DIAGONAL CYCLES AND EULER SYSTEMS II: THE BIRCH AND SWINNERTON-DYER CONJECTURE FOR HASSE-WEIL-ARTIN L-FUNCTIONS

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ABSTRACT. This article establishes new cases of the Birch and Swinnerton-Dyer conjecture in analytic rank 0, for elliptic curves over $\mathbb{Q}$ viewed over the fields cut out by certain self-dual Artin representations of dimension at most 4. When the associated L-function vanishes (to even order ≥ 2) at its central point, two canonical classes in the corresponding Selmer group are constructed and shown to be linearly independent assuming the non-vanishing of a Garrett-Hida p-adic L-function at a point lying outside its range of classical interpolation. The key tool for both results is the study of certain p-adic families of global Galois cohomology classes arising from Gross-Kudla-Schoen diagonal cycles in a tower of triple products of modular curves.

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INTRODUCTION

Let E be an elliptic curve over $\mathbb{Q}$ and let

$$\varrho : G_{\mathbb{Q}} = \mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}_L(V_\varrho) \simeq \mathrm{GL}_n(L)$$

be an Artin representation with coefficients in a finite extension $L \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ of $\mathbb{Q}$, factoring though the Galois group of a finite extension $H/\mathbb{Q}$. The Hasse-Weil-Artin L-series $L(E, \varrho, s)$ of E twisted by ϱ, defined on the right half-plane $\mathrm{Re}(s) > 3/2$ by an absolutely convergent Euler product of degree $2n$, is expected to admit an analytic continuation to the whole complex plane.

Let $E(H)^\varrho := \mathrm{Hom}_{G_{\mathbb{Q}}}(V_\varrho, E(H) \otimes L)$ denote the *ϱ-isotypical component* of the Mordell-Weil group $E(H)$, and define the analytic and algebraic rank of the twist of E by ϱ as

$$r_{\mathrm{an}}(E, \varrho) = \mathrm{ord}_{s=1} L(E, \varrho, s), \qquad r(E, \varrho) = \dim_L E(H)^\varrho.$$

A Galois-equivariant refinement of the Birch and Swinnerton-Dyer conjecture predicts that

$$r_{\mathrm{an}}(E, \varrho) \stackrel{?}{=} r(E, \varrho). \tag{1}$$

The main object of this article is to study conjecture (1) when $\varrho = \varrho_1 \otimes \varrho_2$, where

$$\varrho_1, \varrho_2 : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(L)$$

are odd, irreducible two-dimensional Artin representations satisfying $\det(\varrho_1) = \det(\varrho_2)^{-1}$. This last condition implies that ϱ has real traces, i.e., is isomorphic to its dual, or contragredient, representation.

Thanks to the major advances achieved in the last two decades, the elliptic curve E and the Artin representations ϱ_1 and ϱ_2 are known to be associated to cuspidal newforms

$$f \in S_2(N_f), \qquad g \in S_1(N_g, \chi), \qquad h \in S_1(N_h, \chi^{-1})$$

of weight 2, 1, 1, respectively. The nebentype character $\chi = \det(\varrho_1)$ of g is an odd Dirichlet character whose conductor divides both N_g and N_h.

The L-function $L(E, \varrho, s)$ can be identified with the triple-product convolution $L(f, g, h, s)$, whose analytic continuation and functional equation follow from the work of Garrett, Piatetski-Shapiro and Rallis. In particular, the analytic rank $r_{\mathrm{an}}(E, \varrho)$ is defined unconditionally, suggesting that (1) might repay closer scrutiny in this setting.

It is assumed throughout this work that the level N_f of f is relatively prime to $N_g N_h$. This implies (cf. [Pr, thm. 1.4]) that all the local signs in the functional equation of $L(f, g, h, s) = L(E, \varrho, s)$ are $+1$, hence that the same is true for the global sign, implying that $r_{\mathrm{an}}(E, \varrho)$ is *even*. Subject to these hypotheses, the main results of this article are Theorems A and B below. (Cf. Theorem 6.7 and Theorem 6.13 for somewhat more general statements.)

Theorem A. *If $L(E, \varrho, 1) \neq 0$ then $E(H)^{\varrho} = 0$.*

Theorem A has broad implications for the arithmetic of E over ring class fields of quadratic fields, both imaginary and *real*. For any ring class character ψ of a quadratic field K (of conductor relatively prime to $N_f =: N_E$), let H/K denote the ring class field cut out by it, and define

$$E(H)^{\psi} = \{P \in E(H) \otimes L \text{ such that } P^{\sigma} = \psi(\sigma)P \text{ for all } \sigma \in \mathrm{Gal}\,(H/K)\}.$$

When K is a real quadratic field, assume further that the pair (E, K) satisfies the *non-vanishing hypothesis* of Definition 6.8. This mild hypothesis is expected to always hold, and is satisfied, for example, as soon as E has quadratic twists E' of both possible signs for which $L(E'/K, 1) \neq 0$.

Corollary A1. *If $L(E/K, \psi, 1) \neq 0$ then $E(H)^{\psi} = 0$.*

When K is imaginary quadratic, theorems of this sort were already established by a variety of approaches (cf. [Ko89], [BD97, Th.B], [BD99, Thm.1.2.], [BD05, Cor.4] and [Ne3, Th.A'], for example) using Heegner points on modular or Shimura curves, combined with p-adic techniques and/or congruences between modular forms. Their common feature is that they ultimately rely on Kolyvagin's Euler system of CM points and on some variant (either classical, or p-adic) of the Gross-Zagier formula [GZ86]. The approach followed in this article makes no use of CM points, which accounts for why it extends unconditionally to more general settings, including the case where K is a real quadratic field.

The intimate connection between Corollary A1 for K real quadratic and the theory of *Stark-Heegner points* initiated in [Da] is suggested by the articles [BDD07] and [LRV13], which give a *conditional* proof of Corollary A1 resting on the algebraicity of Stark-Heegner points, essentially by replacing CM points with their conjectural real quadratic counterparts in the proof of [BD99, Thm.1.2]. Stark-Heegner points are still poorly understood, but the hope that the structures developed in the proof of Corollary A1 might shed light on their algebraicity was an important motivation for the present work.

Specialising Theorem A to the case where the representation $\varrho = \varrho_1 \otimes \varrho_2$ is irreducible establishes the Birch and Swinnerton-Dyer conjecture in analytic rank zero, in scenarios lying beyond the scope of both Heegner and Stark-Heegner points. For instance, in the case where the projective representations attached to ϱ_1 and ϱ_2 cut out the same A_5-extension of $\mathbb{Q}$, but where ϱ_1 and ϱ_2 are not twists of each other, Theorem A leads to the following instance of the Birch and Swinnerton-Dyer over quintic fields, which is spelled out in Theorem 6.11.

Corollary A2. *Let E be an elliptic curve over $\mathbb{Q}$, let K be a non-totally real quintic field with Galois group A_5, and assume that the discriminants of E and K are coprime. Then*

$$\mathrm{ord}_{s=1} L(E/K, s) = \mathrm{ord}_{s=1} L(E/\mathbb{Q}, s) \quad \Rightarrow \quad \mathrm{rank}\, E(K) = \mathrm{rank}\, E(\mathbb{Q}).$$

The second main theorem of this article is concerned with the case where $L(E, \varrho, s)$ vanishes at $s = 1$ (and hence to order at least 2). Note that ϱ_1 and ϱ_2 are both *regular*, i.e., there exists $\sigma \in G_{\mathbb{Q}}$ acting on each of these Galois representations with distinct eigenvalues.

Fix an odd prime p not dividing $N := \mathrm{lcm}(N_f, N_g, N_h)$ at which f is not Eisenstein and fix throughout an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $\mathrm{Frob}_p \in G_{\mathbb{Q}}$ denote the Frobenius element at p induced by this embedding. Let L_p denote the completion of L in $\bar{\mathbb{Q}}_p$. Let (α_g, β_g) and (α_h, β_h) be the pairs of eigenvalues of $\varrho_g(\mathrm{Frob}_p)$ and $\varrho_h(\mathrm{Frob}_p)$, respectively. It shall be assumed throughout this article that

$$E \text{ is ordinary at } p, \quad \text{and} \quad \alpha_g \neq \beta_g, \quad \alpha_h \neq \beta_h.$$

Such a prime exists by the Chebotarev density theorem.

Theorem B below relates the p-Selmer group attached to E and ϱ to the behaviour of the Garrett-Hida p-adic L-functions attached to f, g, and h. In order to describe these p-adic L-functions more precisely, let $S^{\mathrm{ord}}_{\tilde\Lambda}(M,\psi)$ denote the space of ordinary Λ-adic modular forms of tame level $M\geq 1$ and tame character ψ with coefficients in a finite flat extension $\tilde\Lambda$ of the Iwasawa algebra $\Lambda=\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$. These spaces are locally free $\tilde\Lambda$-modules of finite rank, and are equipped with a natural $\tilde\Lambda$-linear action of the Hecke operators.

A Hida family $\underline{\phi}$ can be viewed concretely as a formal q-series with coefficients in $\tilde\Lambda$ whose specialisations give rise to a p-adically coherent collection $\{\phi_x\}$ of overconvergent ordinary modular forms indexed by the points x in the rigid analytic space $\tilde\Omega:=\mathrm{Hom}_{\mathrm{cts}}(\tilde\Lambda,\mathbb{C}_p)$. This space is equipped with a finite flat morphism $\mathrm{w}:\tilde\Omega\longrightarrow\Omega$ to the weight space $\Omega:=\mathrm{Hom}_{\mathrm{cts}}(\Lambda,\mathbb{C}_p)$, which is also equipped with a natural embedding $\mathbb{Z}\subset\Omega$. When $k=\mathrm{w}(x)\in\mathbb{Z}$, the form ϕ_x belongs to the space $S^{\mathrm{oc}}_k(M,\psi\omega^{1-k})$ of overconvergent modular forms of weight k, and is classical when $k\geq 2$ by [Hi93, Ch. 7.3, Thm. 3]. When $k=1$, the form ϕ_x may be classical or not.

If $\underline{\phi}$ is a Hida family of eigenforms of tame level M and tame character ψ, let

$$S^{\mathrm{ord}}_{\tilde\Lambda}(M,\psi)[\underline{\phi}]:=\left\{\underline{\varphi}\in S^{\mathrm{ord}}_{\tilde\Lambda}(M,\psi)\quad\text{s.t.}\quad\left|\begin{array}{ll}T_\ell\underline{\varphi}=a_\ell(\underline{\phi})\underline{\varphi}, & \forall\ell\nmid Mp,\\ U_p\underline{\varphi}=a_p(\underline{\phi})\underline{\varphi}, & \end{array}\right.\right\},\tag{2}$$

$$S^{\mathrm{ord}}_{\tilde\Lambda}(M,\psi)^\vee[\underline{\phi}]=\left\{\underline{\gamma}:S^{\mathrm{ord}}_{\tilde\Lambda}(M,\psi)\to\tilde\Lambda\quad\text{s.t.}\quad\left|\begin{array}{ll}\underline{\gamma}(T^*_\ell\underline{\varphi})=a_\ell(\underline{\phi})\underline{\gamma}(\underline{\varphi}), & \forall\ell\nmid Mp,\\ \underline{\gamma}(U^*_p\underline{\varphi})=a_p(\underline{\phi})\underline{\gamma}(\underline{\varphi}) & \underline{\varphi}\in S^{\mathrm{ord}}_{\tilde\Lambda}(M,\psi)\end{array}\right.\right\}\tag{3}$$

denote the $\underline{\phi}$-isotypic subspaces of $S^{\mathrm{ord}}_{\tilde\Lambda}(M,\psi)$ and of its $\tilde\Lambda$-linear dual, in which the latter is endowed with the adjoint action of the Hecke operators, as described in (24) below, the adjoints being taken relative to Poincaré duality.

Let $f_\alpha\in S_2(N_fp)$ denote the unique ordinary p-stabilisation of f, and write $g_\alpha,\ g_\beta\in S_1(N_gp,\chi)$ (resp. $h_\alpha,h_\beta\in S_1(N_hp,\chi^{-1})$) for the pair of ordinary stabilisations of g (resp. h). By [Hi93, Ch. 7.3, Thm. 3] combined with a recent result of Bellaïche and Dimitrov [BeDi], there is a *unique* triple of Hida families, denoted by $(\underline{f},\underline{g},\underline{h})$, which specialises to $(f_\alpha,g_\alpha,h_\alpha)$ in weights $(2,1,1)$. These Hida families are normalised eigenforms

$$\underline{f}\in S^{\mathrm{ord}}_{\Lambda_f}(N_f),\qquad\underline{g}\in S^{\mathrm{ord}}_{\Lambda_g}(N_g,\chi),\qquad\underline{h}\in S^{\mathrm{ord}}_{\Lambda_h}(N_h,\chi^{-1}),$$

with coefficients in suitable finite flat extensions Λ_f, Λ_g and Λ_h of Λ.

