb{C})\big) = \mathrm{H}^0\big(\Gamma, \mathbb{Z}[\mathcal{H}], E(\mathbb{C})\big).$$

Denote by $\Gamma^p$ the group of norm one elements of $(R\otimes_{\mathbb{Z}}\mathbb{Z}[p^{-1}])^\times$ modulo $\mathbb{Z}[p^{-1}]^\times$, and denote by $\mathrm{St}_p(\mathbb{Z})$ the space of locally constant $\mathbb{Z}$-valued function on $\mathbb{P}^1(\mathbb{Q}_p)$ modulo constants. By fixing an isomorphism $\iota_p\colon B\otimes_{\mathbb{Q}}\mathbb{Q}_p \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Q}_p)$ we can let $\Gamma^p$ act on $\mathrm{St}_p(\mathbb{Z})$ using Möbius transformations. Then, the short exact sequence defining the space of harmonic cochains determines an isomorphism

$$\mathrm{H}^0\big(\Gamma^p, \mathrm{St}_p(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[\mathcal{H}], E(\mathbb{C})\big) \cong \mathrm{H}^0\big(\Gamma, \mathbb{Z}[\mathcal{H}], E(\mathbb{C})\big).$$

Remark 1.1. The upshot is that we can interpret $\varphi_{\mathbb{C}}$ as a $\Gamma^p$-invariant measure $\mu_\varphi$ on $\mathbb{P}^1(\mathbb{Q}_p)$ of total mass zero valued in homomorphisms $\mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}[\mathcal{H}], E(\mathbb{C})\big)$ because there is the following canonical identification

$$\mathscr{M}_0\Big(\mathbb{P}^1(\mathbb{Q}_p), \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}[\mathcal{H}], E(\mathbb{C})\big)\Big)^{\Gamma^p} = \mathrm{H}^0\Big(\Gamma^p, \mathrm{St}_p(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[\mathcal{H}], E(\mathbb{C})\Big).$$

To have an interesting $p$-adic integration theory we need our measures to be valued in $p$-adically complete and separated $\mathbb{Z}_p$-modules. However, since $E(\mathbb{C})$ is divisible, we find that

$$\widehat{E(\mathbb{C})} := \lim_{\leftarrow, n}\big(E(\mathbb{C})\otimes_{\mathbb{Z}}\mathbb{Z}/p^n\mathbb{Z}\big) = 0.$$

The strategy is now to consider subsets of $\mathcal{H}$ that are $\Gamma^p$-stable and whose images under the modular parametrization are points defined in controlled algebraic extension of $\mathbb{Q}$. Let $c$ be a positive integer prime to $N$, unramified in $K$, and let $\psi\colon K\to B$ be an embedding of conductor $cp^n$, $n\geqslant 1$, with respect to the fixed Eichler order $R$. The action of $\psi(K)^\times$ on $\mathbb{P}^1(\mathbb{C})\setminus\mathbb{P}^1(\mathbb{R})$ has two fixed points, we denote by $x_\psi$ the one belonging to $\mathcal{H}$ and its $\Gamma^p$-orbit by

$$X_\psi := \Gamma^p\cdot x_\psi \subset \mathcal{H}. \tag{2}$$

Note that $\Gamma^p_\psi := \mathrm{Stab}_{\Gamma^p}(x_\psi)$ is finite since we assumed $p$ inert in $K$. If we denote by $K^c_\infty$ the compositum of all ring class fields of $K$ of conductor $cp^m$ for all $m\geqslant 0$, then the theory of complex multiplication ensures that the image of $X_\psi$ under the modular parametrization $\varphi_{\mathbb{C}}$ belongs to

$E(K_\infty^c)$. Does the the anticyclotomic Iwasawa main conjecture ensure that $\widehat{E(K_\infty^c)} \neq 0$? The upshot is that the modular parametrization $\varphi$ and the $\Gamma^p$-conjugacy class of $\psi$ determine

$$\mu_{\varphi,\psi} \in \mathscr{M}_0\Big(\mathbb{P}^1(\mathbb{Q}_p), \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}[X_\psi], \widehat{E(K_\infty^c)}\big)\Big)^{\Gamma^p}.$$