The Garrett-Hida p-adic L-functions denoted

$$\mathscr{L}_p^{f_\alpha}(\breve{\underline{f}}^*,\breve{\underline{g}},\breve{\underline{h}}),\qquad\mathscr{L}_p^{g_\alpha}(\breve{\underline{f}},\breve{\underline{g}}^*,\breve{\underline{h}}),\qquad\mathscr{L}_p^{h_\alpha}(\breve{\underline{f}},\breve{\underline{g}},\breve{\underline{h}}^*)$$

depend on the choice of a triple $(\breve{\underline{f}},\breve{\underline{g}},\breve{\underline{h}})$ of Λ-*adic test vectors*

$$\breve{\underline{f}}\in S^{\mathrm{ord}}_{\Lambda_f}(N)[\underline{f}],\qquad\breve{\underline{g}}\in S^{\mathrm{ord}}_{\Lambda_g}(N,\chi)[\underline{g}],\qquad\breve{\underline{h}}\in S^{\mathrm{ord}}_{\Lambda_h}(N,\chi^{-1})[\underline{h}],\tag{4}$$

as well as on a triple of *dual test vectors*

$$\breve{\underline{f}}^*\in S^{\mathrm{ord}}_{\Lambda_f}(N)^\vee[\underline{f}]\qquad\breve{\underline{g}}^*\in S^{\mathrm{ord}}_{\Lambda_g}(N,\chi^{-1})^\vee[\underline{g}],\qquad\breve{\underline{h}}^*\in S^{\mathrm{ord}}_{\Lambda_h}(N,\chi)^\vee[\underline{h}].\tag{5}$$

The specialisations $\breve f_x$, $\breve g_y$ and $\breve h_z$ at points of weight $k,\ell,m\in\mathbb{Z}$ belong to the space of overconvergent oldforms of tame level N and weights attached to f_x, g_y and h_z respectively, while $\breve f^*_x$, $\breve g^*_y$ and $\breve h^*_z$ belong to the $\mathbb{C}_p$-linear duals $S^{\mathrm{oc}}_k(N,\omega^{1-k})^\vee[f^*_x]$, $S^{\mathrm{oc}}_\ell(N,\chi^{-1}\omega^{1-\ell})^\vee[g^*_y]$ and $S^{\mathrm{oc}}_m(N,\chi\omega^{1-m})^\vee[h^*_z]$.

The space $S^{\text{n-oc}}_k(N,\chi)$ of nearly overconvergent p-adic modular forms is described for instance in [DR13, §2.4]. Lemma 2.7 of loc.cit. asserts that the image of Hida's ordinary projector $e_{\mathrm{ord}}:=\lim U_p^{n!}$ on $S^{\text{n-oc}}_k(N,\chi)$ is contained in the ordinary part of $S^{\mathrm{oc}}_k(N,\chi)$. Let $d=q\frac{d}{dq}$ be the Atkin-Serre operator on p-adic modular forms, and let $\phi^{[p]}$ denote the *p-depletion* of a classical or p-adic modular form ϕ of weight k, level N and character χ, so that $d^t\phi^{[p]}$ is determined by the fourier expansion

$$d^t\phi^{[p]}(q)=\sum_{p\nmid n}n^ta_n(\phi)q^n.$$

For $t\geq 0$, this modular form belongs to the space $S^{\text{n-oc}}_{k+2t}(N,\chi)$. In particular, the modular form $d^t\breve g_y^{[p]}\times\breve h_z$ is a nearly overconvergent modular form of weight $2t+\ell+m$.

With these basic notions and the Λ-adic test vectors in hand, the Garrett-Hida p-adic L-function attached to $(\underline{\breve{f}}^*, \breve{g}, \underline{\breve{h}})$ is defined (as described in [DR13, Def. 4.4], but adopting a somewhat more flexible point of view) to be the p-adic rigid-analytic function on $\Omega_f \times \Omega_g \times \Omega_h$ satisfying

$$\mathscr{L}_p{}^{f_\alpha}(\underline{\breve{f}}^*, \underline{\breve{g}}, \underline{\breve{h}})(x,y,z) := \breve{f}_x^*(e_{\mathrm{ord}}(d^t \breve{g}_y^{[p]} \times \breve{h}_z)), \qquad t := (k-\ell-m)/2, \tag{6}$$

as (x,y,z) ranges over the dense set of points in $\Omega_f \times \Omega_g \times \Omega_h$ of integral weights (k,ℓ,m), with $k \equiv \ell + m \pmod 2$. Viewing this as a p-adic L-function attached to the triple convolution of the Hida families $\underline{f}$, $\underline{g}$, and $\underline{h}$ is justified by fundamental results of Garrett, Harris-Kudla and Ichino relating the quantity in (6) to the central critical values

$$L(f_x, g_y, h_z, c), \qquad c := (k+\ell+m-2)/2, \quad \text{when } k,\ell,m \geq 1, \qquad k \geq \ell+m,$$

up to a product of local factors which are non-zero for a suitable choice of test vectors. Identical definitions apply to $\mathscr{L}_p{}^{g_\alpha}(\underline{\breve{f}}, \underline{\breve{g}}^*, \underline{\breve{h}})$ and $\mathscr{L}_p{}^{h_\alpha}(\underline{\breve{f}}, \underline{\breve{g}}, \underline{\breve{h}}^*)$, with the difference that their regions of classical interpolation are defined by the inequalities $\ell \geq k+m$ and $m \geq k+\ell$ respectively.

The point $(x_0, y_0, z_0) \in \Omega_f \times \Omega_g \times \Omega_h$ of weight $(2,1,1)$ for which $(f_{x_0}, g_{y_0}, h_{z_0}) = (f_\alpha, g_\alpha, h_\alpha)$ thus lies within the region of interpolation defining $\mathscr{L}_p{}^{f_\alpha}(\underline{\breve{f}}^*, \underline{\breve{g}}, \underline{\breve{h}})$, and its value at this point is given by the formula

$$\mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \breve{g}_\alpha, \breve{h}_\alpha) := \mathscr{L}_p{}^{f_\alpha}(\underline{\breve{f}}^*, \underline{\breve{g}}, \underline{\breve{h}})(x_0,y_0,z_0) = \breve{f}_{x_0}^*(e_{\mathrm{ord}}(\breve{g}_\alpha^{[p]} \times \breve{h}_\alpha)) \tag{7}$$

arising from (6). Ichino's explicit form of Garret's formula asserts that the square of the expression on the right is a simple (non-zero, for a judicious choice of test vectors) multiple of the central critical value $L(E,\varrho,1) = L(f,g,h,1)$. In particular, although $\mathscr{L}_p{}^{f_\alpha}(\underline{\breve{f}}^*, \underline{\breve{g}}, \underline{\breve{h}})$ depends crucially on the Hida families $\underline{f}$, $\underline{g}$ and $\underline{h}$, and hence on the p-stabilisations of g and h that were chosen to define these Hida families, the resulting *values* at the classical point (x_0,y_0,z_0) do not depend in an essential way on the choices α_g and α_h of U_p-eigenvalues, or on the concomitant choice of test vectors. It is therefore not too egregious an abuse of notations to set

$$\mathscr{L}_p{}^{f_\alpha}(f,g,h) := \mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \breve{g}_\alpha, \breve{h}_\alpha) \sim \mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \breve{g}_\alpha, \breve{h}_\beta) \sim \mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \breve{g}_\beta, \breve{h}_\alpha) \sim \mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \breve{g}_\beta, \breve{h}_\beta),$$

where $\sim$ denotes an equality up to an algebraic factor (in the field generated by the Fourier coefficients of f_α, g_α, and h_α) which can be made non-zero with a good choice of test vectors.

The value

$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}_\alpha^*, \breve{h}_\alpha) := \mathscr{L}_p{}^{g_\alpha}(\underline{\breve{f}}, \underline{\breve{g}}^*, \underline{\breve{h}})(x_0,y_0,z_0) = \breve{g}_\alpha^*(e_{\mathrm{ord}}(d^{-1}\breve{f}^{[p]} \times \breve{h}_\alpha)) \tag{8}$$

of $\mathscr{L}_p{}^{g_\alpha}(\underline{\breve{f}}, \underline{\breve{g}}^*, \underline{\breve{h}})$ at the point (x_0,y_0,z_0), which now lies outside the region of classical interpolation, is somewhat more subtle. It depends critically on the choice of stabilisation of g, but [DR13, Lemma 2.17] shows that $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}_\alpha^*, \breve{h}_\alpha)$ is left unchanged upon replacing $\breve{h}_\alpha$ with the other p-stabilisation $\breve{h}_\beta$ of h. Furthermore, a change in the choice of test vector (belonging to the same (f, g_α, h)-isotypic subspaces for the action of the Hecke operators) has the effect of multiplying $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h})$ by explicit local zeta integrals at the places dividing N, of exactly the same type as occur in Ichino's formula for the Garret-Rankin triple product. One is hence justified in adopting the slightly abusive but simpler notations

$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}) := \mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}_\alpha^*, \breve{h}_\alpha) = \mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}_\alpha^*, \breve{h}_\beta), \tag{9}$$

and likewise for the other p-stabilisations of g and of h. The four non-classical p-adic L-values

$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}), \quad \mathscr{L}_p{}^{g_\beta}(\breve{f}, \breve{g}^*, \breve{h}), \quad \mathscr{L}_p{}^{h_\alpha}(\breve{f}, \breve{g}, \breve{h}^*), \quad \mathscr{L}_p{}^{h_\beta}(\breve{f}, \breve{g}, \breve{h}^*) \tag{10}$$

ostensibly bear no simple direct relation to $L(f,g,h,1)$, or to each other. When $L(f,g,h,1)=0$, they should rather be viewed as different p-adic avatars of $L''(f,g,h,1)$.

Let $\mathrm{Sel}_p(E,\varrho) := H^1_{\mathrm{fin}}(\mathbb{Q}, V_p(E) \otimes V_\varrho \otimes_L L_p)$ denote the ϱ-isotypic component of the Bloch-Kato Selmer group of E/H; cf. (154) for the precise definition of this group. The following theorem shows that the above p-adic values convey non-trivial information about $\mathrm{Sel}_p(E,\varrho)$.

Theorem B. *If $L(E,\varrho,1)=0$ and $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}) \neq 0$ for some choice of test vectors, then*

$$\dim_{L_p} \mathrm{Sel}_p(E,\varrho) \geq 2.$$

The same conclusion applies, of course, when any of the four p-adic L-values in (10) is non-zero. Recent joint work with Alan Lauder [DLR] describes an algorithm for calculating the expressions in (8), and hence $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}_\alpha^*, \breve{h}_\alpha)$, and uses this to verify their non-vanishing in many instances of analytic rank two, showing that Theorem B is not vacuous.

Theorem D below formulates a more precise version of Theorem B, in which two canonical elements in $\mathrm{Sel}_p(E, \varrho)$ are associated to the data (f, g_α, h), and shown to be linearly independent under the hypotheses of Theorem B. One of the main contributions of [DLR] and [DR15] is the formulation of a conjecture expressing these two elements as explicit p-adic linear combinations of global points in the Mordell-Weil group attached to $E(H)^\varrho$. (Cf. loc.cit. for the precise statement of this conjecture and a summary of the experimental evidence that has been gathered in its support.)

Theorem B can be compared to the Theorem B of Skinner and Urban [SU], which asserts that the rank of the Selmer group of an elliptic curve is ≥ 2 when the associated L-function vanishes to even order ≥ 2. There is little overlap between both theorems, whose methods of proof are very different. The result of [SU] has the virtue of requiring no non-vanishing hypotheses on an auxiliary p-adic L-function, while Theorem B applies to a different class of Artin representations, typified by the settings described in Corollaries A1 and A2.

The proofs of Theorems A and B rest on a system of global cohomology classes for the Rankin convolution of three modular forms of weights $(2, 1, 1)$, whose construction is one of the main contributions of the present work. These classes arise from *generalised Gross-Kudla-Schoen cycles* in the product of three Kuga-Sato varieties fibered over a classical modular curve, and (crucially) from their *variation in Hida families.*

The prequel [DR13] to the present work initiated the authors' study of generalised Gross-Kudla-Schoen cycles, relating their images under the p-adic Abel-Jacobi map to special values of Garrett-Hida p-adic L-series attached to the triple convolution of three (Hida families of) cusp forms. The p-adic Abel-Jacobi map has the feature—not present in its complex analogue, or in related complex and p-adic heights—of factoring through the restriction to $G_{\mathbb{Q}_p}$ of its p-adic étale counterpart. This turns out to be crucial in arithmetic applications relying on the theory of Euler systems. More precisely, the étale Abel-Jacobi images of generalised diagonal cycles give rise, after applying suitable projections, to distinguished global Galois cohomology classes

$$\kappa(f_x, g_y, h_z) \in H^1(\mathbb{Q}, V_{f_x g_y h_z}(N)), \tag{11}$$

where

- f_x, g_y and h_z are classical specialisations of weights k, ℓ, and m which are *balanced* in the sense that each weight is strictly smaller than the sum of the other two;
- $V_{f_x g_y h_z}(N)$ is the Kummer self-dual twist of the direct sum of several copies of the tensor product of the p-adic representations V_{f_x}, V_{g_y}, V_{h_z} of $G_{\mathbb{Q}}$ attached by Eichler-Shimura and Deligne to these forms, occuring in the middle cohomology of a Kuga-Sato variety $\mathcal{E}_k$ (resp. $\mathcal{E}_\ell$, $\mathcal{E}_m$) of dimension $k-1$ (resp. $\ell - 1$, $m-1$).

The extensions of p-adic Galois representations associated to the classes $\kappa(f_x, g_y, h_z)$ arise *from geometry*; namely, they are realised in the p-adic étale cohomology of an open subvariety of the product of Kuga-Sato varieties $\mathcal{E}_{k,\ell,m} := \mathcal{E}_k \times \mathcal{E}_\ell \times \mathcal{E}_m$, which have good reduction at p. In particular, thanks to the work of Saito their restrictions to a decomposition group at p are known to be *crystalline.*

The main theorem of [DR13] can be interpreted as a direct relationship between the Bloch-Kato p-adic logarithms of the class $\kappa(f_x, g_y, h_z)$ and the special values of Garrett-Hida p-adic L-functions at points which lie *outside* their range of classical interpolation. The concluding paragraphs of the introduction to [DR13] stressed the desirability of deforming this result *along p-adic families* and, in particular, of p-adically interpolating the classes $\kappa(f_x, g_y, h_z)$ themselves as the triple (f_x, g_y, h_z) is made to vary over the classical, balanced specialisations of ordinary Hida families $\underline{f}$, $\underline{g}$ and $\underline{h}$. Such a p-adic interpolation would be described by a global cohomology class

$$\kappa(\underline{f}, \underline{g}, \underline{h}) \overset{?}{\in} H^1(\mathbb{Q}, \mathbb{V}_{\underline{f}\,\underline{g}\,\underline{h}}(N)), \tag{12}$$

where $\mathbb{V}_{\underline{f}\,\underline{g}\,\underline{h}}(N)$ is the direct sum of several copies of the tensor product of Hida's Λ-adic representations $\mathbb{V}_{\underline{f}}$, $\mathbb{V}_{\underline{g}}$, $\mathbb{V}_{\underline{h}}$ attached to $\underline{f}$, $\underline{g}$, and $\underline{h}$, twisted in such a way that for each triple (f_x, g_y, h_z) of classical (balanced or unbalanced) specialisations of $(\underline{f}, \underline{g}, \underline{h})$ there are $G_{\mathbb{Q}}$-equivariant specialisation homomorphisms

$$\nu_{xyz} : \mathbb{V}_{\underline{f}\,\underline{g}\,\underline{h}}(N) \longrightarrow V_{f_x g_y h_z}(N).$$

The putative class $\kappa(\underline{f}, \underline{g}, \underline{h})$ would be uniquely determined by the requirement that its specialisations at balanced triples (f_x, g_y, h_z) should agree, up to normalisation by elementary fudge factors, with the classes $\kappa(f_x, g_y, h_z)$ of (11). Guided by this ideal goal but falling somewhat short of it, Chapter 1 attaches a one-variable cohomology class to a triple $(f, \underline{g}, \underline{h})$ formed by an eigenform f of weight two and trivial nebentype and two Hida families $\underline{g}$ and $\underline{h}$ of tame character χ and χ^{-1}, respectively. This Λ-adic class is described by:

- a finite flat algebra $\Lambda_{f\underline{gh}}$ over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[1 + p\mathbb{Z}_p]]$;
- a $\Lambda_{f\underline{gh}}[G_{\mathbb{Q}}]$-module $\mathbb{V}_{f\underline{gh}}(N)$ whose classical specialisations are of the form $V_{f g_y h_z}(N)$ where the pairs (g_y, h_z) range over the specializations of $\underline{g}$ and $\underline{h}$ at classical points of the *same* weight ℓ and nebentype character at p;
- a cohomology class $\kappa(f, \underline{gh}) \in H^1(\mathbb{Q}, \mathbb{V}_{f\underline{gh}}(N))$.

The class $\kappa(f, \underline{gh})$ gives rise to a collection of global classes in $H^1(\mathbb{Q}, V_{f g_y h_z}(N))$, varying p-adic analytically as (g_y, h_z) varies over pairs of specialisations of $\underline{g}$ and $\underline{h}$ with common weight and nebenytpe character at p. In order to construct $\kappa(f, \underline{gh})$, Section 1.3 introduces a family $\{\Delta_s\}_{s\geq 1}$ of null-homologous *twisted diagonal cycles* on certain twists of finite quotients of the three-folds $X_0(Np) \times X_1(Np^s) \times X_1(Np^s)$, satisfying an explicit compatibility under the natural push-forward maps as s varies. The class $\kappa(f, \underline{gh})$ is then manufactured in §1.4-1.6 as the inverse limit of the cohomology classes one obtains as the image of Δ_s under a suitable projection of the p-adic étale Abel-Jacobi map.

Although $\kappa(f, \underline{gh})$ is constructed by interpolating geometric constructions attached only to the weight two specialisations of $\underline{g}$ and $\underline{h}$, it makes sense to consider its specialisations

$$\kappa(f, g_y, h_z) := \kappa(f, \underline{gh})_{y,z} \in H^1(\mathbb{Q}, V_{f g_y h_z}(N)) \tag{13}$$

at any point (y, z) attached to a pair (g_y, h_z) of classical modular forms of arbitrary weight $\ell \geq 1$ with inverse tame nebentype characters. The relevance of this class to the proof of Theorems B and C arises when f is the weight 2 newform associated to the elliptic curve E, and $g_y = g_\alpha$ and $h_z = h_\alpha$ are (ordinary) p-stabilisations of the forms g and h associated to the Artin representations ϱ_1 and ϱ_2.

Since $g_\alpha \neq g_\beta$ and $h_\alpha \neq h_\beta$, the construction described above yields four relevant cohomology classes, denoted

$$\kappa(f, g_\alpha, h_\alpha), \quad \kappa(f, g_\alpha, h_\beta), \quad \kappa(f, g_\beta, h_\alpha), \quad \kappa(f, g_\beta, h_\beta) \quad \in H^1(\mathbb{Q}, V_{fgh}(N)), \tag{14}$$

where again $V_{fgh}(N)$ is just a direct sum of several copies of $V_p(E) \otimes V_\varrho \otimes_L L_p$. These classes are called the *generalised Kato classes* attached to the triple (f, g, h) of modular forms of weights $(2, 1, 1)$. Justifying this terminology is the fact that the classes of (14) were first constructed and studied by Kato in the special case where both g and h are weight one Eisenstein series, where they form the basis for his remarkable proof of the Birch and Swinnerton-Dyer conjecture in analytic rank zero for L-functions of elliptic curves twisted by Dirichlet characters.

The Kummer-self-dual representation V_{fgh} is still crystalline, but there is no a priori reason for the classes in (14) to be crystalline at p because the weights $(2, 1, 1)$ of the triple (f, g, h) are *unbalanced*, with f being the eigenform of *dominant weight* in the sense of [DR13] and thus the points in weight space attached to the classes in (14) lie outside the range of "geometric interpolation" defining $\kappa(f, \underline{gh})$.

Theorem 6.4 of §5.2 establishes a reciprocity law relating the image of the classes in (14) under the Bloch-Kato *dual exponential map* to the central critical value $L(E, \varrho, 1)$. It is the main ingredient in the proof of Theorem A.

Theorem C. *The generalised Kato class $\kappa(f, g_\alpha, h_\alpha)$ is crystalline at p if and only if $L(E, \varrho, 1) = 0$.*

When $L(E, \varrho, 1) \neq 0$, Theorem C combined with the regularity of V_g and V_h at p implies that the cohomology classes in (14) are linearly independent, more precisely, that their natural images generate the "singular quotient" of the local cohomology at p. A standard argument involving the global Poitou-Tate exact sequence then implies that the image of the Selmer group of V_{fgh} maps to zero in the local cohomology $H^1(\mathbb{Q}_p, V_{fgh})$ at p. The fact that the Mordell-Weil group of global points always injects into the group of local points then shows the triviality of the ϱ-isotypical component of the Mordell-Weil group of E, thereby proving Theorem A.

One also expects (as a consequence of the Shafarevich-Tate conjecture) that $\mathrm{Sel}_p(E,\varrho)$ is trivial when $E(H)_L^{\varrho}$ is, but the proof of Theorem A described above does not yield the finiteness of the associated Selmer group or Shafarevich-Tate group. The Gross-Schoen diagonal cycles can be viewed as a special instance of a broader class of examples, namely the "Euler systems of Garrett-Rankin-Selberg type" described in [BCDDPR], which also encompass Kato's Euler system of Beilinson elements and the Euler system of Beilinson-Flach elements. In both these settings, finiteness results for the Shafarevich-Tate groups have been obtained by Kato [Ka98] and Lei, Loeffler, and Zerbes [LLZ1] respectively. It would be of great interest to extend these authors' refinements to the setting of diagonal cycles. Recent work of Yifeng Liu makes striking progress in a closely related setting, where diagonal cycles are replaced by twisted variants on the product of a modular curve and a Hilbert modular surface [Liu1] and on a Hilbert modular threefold [Liu2], and the Selmer groups that are studied are those attached to Galois representations of weights $(2,2,2)$ (in the region of "balanced weights", which can thus be treated without recourse to the p-adic deformation of Abel-Jacobi images of cycles).

When $L(E,\varrho,1)=0$, and hence $r_{\mathrm{an}}(E,\varrho)\geq 2$, Theorem C implies that the four classes in (14) belong to the Selmer group $\mathrm{Sel}_p(E,\varrho)$. The submodule of $\mathrm{Sel}_p(E,\varrho)$ generated by these four classes is expected to be non-trivial precisely when $r_{\mathrm{an}}(E,\varrho)=2$, i.e., when $L''(E,\varrho,1)\neq 0$. The following result, whose proof is described in Theorem 6.13 of Section 6.4, involves $\mathscr{L}_p{}^{g_\alpha}(\breve{f},\breve{g}^*,\breve{h})$ rather than $L''(f,g,h,1)$, and directly implies Theorem B above.

Theorem D. *If $L(E,\varrho,1)=0$ and $\mathscr{L}_p{}^{g_\alpha}(\breve{f},\breve{g}^*,\breve{h})\neq 0$ for some choice of test vectors, there exist $G_{\mathbb{Q}}$-equivariant projections $j_\alpha, j_\beta : V_{fgh}(N)\to V_p(E)\otimes V_\varrho$ such that the classes $\kappa_{\alpha\alpha}=j_\alpha(\kappa(f,g_\alpha,h_\alpha))$ and $\kappa_{\alpha\beta}=j_\beta(\kappa(f,g_\alpha,h_\beta))$ are linearly independent in $\mathrm{Sel}_p(E,\varrho)$.*

The canonical nature of the elements in $\mathrm{Sel}_p(E,\varrho)$ described in Theorem D is key to the formulation of the conjectures of [DLR] and [DR15], which seem to admit no direct counterpart for the Selmer classes constructed in [SU, Thm. B]. To further compare Theorem D with the results of [SU], note that generalised Kato classes can only contribute to the pro-p Selmer group $\mathrm{Sel}_p(E/\mathbb{Q})$ of E over $\mathbb{Q}$ when ϱ contains the trivial representation as a constituent, which occurs precisely when g and $h=\bar{g}$ are dual to each other, so that $\varrho = 1\oplus \mathrm{Ad}^0(\varrho_g)$, where $\mathrm{Ad}^0(\varrho_g)$ denotes the three dimensional *adjoint* of ϱ_g, consisting of the trace zero endomorphisms of ϱ_g. In the scenario where $L(E/\mathbb{Q},s)$ admits a double zero at the center, the elliptic Stark conjectures of [DLR] predict that all of the p-adic L-values in (10) vanish and that the generalised Kato classes attached to $(f,g,\bar{g})$ never generate the Selmer group or Mordell-Weil group of E. However, they are expected to generate a one-dimensional subspace of $\mathrm{Sel}_p(E/\mathbb{Q})$ whenever $L(E,\mathrm{Ad}^0(\varrho_g),1)\neq 0$, consisting of the classes whose restriction to $\mathbb{Q}_p$ lies in the kernel of the logarithm map $\log_p : E(\mathbb{Q}_p)\longrightarrow \mathbb{Q}_p$. The non-triviality of such a class only implies that $\dim \mathrm{Sel}_p(E/\mathbb{Q})\geq 1$, but it does imply that $\mathrm{rank}(E(\mathbb{Q}))\geq 2$ under the further (highly non-trivial) assumption that $Ш(E/\mathbb{Q})$ is finite. See [DR15, §4.5] for a more detailed discussion of this intriguing scenario.