Let $\widehat{K}_p^\times$ denote the maximal torsion-free quotient of $K_p^\times$ and $\mathrm{St}_p^{\mathrm{ct}}(\widehat{K}_p^\times)$ the continuous Steinberg representation with coefficients in $\widehat{K}_p^\times$. By [FG23b, Lemma 4.2] the following map, induced by the inclusion $\mathrm{St}_p(\widehat{K}_p^\times) \subset \mathrm{St}_p^{\mathrm{ct}}(\widehat{K}_p^\times)$, is an isomorphism

$$\mathrm{H}^0\Big(\Gamma^p, \mathrm{St}_p^{\mathrm{ct}}(\widehat{K}_p^\times) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[X_\psi], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big) \xrightarrow{\sim} \mathrm{H}^0\Big(\Gamma^p, \mathrm{St}_p(\widehat{K}_p^\times) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[X_\psi], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big)$$

and hence we can also integrate continuous functions with respect to the measure $\mu_{\varphi,\psi}$. Let $\mathbb{Z}_p[\mathcal{H}_p]^0$ denote the free $\mathbb{Z}_p$-module of degree-zero divisors on Drinfeld's upper-half plane $\mathcal{H}_p$ which sits in the short exact sequence

$$0 \longrightarrow \mathbb{Z}_p[\mathcal{H}_p]^0 \longrightarrow \mathbb{Z}_p[\mathcal{H}_p] \longrightarrow \mathbb{Z}_p \longrightarrow 0. \tag{3}$$

There is a homomorphism

$$\Phi_p \colon \mathbb{Z}_p[\mathcal{H}_p]^0 \longrightarrow \mathrm{St}_p^{\mathrm{ct}}(\widehat{K}_p^\times), \qquad [x]-[y] \mapsto \left[t \mapsto \left(\frac{t-x}{t-y}\right)\right],$$

and we can consider the pushforward

$$(\Phi_p)_*\mu_{\varphi,\psi} \in \mathrm{H}^0\Big(\Gamma^p, \mathbb{Z}_p[\mathcal{H}_p]^0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[X_\psi], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big).$$

We are interested in the first 4 terms of the long exact sequence in cohomology induced by (3):

$$\begin{array}{c} \mathrm{H}^0\Big(\Gamma^p, \mathbb{Z}_p[X_\psi], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big) \hookrightarrow \mathrm{H}^0\Big(\Gamma^p, \mathbb{Z}_p[\mathcal{H}_p \times X_\psi], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big) \\ \swarrow \\ \mathrm{H}^0\Big(\Gamma^p, \mathbb{Z}_p[\mathcal{H}_p]^0 \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[X_\psi], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big) \longrightarrow \mathrm{H}^1\Big(\Gamma^p, \mathbb{Z}_p[X_\psi], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big). \end{array}$$

Shapiro's lemma gives

$$\mathrm{H}^i\Big(\Gamma^p, \mathbb{Z}_p[X_\psi], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big) \cong \mathrm{H}^i\Big(\Gamma^p_\psi, \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big) \qquad \forall\, i \geqslant 0,$$

so if $p$ is does not divide the cardinality of the stabilizer $\Gamma^p_\psi$, then the obstruction to lifting our class vanishes. Since the cohomology group measuring the indeterminacy in the choice of the lift is Eisenstein, we obtain uniqueness by requiring the lift to belong to the isotypic component determined by the system of eigenvalues attached to $E_{/\mathbb{Q}}$. The upshot is the existence of a canonical element

$$⨍\!\!\int \omega_{\varphi,\psi} \in \mathrm{Hom}_{\mathbb{Z}_p}\Big(\mathbb{Z}_p[\Gamma^p \backslash (\mathcal{H}_p \times X_\psi)], \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times\Big).$$

The action of $\psi(K)^\times$ on $\mathcal{H}_p$ has two fixed points $\tau_\psi, \overline{\tau}_\psi$, we choose $\tau_\psi$ in some precise way. Then the pair $(\tau_\psi, x_\psi)$ determines a "CM point" of the mock quaternionic surface

$$\mathrm{MS}_B := \Gamma^p \backslash (\mathcal{H}_p \times \mathcal{H})$$

depending only on the $\Gamma^p$-conjugacy class of $\psi$. For any positive integer $c$ we denote by $\mathrm{MS}_B(c)$ the collection of all points arising from embeddings of prime to $p$ conductor $c$. Then, we can rephrase what we have done so far by claiming the existence of a mock plectic parametrization

$$\times\!\!\!\!\!\!\int\!\!\int \omega_\varphi \colon \mathrm{MS}_B(c) \longrightarrow \widehat{E(K_\infty^c)} \otimes_{\mathbb{Z}_p} \widehat{K}_p^\times.$$

Definition 1.2. The mock plectic point attached to the $\Gamma^p$-conjugacy class of an embedding $\psi$ and the modular parametrization $\varphi$ is

$$\mathrm{P}_{\varphi,\psi} := \times\!\!\!\!\!\!\int^{\tau_\psi}\int^{x_\psi} \omega_{\varphi,\psi}.$$

## References

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