Underlying the crucial reciprocity law relating the local behaviour at p of the global classes in (14) to p-adic L-values is a precise relationship between the p-adic families $\kappa(f,\underline{gh})$ of global cohomology classes (more precisely, their restriction to a decomposition group at p) and the Garrett-Hida p-adic L-functions alluded to above. See Theorem 5.3 for the precise statement. Such relationships are part of a long tradition of *explicit reciprocity laws* going back to the seminal work of Coates and Wiles, and even further to the work of Kummer, Iwasawa and Kubota-Leopoldt on the arithmetic of cyclotomic fields. For more on how the approach of this paper fits into the larger perspective of Euler systems and explicit reciprocity laws, notably with Kato's work on Beilinson elements and the work of Bertolini and the authors on Beilinson-Flach elements, the reader is invited to consult the survey [BCDDPR].

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1. Global Λ-adic cohomology classes

1.1. **Cycles, Chow groups, and correspondences.** This section collects a few general facts and notations concerning algebraic cycles and correspondences for future reference.

Given a smooth proper irreducible variety W of dimension d over a field F, and any integer $c \geq 0$, let $\mathrm{C}^c(W)(F)$ denote the group of codimension c cycles in W defined over F with coefficients in $\mathbb{Z}_p$, and let

$$\mathrm{CH}^c(W)(F) = \mathrm{C}^c(W)_{/\sim}$$

denote the Chow group of rational equivalence classes of such cycles. The field F is sometimes supressed from the notation when this leads to no ambiguity.

The standard conventions are employed for Tate twists: if H is a $\mathbb{Z}_p[G_{\mathbb{Q}}]$-module and k is an integer, then $H(k) = H \otimes \epsilon_{\mathrm{cyc}}^k$, where

$$\epsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^\times \tag{15}$$

denotes the p-adic cyclotomic character.

If Δ is a cycle in $\mathrm{C}^c(W)$, the same symbol Δ will often be used as well to denote the class of this cycle in the Chow group $\mathrm{CH}^c(W)$. There is a fundamental *p-adic étale cycle class map* over F

$$\mathrm{cl}_F : \mathrm{CH}^c(W)(F) \longrightarrow H^{2c}_{\mathrm{et}}(W, \mathbb{Z}_p(c)), \tag{16}$$

which is described in [Mi, §23] when F is algebraically closed. For more general fields F (such as number fields), the cycle class map is described in [Ja, eqn. (0.3)], and its target is to be interpreted as the *continuous étale cohomology* of loc.cit. After choosing an embedding of F into an algebraic closure $\bar{F}$, the symbol $\overline{W}$ is used to denote the variety over $\bar{F}$ deduced from W by extension of scalars, and

$$\mathrm{cl} : \mathrm{CH}^c(W)(F) \longrightarrow H^{2c}_{\mathrm{et}}(\overline{W}, \mathbb{Z}_p(c)) \tag{17}$$

denotes the natural map that is deduced from (16). A cycle in $\mathrm{CH}^c(W)$ which is in the kernel of cl is said to be *null-homologous*, and the subgroup of null-homologous cycles defined over F is denoted $\mathrm{CH}^c(W)_0(F)$. The Hochschild-Serre spectral sequence

$$H^i(F, H^j_{\mathrm{et}}(\overline{W}, \mathbb{Z}_p(c)) \Rightarrow H^{i+j}_{\mathrm{et}}(W, \mathbb{Z}_p(c))$$

of [Ja] relating continuous étale cohomology with the continuous group cohomology of $G_F = \mathrm{Gal}\,(\bar{F}/F)$ gives rise to the p-adic étale Abel-Jacobi map

$$\mathrm{AJ}_{\mathrm{et}} : \mathrm{CH}^c(W)_0(F) \longrightarrow H^1(F, H^{2c-1}_{\mathrm{et}}(\bar{W}, \mathbb{Z}_p)(c)), \tag{18}$$

as described, for instance, in the penultimate displayed equation of [Ja]. This map plays a key role in the constructions of the global cohomology classes alluded to in the introduction.

A *correspondence* between two varieties W_1, W_2 of dimension d is an element $T \in \mathrm{Corr}(W_1, W_2) := \mathrm{C}^d(W_1 \times W_2)$. As indicated, the symbol T is used to denote both the cycle and its class in $\mathrm{CH}^d(W_1 \times W_2)$. Let $T^* \in \mathrm{C}^d(W_2 \times W_1)$ denote the transpose correspondence from W_2 to W_1. An element $T \in \mathrm{Corr}(W_1, W_2)$ gives rise to a homomorphism from $\mathrm{CH}^c(W_1)$ to $\mathrm{CH}^c(W_2)$ for any $c \geq 0$ by the rule $T(\Delta) := \pi_{W_2,*}(T \cdot \pi^*_{W_1}(\Delta))$, where $\cdot$ denotes the intersection product on $W_1 \times W_2$, the maps π_{W_1} and π_{W_2} denote the natural projections of $W_1 \times W_2$ to its first and second factors, and $\pi^*_{W_1}$ and $\pi_{W_2,*}$ denote the induced pull-back and proper pushforward maps associated to π_{W_1} and π_{W_2} respectively.

When $W = W_1 = W_2$, set $\mathrm{Corr}(W) := \mathrm{CH}^d(W \times W)$. There is a natural identification

$$H^{2d}(\overline{W} \times \overline{W}, \mathbb{Q}_p)(d) = \oplus_{i=0}^{2d} \mathrm{End}(H^i(\overline{W}, \mathbb{Q}_p))$$

arising from the Künneth decomposition combined with Poincaré duality, and the image of a correspondence $T \in \mathrm{Corr}(W)$ under the map cl of (17) gives rise to a degree 0 endomorphism of the cohomology of W, which is compatible with the action on Chow groups via the cycle class map. Given a correspondence T on W, the same symbol is routinely used to describe this induced endomorphism on Chow groups and on cohomology. The transpose correspondence T^* gives rise to the adjoint endomorphisms on cohomology relative to Poincaré duality.

1.2. **Modular curves.** Let $M \geq 1$ be a positive integer and $X_0(M)$ and $X_1(M)$ denote the classical modular curves over $\mathbb{Q}$ attached to the Hecke congruence subgroups $\Gamma_0(M)$ and $\Gamma_1(M)$, respectively.

Several distinct models over $\mathbb{Q}$ of $X_1(M)$ occur in the literature; in what follows, the affine curve $Y_1(M)$ obtained by removing the cusps from $X_1(M)$ is taken to be the coarse moduli space of pairs (A, i) where A is an elliptic curve and $i : \mu_M \to A$ is an isomorphism between the group scheme μ_M and a closed finite flat subgroup scheme of A. This model of $X_1(M)$ is characterized as being the one whose field of rational functions $\mathbb{Q}(X_1(M))$ is the subfield of $\mathbb{C}(X_1(M))$ given by those functions whose q-expansion at the cusp $i\infty$ have rational coefficients.

Fix an integer $N \geq 3$ and an odd prime $p \nmid N$. For any $s \geq 1$, let $X_s := X_1(Np^s)$. It will be convenient to view X_s as classifying triples (A, i_N, i_p) where $i_N : \mu_N \to A$ and $i_p : \mu_{p^s} \to A$ are embeddings of finite group schemes. It will also be useful to consider the curve $X_s^\dagger/\mathbb{Q}$ which arises as the coarse moduli space associated to the problem of classifying triples (A, i_N, P) where (A, i_N) is as above and $P \in A$ is a point of exact order p^s. The curves X_s and $X_s^\dagger$ are not naturally isomorphic over $\mathbb{Q}$, but are twists of each over the cyclotomic field of p^s-th roots of unity. See below for more details.

The curves X_s are endowed with the following additional structures:

Diamond operators. Let

$$j_s : X_s \longrightarrow X_0(Np^s) \tag{19}$$

be the forgetful map sending a triple (A, i_N, i_p) to (A, C), where $C = \langle \mathrm{Im}(i_N), \mathrm{Im}(i_p)\rangle$ is the subgroup generated by the images of i_N and i_p. It is a Galois cover, whose Galois group is identified with the group

$$G_s^{(N)} = \{\langle d\rangle, d \in (\mathbb{Z}/Np^s\mathbb{Z})^\times\} \simeq (\mathbb{Z}/Np^s\mathbb{Z})^\times/\langle \pm 1\rangle$$

of diamond operators, acting on X_s by the rule $\langle d\rangle(A, i_N, i_p) := (A, d \cdot i_N, d \cdot i_p)$. Let

$$\Gamma_s \subset G_s \subset G_s^{(N)}, \qquad \Gamma_s = 1 + p(\mathbb{Z}/p^s\mathbb{Z}), \quad G_s = (\mathbb{Z}/p^s\mathbb{Z})^\times$$

be the natural subgroups, and write

$$\Lambda_s := \mathbb{Z}_p[\Gamma_s], \qquad \tilde{\Lambda}_s := \mathbb{Z}_p[G_s], \qquad \tilde{\Lambda}_s^{(N)} := \mathbb{Z}_p[G_s^{(N)}] \tag{20}$$

for the associated group rings with $\mathbb{Z}_p$-coefficients. The corresponding completed group rings are

$$\Lambda := \varprojlim \Lambda_s, \qquad \tilde{\Lambda} := \varprojlim \tilde{\Lambda}_s, \qquad \tilde{\Lambda}^{(N)} := \varprojlim \tilde{\Lambda}_s^{(N)}. \tag{21}$$

All of the group rings in (20) and (21) act naturally on spaces of modular forms on $\Gamma_1(Np^s)$ as well as on the various cohomology groups attached to X_s.

Given $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ and $b \in (\mathbb{Z}/p^s\mathbb{Z})^\times$, it will sometimes be convenient to denote by $\langle a; b\rangle$ the diamond operator $\langle d\rangle$, where $d \in (\mathbb{Z}/Np^s\mathbb{Z})^\times$ is congruent to a modulo N and to b modulo p^s. This automorphism acts on X_s by the rule $\langle a; b\rangle(A, i_N, i_p) = (A, a \cdot i_N, b \cdot i_p)$.

Let ζ_s be a primitive p^s-th root of unity and let $\epsilon_s : G_{\mathbb{Q}} \longrightarrow (\mathbb{Z}/p^s\mathbb{Z})^\times$ denote the mod p^s cyclotomic character, factoring through $\mathrm{Gal}\,(\mathbb{Q}(\zeta_s)/\mathbb{Q})$. The curve $X_s^\dagger$ introduced above is isomorphic to the twist of X_s by the cocycle

$$\langle \epsilon_s^{-1}\rangle : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(X_s), \quad \langle \epsilon_s^{-1}\rangle(\sigma) := \langle 1; \epsilon_s^{-1}(\sigma)\rangle, \tag{22}$$

and therefore X_s is isomorphic to $X_s^\dagger$ over $\mathbb{Q}(\zeta_s)$.

Duality. The (étale and de Rham) cohomology groups of X_s are endowed with perfect pairings

$$\langle\ ,\ \rangle_s : H^1_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p) \times H^1_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p(-1), \qquad H^1_{\mathrm{dR}}(X_s/\mathbb{Q}) \times H^1_{\mathrm{dR}}(X_s/\mathbb{Q}) \longrightarrow \mathbb{Q} \tag{23}$$

arising from Poincaré duality. The symbol $\langle\ ,\ \rangle_s$ is used to denote both the étale and de Rham Poincaré pairing, relying on the context will make it clear which is being used. The de Rham pairing is defined on classes of differentials of the second kind by the usual formula

$$\langle \omega_1, \omega_2\rangle_s := \sum_{P \in X_s} \mathrm{res}_P(F_{\omega_1,P} \cdot \omega_2),$$

where the sum is taken over the closed points P of X_s, and $F_{\omega_1,P}$ denotes a local primitive of ω_1 at P.

Hecke operators. There is a commuting algebra of Hecke correspondences T_n indexed by the integers prime to Np (defined for instance as in [DS05, Ch. 5]). For all $s \geq 1$, write $\mathbb{T}_0(Np^s)$ and $\mathbb{T}_1(Np^s)$ for the $\mathbb{Q}$-algebra generated by these Hecke operators, together with the diamond operators in the latter case, in the ring of correspondences of the curves $X_0(Np^s)$ and X_s respectively. These "good" Hecke operators commute with their adjoints relative to the pairings $\langle\ ,\ \rangle_s$ and, more generally, relative to the Poincaré duality on any modular curve. More precisely,

$$T_n^* = \langle n\rangle T_n, \qquad \text{for all } n \text{ with } \gcd(n, Np) = 1. \tag{24}$$

Atkin-Lehner automorphisms. Fix a norm-compatible collection $\{\zeta_s\}_{s\geq 1}$ of primitive roots of unity of order p^s. Associated to ζ_s is the Atkin-Lehner automorphism denoted w_s which is defined on triples (A, i_N, i_p) by the rule

$$w_s(A, i_N, i_p) = (A/C_p, i_N, i'_p), \qquad C_p := \langle i_p(\zeta_s)\rangle,$$

where $i'_p(\zeta_s) = x' \pmod{C_p}$ and x' is a point on A of order p^s satisfying $\langle i(\zeta_s), x'\rangle_{\mathrm{Weil}} = \zeta_s$. Likewise, the choice of a primitive N-th root of unity ζ_N gives rise to a similar Atkin-Lehner operation at N, which shall be denoted w, and is defined on any of the curves X_s. The automorphisms w_s and w do not commute with the diamond and Hecke operators, and instead satisfy

$$w_s\langle a; b\rangle = \langle a; b^{-1}\rangle w_s, \qquad w_s T_n = \langle 1; n\rangle T_n w_s, \qquad w_s^2 = \langle -p^s; 1\rangle, \tag{25}$$

$$w\langle a; b\rangle = \langle a^{-1}; b\rangle w, \qquad wT_n = \langle n; 1\rangle T_n w, \qquad w^2 = \langle 1; -N\rangle. \tag{26}$$

Moreover, the operators w_s are not defined over $\mathbb{Q}$, but only over $\mathbb{Q}(\zeta_s)$ and $\mathbb{Q}(\zeta_N)$ respectively. More precisely, for all $\sigma \in G_{\mathbb{Q}}$,

$$(w_s)^\sigma = \langle 1; \epsilon_s^{-1}(\sigma)\rangle\, w_s, \qquad (w)^\sigma = \langle \epsilon_N^{-1}(\sigma); 1\rangle\, w, \tag{27}$$

where ϵ_N denotes the mod N cyclotomic character. The actions of w and w_s on differentials, functions, and de Rham cohomology via pullback will commonly be denoted by w and w_s rather than by the more strictly correct (but more notationally cumbersome) w^* and w_s^*.

The U_p operator. The Hecke operator U_p on X_s is realised by a correspondence of bi-degree p, which is given on geometric points by the rule

$$U_p(A, i_N, i_p) = \sum_{\varphi: A\longrightarrow A'} (A', \varphi\circ i_N, \varphi\circ i_p), \tag{28}$$

where the sum is taken over the p distinct isogenies $\varphi : A \longrightarrow A'$ of degree p whose kernel is not equal to $p^{s-1}\cdot \mathrm{Im}(i_p)$. The Hecke operator U_p acts on modular forms of weight two and its action on differentials of the form $f(\tau)d\tau = \sum_{n\geq 1} a_n q^n \frac{dq}{q}$ is given by the familiar rule

$$U_p(f(\tau)d\tau) = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{t+j}{p}\right) d\tau = \sum_{n\geq 1} a_{np}q^n. \tag{29}$$

Note that while U_p commutes with the diamond operators, it does not commute with its adjoint U_p^* relative to the Poincaré pairings $\langle\ ,\ \rangle_s$, which is given by the formula

$$U_p^* = w_s U_p w_s^{-1}.$$

A modular form (or cohomology class) ϕ on X_s is said to be *ordinary* if there exists a polynomial $P(T) \in \bar{\mathbb{Q}}_p[T]$ such that $P(U_p)(\phi) = 0$ and $P(0)$ is a p-adic unit. If instead ϕ has a similar property relative to U_p^*, it is designated as *anti-ordinary.*

The ordinary and anti-ordinary projectors, which play a key role in the constructions of this article, are defined as

$$e_{\mathrm{ord}} := \lim_{n\to\infty} U_p^{n!}, \qquad e^*_{\mathrm{ord}} := \lim_{n\to\infty} (U_p^*)^{n!}. \tag{30}$$

Projective systems: The curves X_s form projective systems in two different ways, relative to the collections of degeneracy maps

$$\varpi_1, \varpi_2 : X_{s+1} \to X_s, \qquad \left\{\begin{array}{l} \varpi_1(A, i_N, i_p) = (A, i_N, p\cdot i_p), \\ \varpi_2(A, i_N, i_p) = (A, i_N, i_p)/C, \quad C := i_p(\mu_p), \end{array}\right. \tag{31}$$

where the quotient of the triple (A, i_N, i_p) by the subgroup scheme C is to be understood in the obvious sense. The degeneracy maps ϖ_1 and ϖ_2 are of degree p^2 if $s \geq 1$, and of degree $(p^2 - 1)$ if $s = 0$. With respect to the analytic uniformisation by the upper-half plane, the map ϖ_1 corresponds to the natural map $\Gamma_{s+1}\cdot\tau \mapsto \Gamma_s\cdot\tau$, while ϖ_2 corresponds to $\Gamma_{s+1}\cdot\tau\mapsto\Gamma_s\cdot(p\tau)$. Both maps commute with the good Hecke operators and the diamond operators, and factor through the natural projection $\mu : X_{s+1}\longrightarrow X_s^\flat$, where $X_s^\flat$ denotes the modular curve associated to $\Gamma_1(Np^s)\cap\Gamma_0(p^{s+1})$. More precisely, they can be written as $\varpi_1=\pi_1\circ\mu$, $\varpi_2=\pi_2\circ\mu$, and fit into the diagram below in which the vertical map is a Galois cover (with Galois group $\mathbb{Z}/p\mathbb{Z}$, generated by suitable diamond operators) and the two horizontal maps are non-Galois morphisms of degree p:

$$\begin{tikzcd} X_{s+1} \arrow[d, "\mu"'] \arrow[dr, "\varpi_1,\varpi_2"] & \\ X_s^\flat \arrow[r, "\pi_1,\pi_2"'] & X_s. \end{tikzcd} \tag{32}$$

The Hecke operator U_p acting on $\Omega_1(X_s)$ or on $H^1_{\mathrm{dR}}(X_s)$ is described by the standard formula

$$U_p=(\pi_2)_*\pi_1^*, \tag{33}$$

as can be verified directly from the formula for U_p in terms of the coordinate τ given in (29) (cf. also, for example, [DS05, Ex. 7.9.3]). It follows from this that

$$(\varpi_2)_*\varpi_1^*=(\pi_2)_*\mu_*\mu^*\pi_1^*=pU_p. \tag{34}$$

The endomorphism in (34) is the diagonal arrow in the commutative diagram:

$$\begin{tikzcd} H^1_{\mathrm{dR}}(X_s) \arrow[r, "\varpi_1^*"] \arrow[d, "(\varpi_2)_*"'] \arrow[dr, "pU_p"] & H^1_{\mathrm{dR}}(X_{s+1}) \arrow[d, "(\varpi_2)_*"] , \\ H^1_{\mathrm{dR}}(X_{s-1}) \arrow[r, "\varpi_1^*"'] & H^1_{\mathrm{dR}}(X_s) \end{tikzcd}$$

and it is apparent from this description that π_1^*, ϖ_1^*, $(\pi_2)_*$ and $(\varpi_2)_*$ commute with U_p, while π_2^*, ϖ_2^*, $(\pi_1)_*$ and $(\varpi_1)_*$ commute with the adjoint U_p^*. In particular, the Hecke operator U_p acts naturally on the inverse limit of the (étale or de Rham) cohomology groups $H^1(X_s)$ taken relative to the maps $(\varpi_2)_*$. Whenever inverse limits of $H^1(X_s)$ are discussed, it should always be understood that the transition maps are given by the pushforward maps $(\varpi_2)_*$.

Group ring-valued pairings. The action of the diamond operators on X_s can be combined with Poincaré duality to define group-ring valued pairings

$$\langle\ ,\ \rangle_{\Gamma_s} : H^1_{\mathrm{et}}(\bar X_s,\mathbb{Z}_p)\times H^1_{\mathrm{et}}(\bar X_s,\mathbb{Z}_p)\longrightarrow\mathbb{Z}_p(-1)[\Gamma_s],\quad H^1_{\mathrm{dR}}(X_s/\mathbb{Q})\times H^1_{\mathrm{dR}}(X_s/\mathbb{Q})\longrightarrow\mathbb{Q}[\Gamma_s] \tag{35}$$

by the formula

$$\langle a,b\rangle_{\Gamma_s}=\sum_{\sigma\in\Gamma_s}\langle a^\sigma,b\rangle_s\cdot\sigma^{-1}=\sum_{\sigma\in\Gamma_s}\langle a,b^\sigma\rangle_s\cdot\sigma.$$

As before, the same symbols are reserved for the pairings arising from Poincaré duality on de Rham or étale cohomology, leaving it to the context to determine which is being used. The pairings $\langle\ ,\rangle_{\Gamma_s}$ are Λ_s-linear and anti-linear (with respect to the involution sending a group-like element to its inverse) in the first and second argument respectively.

The pairings $\langle\ ,\rangle_{\Gamma_s}$ and $\langle\ ,\rangle_s$ are related by the formula, valid for all characters ϵ of Γ_s,

$$\epsilon(\langle a,b\rangle_{\Gamma_s})=\langle\theta_\epsilon a,b\rangle_s=\langle a,\theta_{\epsilon^{-1}}b\rangle_s=d_s^{-1}\langle\theta_\epsilon a,\theta_{\epsilon^{-1}}b\rangle_s, \tag{36}$$

where

$$\theta_\epsilon:=\sum_{\sigma\in\Gamma_s}\epsilon(\sigma)\cdot\sigma^{-1}\in\mathbb{C}_p[\Gamma_s]. \tag{37}$$

The first identity in (36) is a direct consequence of the definitions, while the second follows from the identity $\theta_\epsilon^2=d_s\theta_\epsilon$. A convenient modification of the $\langle\ ,\ \rangle_{\Gamma_s}$ pairings (in both their étale and de Rham incarnations) is given by setting

$$[a,b]_{\Gamma_s}:=\langle a,ww_s\cdot U_p^s\cdot b\rangle_{\Gamma_s}. \tag{38}$$

All the Hecke operators on X_s, including U_p, are self-adjoint relative to this modified pairing, which is also Λ_s-linear relative to both its arguments. In addition, the pairings $[\ ,\ ]_{\Gamma_s}$ have the virtue of being compatible with the maps ϖ_{2*} and the ring homomorphisms

$$p_{s+1} : \mathbb{C}_p[\Gamma_{s+1}] \longrightarrow \mathbb{C}_p[\Gamma_s]$$

induced from the natural homomorphism $\Gamma_{s+1} \longrightarrow \Gamma_s$, in the following sense:

Lemma 1.1. *For either* $\square \in \{\mathrm{et}, \mathrm{dR}\}$*, the diagram below commutes:*

$$\begin{tikzcd} H^1_\square(X_{s+1}) \times H^1_\square(X_{s+1}) \arrow[r, "{[\ ,\ ]_{\Gamma_{s+1}}}"] \arrow[d, "\varpi_{2*}\times\varpi_{2*}"'] & \mathbb{C}_p[\Gamma_{s+1}] \arrow[d, "p_{s+1}"] \\ H^1_\square(X_s) \times H^1_\square(X_s) \arrow[r, "{[\ ,\ ]_{\Gamma_s}}"] & \mathbb{C}_p[\Gamma_s] \end{tikzcd}$$

Proof. This follows from a direct calculation (in which the symbol $\langle\ ,\ \rangle_{X_s^\flat}$ refers to the Poincaré pairing on the curve $X_s^\flat$):

$$\begin{aligned} p_{s+1}([\eta_{s+1}, \phi_{s+1}]_{\Gamma_{s+1}}) &= \sum_{\sigma\in\Gamma_s} \langle \sigma\mu_*\eta_{s+1}, \mu_* w w_{s+1} U_p^{s+1}\phi_{s+1}\rangle_{X_s^\flat} \cdot \sigma^{-1} \\ &= \sum_{\sigma\in\Gamma_s} \langle \sigma\mu_*\eta_{s+1}, w w_{s+1} U_p^{s}\pi_1^*\pi_{2*}\mu_*\phi_{s+1}\rangle_{X_s^\flat} \cdot \sigma^{-1} \\ &= \sum_{\sigma\in\Gamma_s} \langle \sigma\mu_*\eta_{s+1}, \pi_2^* w w_s U_p^s\phi_s\rangle_{X_s^\flat} \cdot \sigma^{-1} = \sum_{\sigma\in\Gamma_s} \langle \sigma\pi_{2*}\mu_*\eta_{s+1}, w w_s U_p^s \phi_s\rangle_{X_s} \cdot \sigma^{-1} \\ &= \sum_{\sigma\in\Gamma_s} \langle \sigma\eta_s, w w_s U_p^s\phi_s\rangle_{X_s} \cdot \sigma^{-1} = [\eta_s, \phi_s]_{\Gamma_s}. \end{aligned}$$

$\square$

Define

$$H^1_\square(X_\infty) := \varprojlim_s H^1_\square(X_s),$$

the inverse limit being taken relative to the maps $(\varpi_2)_*$. Lemma 1.1 shows that the pairings $[\ ,\ ]_{\Gamma_s}$ can be packaged into Λ-adic pairings with values in the appropriate completed group rings

$$[\ ,\ ]_\Gamma : H^1_{\mathrm{dR}}(X_\infty) \times H^1_{\mathrm{dR}}(X_\infty) \longrightarrow \mathbb{C}_p[[\Gamma]], \tag{39}$$

$$[\ ,\ ]_\Gamma : H^1_{\mathrm{et}}(X_\infty, \mathbb{Z}_p) \times H^1_{\mathrm{et}}(X_\infty, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p[[\Gamma]](-1). \tag{40}$$

The étale incarnation of $[\ ,\ ]_\Gamma$ is even $G_\mathbb{Q}$-equivariant, after endowing $\mathbb{Z}_p[[\Gamma]]$ with the tautological $G_\mathbb{Q}$-action whereby $\sigma \in G_\mathbb{Q}$ acts as multiplication by the group-like element $\epsilon_{\mathrm{cyc}}(\sigma) \in \mathbb{Z}_p^\times$.

1.3. **Twisted diagonal cycles.** For any integer $s \geq 1$ define

$$W_{s,s} := X_0(Np) \times X_1(Np^s) \times X_1(Np^s) = X_0(Np) \times X_s \times X_s.$$

This three-fold is equipped with a natural action of the group $G_s^{(N)} \times G_s^{(N)}$ via the diamond operators acting on the second and third factors. Let

$$\delta : X_s \longrightarrow X_s \times X_s \times X_s, \qquad X_s^\flat \longrightarrow X_s^\flat \times X_s^\flat \times X_s^\flat$$

denote the natural diagonal embeddings.

Fix a system $\{\zeta_s\}$ of compatible p^s-th roots of unity and set

$${}^\flat\Delta_{s,s} := (j_1 \circ \varpi_2^{s-1} \circ w_s, \mathrm{Id}, w_s)_* \delta_*(X_s) \in \mathrm{C}^2(W_{s,s})(\mathbb{Q}(\zeta_s)), \tag{41}$$

where $\varpi_2^{s-1} := \varpi_2 \circ \cdots \circ \varpi_2 : X_s \to X_1$ and $j_1 : X_1 \to X_0(Np)$ are the maps introduced in (31) and (19) respectively. This codimension two cycle is just a copy of the curve X_s embedded diagonally in the three-fold product X_s^3, but with a twist by w_s in the first and third factors, projected to $W_{s,s}$ in the natural way. It is the twist by w_s which causes ${}^\flat\Delta_{s,s}$ to only be defined over $\mathbb{Q}(\zeta_s)$ in general. The cycle ${}^\flat\Delta_{s,s}$ will be referred to as the *twisted diagonal* of level s. Let

$$(\mathrm{Id}, U_p, \mathrm{Id}) : \mathrm{C}^2(W_{s,s}) \longrightarrow \mathrm{C}^2(W_{s,s})$$

be the endomorphism induced by the correspondence U_p acting on the middle factor of the triple product $W_{s,s}$ and whose action on cycles is spelled out in the proof of Proposition 1.2 below. The correspondence $(\mathrm{Id}, U_p, \mathrm{Id})$ is realised as a copy of $X_0(Np) \times X_s^\flat \times X_s$ embedded in $W_{s,s}^2$ via the map

$$(\mathrm{Id}_1, \pi_1, \mathrm{Id}_3, \mathrm{Id}_1, \pi_2, \mathrm{Id}_3) : X_0(Np) \times X_s^\flat \times X_s \longrightarrow (X_0(Np) \times X_s \times X_s) \times (X_0(Np) \times X_s \times X_s),$$

where Id_1 and Id_3 refer to the identity maps on the first and third factors respectively in the direct product, and $\pi_1, \pi_2 : X_s^\flat \longrightarrow X_s$ are defined on the second factor. A direct calculation using the definition of the action of correspondences shows that

$$(\mathrm{Id}, U_p, \mathrm{Id})_* \delta_*(X_s) = (\pi_1, \pi_2, \pi_1)_* \delta_*(X_s^\flat), \qquad (\mathrm{Id}, U_p^*, \mathrm{Id})_* \delta_*(X_s) = (\pi_2, \pi_1, \pi_2)_* \delta_*(X_s^\flat). \tag{42}$$

The next proposition studies the compatibilities of the cycles ${}^\flat\Delta_{s,s}$ for varying s under the maps

$$\varpi_{22} = (\mathrm{Id},\ \varpi_2,\ \varpi_2) : W_{s+1,s+1} \longrightarrow W_{s,s}, \qquad s \geq 1 \tag{43}$$

induced by the degeneracy maps of (31).

Proposition 1.2. *For all $s \geq 1$,* $\quad \varpi_{22,*}({}^\flat\Delta_{s+1,s+1}) = p \cdot (\mathrm{Id}, U_p, \mathrm{Id})_*({}^\flat\Delta_{s,s})$ *in* $\mathrm{C}^2(W_{s,s})$.

Proof. The commutativity of the diagram

$$\begin{tikzcd}
X_{s+1} \arrow[r, "\delta"] \arrow[d, "\mu"'] & X_{s+1} \times X_{s+1} \times X_{s+1} \arrow[rr, "{(j_1\varpi_2^s w_{s+1},\ \mathrm{Id},\ w_{s+1})}"] \arrow[d, "{(\mu,\mu,\mu)}"'] & & X_0(Np) \times X_{s+1} \times X_{s+1} \arrow[d, "{(\mathrm{Id},\mu,\mu)}"] \\
X_s^\flat \arrow[r, "\delta"] & X_s^\flat \times X_s^\flat \times X_s^\flat \arrow[rr, "{(j_1\varpi_2^{s-1}\pi_2 w_{s+1},\ \mathrm{Id},\ w_{s+1})}"] \arrow[d, "{(\pi_1,\pi_2,\pi_1)}"'] & & X_0(Np) \times X_s^\flat \times X_s^\flat \arrow[d, "{(\mathrm{Id},\pi_2,\pi_2)}"] \\
& X_s \times X_s \times X_s \arrow[rr, "{(j_1\varpi_2^{s-1} w_s,\ \mathrm{Id},\ w_s)}"] & & X_0(Np) \times X_s \times X_s,
\end{tikzcd}$$

in which the composition of the rightmost vertical arrows is the map ϖ_{22}, implies that

$$\begin{aligned}
\varpi_{22*}({}^\flat\Delta_{s+1,s+1}) &= (\mathrm{Id}, \pi_2, \pi_2)_* \circ (\mathrm{Id}, \mu, \mu)_* \circ (j_1\varpi_2^s w_{s+1}, \mathrm{Id}, w_{s+1})_* \circ \delta_*(X_{s+1}) \\
&= (j_1\varpi_2^{s-1} w_s, \mathrm{Id}, w_s)_* \circ (\pi_1, \pi_2, \pi_1)_* \circ \delta_* \circ \mu_*(X_{s+1}) \\
&= p \cdot (j_1\varpi_2^{s-1} w_s, \mathrm{Id}, w_s)_* \circ (\pi_1, \pi_2, \pi_1)_* \circ \delta_*(X_s^\flat) \\
&= p \cdot (j_1\varpi_2^{s-1} w_s, \mathrm{Id}, w_s)_* \circ (\mathrm{Id}, U_p, \mathrm{Id})_* \delta_*(X_s),
\end{aligned}$$

where the last equality follows from (42). It follows that

$$\begin{aligned}
\varpi_{22*}({}^\flat\Delta_{s+1,s+1}) &= p \cdot (\mathrm{Id}, U_p, \mathrm{Id})_* (j_1\varpi_2^{s-1} w_s, \mathrm{Id}, w_s)_* \circ \delta_*(X_s) \\
&= p \cdot (\mathrm{Id}, U_p, \mathrm{Id})_*({}^\flat\Delta_{s,s}),
\end{aligned}$$

as was to be shown. □

Proposition 1.2 shows that, after *formally* regularizing the cycles ${}^\flat\Delta_{s,s}$ by the rule

$${}^\flat\Delta_{s,s}^{\mathrm{reg}} := \frac{1}{p^s} \cdot (\mathrm{Id}, U_p, \mathrm{Id})^{-s}({}^\flat\Delta_{s,s}), \tag{44}$$

the family $\{{}^\flat\Delta_{s,s}^{\mathrm{reg}}\}$ becomes part of a *projective system* under the push-forward maps $\varpi_{22,*}$. The factor $\frac{1}{p^s}$ arising in (44) is problematic, introducing unwanted denominators which present a genuine obstruction to piecing the cycles ${}^\flat\Delta_{s,s}$ into a "two-variable p-adic family". This is what motivates the passage to a suitable quotient of $W_{s,s}$ by a group of automorphisms of (essentially) p-power order, which will now be explained.

Given $s \geq 1$, let D_s denote the "diagonal at N" and "anti-diagonal at p" subgroup of $G_s^{(N)} \times G_s^{(N)}$ consisting of elements of the form $(\langle a; b\rangle, \langle a; b^{-1}\rangle)$ with $(a, b) \in (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/p^s\mathbb{Z})^\times$, and let

$$\mathrm{pr}_s : W_{s,s} \longrightarrow W_s := W_{s,s}/D_s$$

be the natural projection from $W_{s,s}$ to its quotient by the action of this group.

By (25), the action of D_s on $W_{s,s}$ preserves the cycle ${}^\flat\Delta_{s,s}$. Identifying ${}^\flat\Delta_{s,s}$ with the curve X_s, the element $(\langle a; b\rangle, \langle a; b^{-1}\rangle) \in D_s$ acts as the diamond operator $\langle a; b\rangle$ on this curve. Let ${}^\flat\Delta_s \in$

$\mathrm{C}^2(W_s)(\mathbb{Q}(\zeta_s))$ be the natural image of ${}^\flat\Delta_{s,s}$ in the quotient W_s. Since pr_s induces a morphism of degree $d_s := \frac{1}{2}\varphi(N)(p-1)p^{s-1}$ from ${}^\flat\Delta_{s,s}$ to ${}^\flat\Delta_s$, the latter cycle is defined by the relation

$$(\mathrm{pr}_s)_*({}^\flat\Delta_{s,s}) =: d_s \cdot ({}^\flat\Delta_s) \text{ in } \mathrm{C}^2(W_s)(\mathbb{Q}(\zeta_s)). \tag{45}$$

Note that the curve ${}^\flat\Delta_s$ is isomorphic to an embedded copy of $X_0(Np^s)$ in the three-fold W_s, but that the corresponding closed immersion $X_0(Np^s) \simeq {}^\flat\Delta_s \hookrightarrow W_s$ is only defined over $\mathbb{Q}(\zeta_s)$.

By a slight abuse of notation, let

$$\varpi_{11}, \varpi_{22} : W_{s+1} = W_{s+1,s+1}/D_{s+1} \longrightarrow W_s = W_{s,s}/D_s \tag{46}$$

denote the natural projection map "from level $s+1$ to level s" induced from the maps ϖ_{11} and ϖ_{22} above, by passing to the quotient via pr_{s+1} and pr_s.

Proposition 1.3. *For all $s \geq 1$,* $\quad \varpi_{22,*}({}^\flat\Delta_{s+1}) = (\mathrm{Id}, U_p, \mathrm{Id})_*({}^\flat\Delta_s)$.

Proof. Proposition 1.2 asserts that $\varpi_{22,*}({}^\flat\Delta_{s+1,s+1}) = p \cdot (\mathrm{Id}, U_p, \mathrm{Id})_*({}^\flat\Delta_{s,s})$ in $\mathrm{C}^2(W_{s,s})$. Applying the pushforward map induced by $\mathrm{pr}_s : W_{s,s} \longrightarrow W_s$ to this identity, using the fact that $\mathrm{pr}_s \circ \varpi_{22} = \varpi_{22} \circ \mathrm{pr}_{s+1}$, and that pr_s also commutes with $(\mathrm{Id}, U_p, \mathrm{Id})$, one obtains

$$\varpi_{22,*}\,\mathrm{pr}_{s+1,*}({}^\flat\Delta_{s+1,s+1}) = p \cdot (\mathrm{Id}, U_p, \mathrm{Id})\,\mathrm{pr}_{s,*}({}^\flat\Delta_{s,s}).$$

The sought-for equality then follows from (45), because $\mathrm{C}^2(W_s)$ is torsion free. □

It would also have been possible to adapt the proof of Proposition 1.2 to the cycles ${}^\flat\Delta_s$ on the quotients W_s to show directly that Proposition 1.3 holds at the level of cycles. In reducing Proposition 1.3 to Proposition 1.2, it was expedient to work with the groups $\mathrm{C}^2(W_{s,s})$ and $\mathrm{C}^2(W_s)$ of codimension 2 cycles in $W_{s,s}$ and W_s, rather than the Chow groups $\mathrm{CH}^2(W_{s,s})$ and $\mathrm{CH}^2(W_s)$ of rational equivalence classes, because the latter are not necessarily torsion free. This is no longer necessary, and from now on ${}^\flat\Delta_{s,s}$ and ${}^\flat\Delta_s$ will be identified with their classes in $\mathrm{CH}^2(W_{s,s})$ and $\mathrm{CH}^2(W_s)$, respectively.

These cycle classes can be made null-homologous by applying to them a suitable correspondence, denoted

$$\varepsilon_{s,s} := (\varepsilon, \varepsilon_s, \varepsilon_s) : X_0(Np) \times X_s \times X_s, \tag{47}$$

where

- ε is a correspondence on $X_0(Np)$ with coefficients in $\mathbb{Z}_p$ which annihilates $H^0(X_0(Np))$ and $H^2(X_0(Np))$ but fixes the f-isotypic subspace $H^1(X_0(Np))[f]$ of the introduction (defined as an eigenspace only for the good Hecke operators T_ℓ with $\ell \nmid Np$). To fix ideas, recall that the prime p is assumed to be a non-Eisenstein prime for f, meaning that f is not congruent to an Eisenstein series modulo p. There is therefore an auxiliary prime $\ell \nmid Np$ for which $\ell + 1 - a_\ell(f)$ lies in $\mathbb{Z}_p^\times$. Set
$$\varepsilon := (\ell + 1 - T_\ell)/(\ell + 1 - a_\ell(f)).$$
- ε_s is the correspondence on X_s defined by setting $\varepsilon_s := 2^{-1} \cdot (1 - \langle -1; 1 \rangle)$, where the standard convention has been adopted of identifying a morphism $\varphi : V \longrightarrow W$ between varieties V and W with the correspondence on $V \times W$ associated to its graph.

The *modified twisted diagonal cycle* of level s is then defined by

$$\Delta_{s,s} := \varepsilon_{s,s}({}^\flat\Delta_{s,s}) \in \mathrm{CH}^2(W_{s,s}). \tag{48}$$

Proposition 1.4. *For all $s \geq 1$, the class of $\Delta_{s,s}$ lies in* $\mathrm{CH}^2(W_{s,s})_0$.

Proof. In order to lighten the notations in the proof, write $H^i(V)$ as a shorthand for the étale cohomology $H^i_{\mathrm{et}}(\overline{V}, \mathbb{Z}_p)$ of V with coefficients in $\mathbb{Z}_p$, and write $W_{s,s} = C_1 \times C_2 \times C_3$. This variety is a product of curves, whose integral cohomology is torsion-free. By the Künneth decomposition theorem (cf. [Mi, Thm. 22.4]) $H^4(W_{s,s})$ can therefore be expressed as a sum of three terms

$$H^2(C_1) \otimes H^2(C_2 \times C_3) \;+\; H^2(C_2) \otimes H^2(C_1 \times C_3) \;+\; H^2(C_3) \otimes H^2(C_1 \times C_2).$$

By (48), $\mathrm{cl}(\Delta_{s,s}) = \varepsilon_{s,s}(\mathrm{cl}({}^\flat\Delta_{s,s}))$. The correspondence ε annihilates the first term in the above sum, while ε_s annihilates the second and third term, since it annihilates $H^2(X_s)$. It follows that $\varepsilon_{s,s}$ annihilates the target of the cycle class map and hence *a fortiori* that $\Delta_{s,s}$ is null-homologous. □

Remark 1.5. The element $\varepsilon_{s,s}$ has the virtue of annihilating $H^4_{\mathrm{dR}}(W_{s,s})$ while fixing much of the interesting part of the middle cohomology $H^3_{\mathrm{dR}}(W_{s,s})$. More precisely, if $\eta_{\breve{f}}$ is any class in $H^1_{\mathrm{dR}}(X_0(Np))[f]$, and $\eta_{\breve{g}} \in H^1_{\mathrm{dR}}(X_s)[g]$ and $\eta_{\breve{h}} \in H^1_{\mathrm{dR}}(X_s)[h]$ are classes in the isotypic subspaces attached to eigenforms g and h with nebentypus characters whose prime-to-p parts are the odd Dirichlet characters χ and χ^{-1} respectively, then $\varepsilon_{s,s}(\eta_{\breve{f}} \otimes \eta_{\breve{g}} \otimes \eta_{\breve{h}}) = \eta_{\breve{f}} \otimes \eta_{\breve{g}} \otimes \eta_{\breve{h}}$.

Recall that the group $G_s^{(N)} \times G_s^{(N)}$ acts freely on $W_{s,s}$, and that this action descends to an action on W_s factoring through the quotient $(G_s^{(N)} \times G_s^{(N)})/D_s \simeq G_s^{(N)}$. More generally, the linear endomorphism of $\mathrm{CH}^*(W_{s,s})$ induced by any element $\varepsilon \in \mathbb{Z}[G_s^{(N)} \times G_s^{(N)}]$ descends to a linear map on $\mathrm{CH}^*(W_s)$ which shall be denoted by the same symbol. With these notations, letting Δ_s denote the natural image of $\Delta_{s,s}$ in the quotient W_s, the classes of the cycles Δ_s and ${}^\flat\Delta_s$ are related by

$$\Delta_s = \varepsilon_{s,s}({}^\flat\Delta_s) \in \mathrm{CH}^2(W_s)_0(\mathbb{Q}(\zeta_s)). \tag{49}$$

Moreover, similarly as above it again holds that

$$(\mathrm{pr}_s)_*(\Delta_{s,s}) = d_s \cdot \Delta_s \quad \text{ and } \quad \varpi_{22,*}(\Delta_{s+1}) = (\mathrm{Id}, U_p, \mathrm{Id})_*(\Delta_s). \tag{50}$$

The first equality in (50) follows exactly as in (45), while the second is a consequence of Proposition 1.3 because the correspondences $\varepsilon_{s+1,s+1}$ and $\varepsilon_{s,s}$ commute with the diamond operators, with $(\mathrm{Id}, U_p, \mathrm{Id})$, and with the projections $\varpi_{22,*}$ in the sense that $\varepsilon_{s,s} \circ \varpi_{22,*} = \varpi_{22,*} \circ \varepsilon_{s+1,s+1}$.

The cycles $\Delta_{s,s}$ and Δ_s are only defined over $\mathbb{Q}(\zeta_s)$; the following proposition describes the action of the Galois group $\mathrm{Gal}\,(\mathbb{Q}(\zeta_s)/\mathbb{Q}) = (\mathbb{Z}/p^s\mathbb{Z})^\times$ on these cycles.

Proposition 1.6. *For all $\sigma \in G_{\mathbb{Q}}$, $\sigma(\Delta_{s,s}) = (\mathrm{Id}, \mathrm{Id}, \langle 1; \epsilon_s^{-1}(\sigma)\rangle)(\Delta_{s,s})$.*

Proof. Equation (27) implies that

$$\sigma({}^\flat\Delta_{s,s}) = (\mathrm{Id}, \mathrm{Id}, \langle 1; \epsilon_s^{-1}(\sigma)\rangle)({}^\flat\Delta_{s,s}). \tag{51}$$

Note the the correspondence $\varepsilon_{s,s}$ is defined over $\mathbb{Q}$ and hence commutes with the action of $G_{\mathbb{Q}}$ on cycles. It also commutes with the diamond operators and hence with the morphism $(\mathrm{Id}, \mathrm{Id}, \langle 1; \epsilon_s^{-1}(\sigma)\rangle)$. Applying it to (51) and invoking equation (48), one obtains that $\sigma(\Delta_{s,s}) = (\mathrm{Id}, \mathrm{Id}, \langle 1; \epsilon_s^{-1}(\sigma)\rangle)(\Delta_{s,s})$. Proposition 1.6 follows because pr_s is defined over $\mathbb{Q}$. $\square$

In light of Proposition 1.6, the varieties $W_{s,s}^\dagger$ and $W_s^\dagger$ are defined to be the twists of $W_{s,s}$ and W_s respectively by the cocycle $\sigma \in G_{\mathbb{Q}} \mapsto \langle \epsilon_s^{-1}\rangle(\sigma) = (1, 1, \langle 1; \epsilon_s^{-1}(\sigma)\rangle)$. Note that $W_{s,s}^\dagger = X_0(Np) \times X_s \times X_s^\dagger$ and $W_s^\dagger = W_{s,s}^\dagger / D_{N,p}^\dagger$, where $D_{N,p}^\dagger$ is the subgroup of diamond operators acting "diagonally at N" and "anti-diagonally at p" on $W_{s,s}^\dagger$, defined in a similar way as for D_s.

Thanks to Proposition 1.6, the cycle Δ_s can be viewed as a codimension two cycle in $W_s^\dagger$ with coefficients in $\mathbb{Z}_p$ which is defined over $\mathbb{Q}$, and whose associated class in the Chow group gives rise to a canonical element

$$\Delta_s \in \mathrm{CH}^2(W_s^\dagger)_0(\mathbb{Q}). \tag{52}$$

1.4. **Global cohomology classes.** Let

$$\mathrm{AJ}_{\mathrm{et}} : \mathrm{CH}^2(W_s^\dagger)_0(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, H^3_{\mathrm{et}}(\bar{W}_s^\dagger, \mathbb{Z}_p)(2)), \tag{53}$$

denote the p-adic étale Abel-Jacobi map arising from (18). Of crucial importance for the results of this paper are the images of the cycles Δ_s under this map:

$$\kappa_s^{(1)} := \mathrm{AJ}_{\mathrm{et}}(\Delta_s) \in H^1(\mathbb{Q}, H^3_{\mathrm{et}}(\bar{W}_s^\dagger, \mathbb{Z}_p)(2)) = H^1\left(\mathbb{Q}, H^3_{\mathrm{et}}(\bar{W}_s, \mathbb{Z}_p)(2\,\langle \epsilon_s^{-1}\rangle)\right). \tag{54}$$

As was already observed, the $\mathbb{Z}_p$-modules $H^3_{\mathrm{et}}(\bar{W}_{s,s}^\dagger, \mathbb{Z}_p)$ and $H^3_{\mathrm{et}}(\bar{W}_s^\dagger, \mathbb{Z}_p)$ are equipped with a natural structure of modules over the rings $\mathbb{Z}_p[G_s^{(N)} \times G_s^{(N)}]$ and $\mathbb{Z}_p[(G_s^{(N)} \times G_s^{(N)})/D_s] \simeq \mathbb{Z}_p[G_s^{(N)}]$ via the action of the diamond operators.

Lemma 1.7. *For every $d \geq 0$, the group $H^d_{\mathrm{et}}(\bar{W}_{s,s}^\dagger, \mathbb{Z}_p)$ is free over $\mathbb{Z}_p$ and admits a canonical direct sum decomposition (the Künneth formula):*

$$H^d_{\mathrm{et}}(\bar{W}_{s,s}^\dagger, \mathbb{Z}_p) = \bigoplus_{\substack{i+j+k=d \\ 0 \leq i,j,k \leq 2}} H^i_{\mathrm{et}}(\bar{X}_0(Np), \mathbb{Z}_p) \otimes H^j_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p) \otimes H^k_{\mathrm{et}}(\bar{X}_s^\dagger, \mathbb{Z}_p). \tag{55}$$

Proof. This follows from e.g. Theorem 22.4 of [Mi], in light of the fact that the integral cohomology of smooth projective curves is torsion-free. □

The natural projection $\mathrm{pr}_s : W^\dagger_{s,s} \longrightarrow W^\dagger_s$ induces functorial maps between the corresponding étale cohomology groups in both directions

$$\mathrm{pr}_s^* : H^3_{\mathrm{et}}(\bar{W}^\dagger_s, \mathbb{Z}_p) \longrightarrow H^3_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)^{D_s}, \qquad \mathrm{pr}_{s,*} : H^3_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)_{D_s} \longrightarrow H^3_{\mathrm{et}}(\bar{W}^\dagger_s, \mathbb{Z}_p),$$

the second map being related to the first by taking duals and invoking Poincaré duality. These two maps become isomorphisms after tensoring with $\mathbb{Q}_p$, but working integrally requires a bit more care.

Recall the correspondence $\varepsilon_{s,s}$ on $W_{s,s}$ and W_s introduced in (47). By an abuse of notation, continue to denote with the same symbol the correspondence on $W^\dagger_{s,s}$ and $W^\dagger_s$ defined exactly in the same way, taking into account that both X_s and $X^\dagger_s$ are equipped with a canonical action of diamond operators.

Lemma 1.8. *The kernels and cokernels of the maps* pr_s^* *and* $\mathrm{pr}_{s,*}$ *are annihilated by the element* $\varepsilon_{s,s}$.

Proof. The Hochschild-Serre spectral sequence $H^p(D_s, H^q_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)) \Rightarrow H^{p+q}_{\mathrm{et}}(\bar{W}^\dagger_s, \mathbb{Z}_p)$ shows that pr_s^* sits in the middle of an exact sequence

$$\mathcal{K} \longrightarrow H^3_{\mathrm{et}}(\bar{W}^\dagger_s, \mathbb{Z}_p) \xrightarrow{\mathrm{pr}_s^*} H^3_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)^{D_s} \longrightarrow H^2(D_s, H^2_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)),$$

where the kernel $\mathcal{K}$ is canonically a subquotient of $H^2(D_s, H^1_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)) \oplus H^1(D_s, H^2_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p))$. Since $\varepsilon_{s,s}$ annihilates each term in the Künneth decomposition (55) of $H^d_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)$ when $0 \le d \le 2$, the assertion of the lemma for pr_s^* follows. The result for $\mathrm{pr}_{s,*}$ then follows by taking duals. □

Since the projector $\varepsilon_{s,s}$ of (47) is integrally defined over $\mathbb{Z}_p$, Lemma 1.8 makes it possible to define classes

$$\kappa_s^{(2)} := \varepsilon_{s,s}\, \mathrm{pr}_{s,*}^{-1}(\varepsilon_{s,s}\kappa_s^{(1)}) \in H^1(\mathbb{Q}, H^3_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)(2))_{D_s} = H^1(\mathbb{Q}, H^3_{\mathrm{et}}(\bar{W}_{s,s}, \mathbb{Z}_p)(2\,\langle \epsilon_s^{-1}\rangle))_{D_s}, \tag{56}$$

where the first application of $\varepsilon_{s,s}$ ensures the existence of the inverse image, and the second guarantees its uniqueness. Let

$$V_0(Np) = H^1_{\mathrm{et}}(\bar{X}_0(Np), \mathbb{Z}_p)(1), \qquad V_1(Np^s) = H^1_{\mathrm{et}}(\bar{X}_s, \mathbb{Z}_p)(1), \tag{57}$$

denote the Tate modules of the jacobians of $X_0(Np)$ and X_s, and let

$$V_1(Np^s)^\dagger = H^1_{\mathrm{et}}(\bar{X}^\dagger_s, \mathbb{Z}_p)(1) = V_1(Np^s)(-1)(\langle \epsilon_s^{-1}\rangle).$$

Define also

$$V_{s,s} := V_0(Np) \otimes \left(V_1(Np^s) \otimes V_1(Np^s)^\dagger\right), \qquad V_s := V_0(Np) \otimes \left(V_1(Np^s) \otimes_{\tilde{\Lambda}_s} V_1(Np^s)^\dagger\right). \tag{58}$$

The projection from $H^3_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)$ onto the $(1,1,1)$-component in the Künneth decomposition (55) induces a $G_{\mathbb{Q}}$-equivariant map on the D_s-coinvariants:

$$\mathrm{pr}_{111} : H^3_{\mathrm{et}}(\bar{W}^\dagger_{s,s}, \mathbb{Z}_p)(2)_{D_s} \longrightarrow (V_{s,s})_{D_s} = V_s.$$

The classes $\kappa_s^{(3)}$ are now defined by setting

$$\kappa_s^{(3)} := \mathrm{pr}_{111}(\kappa_s^{(2)}) \in H^1(\mathbb{Q}, V_s). \tag{59}$$

By a slight abuse of notation, the symbol $\varpi_{22,*}$ is also used to denote the proper push-forward maps $\varpi_{22,*} : V_{s+1} \longrightarrow V_s$ induced on étale cohomology by the projections ϖ_{22} of (46). Denote also by $(1 \otimes U_p \otimes 1)$ the endomorphism acting on V_s via the action of U_p on the second factor in (59).

Proposition 1.9. *For all* $s \ge 1$, $\quad \varpi_{22,*}(\kappa_{s+1}^{(3)}) = (1 \otimes U_p \otimes 1)(\kappa_s^{(3)})$.

Proof. The classes $\kappa_s^{(3)}$ arise from the cycles Δ_s by applying succesively the étale Abel-Jacobi map, the inverse of the push-forward map pr_{s*}^{-1} induced by the projection pr_s, the correspondence $\varepsilon_{s,s}$ constructed using diamond operators and the projection pr_{111} arising from the Künneth decomposition. It is immediate to verify that ϖ_{22*} commutes with these four maps. Since in addition ϖ_{22*} commutes with $(\mathrm{Id}, U_p, \mathrm{Id})$ as proved in the paragraph following (34), the result follows from (50). □

In order to make the classes $\kappa_s^{(3)}$ fit into families that are compatible under the natural projections, and take values in manageable $\tilde{\Lambda} := \lim \tilde{\Lambda}_s$-modules (locally free of finite rank, say) it is desirable to cut down the modules V_s by applying Hida's ordinary projector $e_{\rm ord} = \lim U_p^{n!}$ to them. Write

$$V_1^{\rm ord}(Np^s) := e_{\rm ord} V_1(Np^s), \qquad V_1^{\rm ord}(Np^s)^\dagger := e_{\rm ord} V_1(Np^s)^\dagger,$$
$$V_s^{\rm ord} := V_0(Np) \otimes (V_1^{\rm ord}(Np^s) \otimes_{\tilde{\Lambda}_s} V_1^{\rm ord}(Np^s)^\dagger)$$

(60) and set $\kappa_s := (\mathrm{Id}, U_p, \mathrm{Id})^{-s}(1 \otimes e_{\rm ord} \otimes e_{\rm ord})\kappa_s^{(3)} \in H^1(\mathbb{Q}, V_s^{\rm ord}).$

Note that (60) is well-defined because U_p acts invertibly on $V_1^{\rm ord}(Np^s)$. Define the $\tilde{\Lambda}^{(N)}[G_{\mathbb{Q}}]$-modules

$$\mathbb{V}_1^{\rm ord}(Np^\infty) := \varprojlim_s V_1^{\rm ord}(Np^s), \quad \mathbb{V}_1^{\rm ord}(Np^\infty)^\dagger := \varprojlim_s V_1^{\rm ord}(Np^s)^\dagger \quad \text{and} \tag{61}$$

$$\mathbb{V}_\infty^{\rm ord} := \varprojlim_s V_s^{\rm ord} \simeq V_0(Np) \otimes \left(\mathbb{V}_1^{\rm ord}(Np^\infty) \otimes_{\tilde{\Lambda}} \mathbb{V}_1^{\rm ord}(Np^\infty)^\dagger\right) \tag{62}$$

where the inverse limits are taken relative to the maps induced from ϖ_2 and ϖ_{22} by covariant functoriality on the étale cohomology of the towers of curves X_s and of threefolds W_s.

Since U_p commutes with ϖ_{2*}, it follows from Proposition 1.9 that the classes κ_s form a system of global classes satisfying the crucial compatibility $\varpi_{22,*}(\kappa_{s+1}) = \kappa_s$. Hence the classes κ_s can be pieced together into the Λ-adic class

$$\kappa_\infty \quad := \quad \varprojlim_s \kappa_s \in H^1(\mathbb{Q}, \mathbb{V}_\infty^{\rm ord}). \tag{63}$$

The next two sections explain how the $\tilde{\Lambda}^{(N)}[G_{\mathbb{Q}}]$- module $\mathbb{V}_\infty^{\rm ord}$ realises one-parameter families of p-adic Galois representations interpolating the Kummer self-dual twists of the triple tensor product of the Galois representations associated to f, g and h, as (g,h) range over certain pairs of classical specialisations of common weight ℓ of *Hida families* of tame level Np. The classes κ_∞ likewise will give rise to p-adic families of Galois cohomology classes with values in these representations.

1.5. **Modular forms and Galois representations in families.**

1.5.1. *Classical modular forms and Galois representations.* Let $S_k(M)_R = S_k(\Gamma_1(M))_R$ denote the space of classical modular forms of weight $k \geq 1$ and level $M \geq 1$ with coefficients in some ring R.

If χ is a Dirichlet character mod M, let $S_k(M,\chi)_R$ denote the subspace of $S_k(M)_R$ on which diamond operators act through χ. Let

$$\phi = q + \sum_{n\geq 2} a_n(\phi) q^n \quad \in \quad S_k(N_\phi p^s, \chi)_{\mathbb{C}} \tag{64}$$

be a normalized cuspform of weight $k \geq 1$, level $N_\phi p^s$, $s \geq 1$ and character $\chi : (\mathbb{Z}/N_\phi p^s\mathbb{Z})^\times \to \mathbb{C}^\times$. Here p is taken to be a prime number not dividing N_ϕ. Write $\chi = \chi_0 \chi_p$ as the product of Dirichlet characters of conductor dividing N_ϕ and p^s, respectively.

Assume ϕ is an eigenform (with respect to all good Hecke operators T_ℓ, $\ell \nmid N_\phi p$) and let K_ϕ denote the finite extension of $\mathbb{Q}_p$ generated by the fourier coefficients $a_n(\phi)$ of ϕ. Write $\mathcal{O}_\phi$ for its valuation ring. Eichler-Shimura, Deligne and Serre associated to ϕ a two-dimensional representation

$$\varrho_\phi : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(V_\phi) \simeq \mathrm{GL}_2(\mathcal{O}_\phi)$$

which is unramified at all primes not dividing $N_\phi p$ and satisfies $\mathrm{Tr}(\varrho_\phi(\mathrm{Frob}_\ell)) = a_\ell(\phi)$ for any such $\ell \nmid N_\phi p$, where Frob_ℓ denotes an arithmetic Frobenius element at ℓ.

If ϕ is ordinary at p, then the Hecke polynomial $T^2 - a_p(\phi)T + \chi(p)p^{k-1}$ has at least one root which is a p-adic unit. If $k \geq 2$ or $\chi_p \neq 1$, this root is unique. Fix such a root, call it α_ϕ, and let $\psi_\phi : G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}_\phi^\times$ be the unramified character such that $\psi_\phi(\mathrm{Frob}_p) = \alpha_p(\phi) \in \mathcal{O}_\phi^\times$. By [W88, Theorem 2], the restriction of V_ϕ to a decomposition group at p takes the form

$$V_{\phi|G_{\mathbb{Q}_p}} : \begin{pmatrix} \psi_\phi^{-1}\chi\epsilon_{\rm cyc}^{k-1} & * \\ 0 & \psi_\phi \end{pmatrix} \tag{65}$$

on a suitable basis. Here $\epsilon_{\rm cyc} = \varprojlim_s \epsilon_s$ is the cyclotomic character introduced in (15).

Definition 1.10. We let V_ϕ^+ denote the one-dimensional $G_{\mathbb{Q}_p}$-submodule of V_ϕ on which $G_{\mathbb{Q}_p}$ acts via $\psi_\phi^{-1}\chi\epsilon_{\rm cyc}^{k-1}$, and write V_ϕ^- for the unramified quotient on which $G_{\mathbb{Q}_p}$ acts via ψ_ϕ.

When $k\geq 2$, the representation V_ϕ arises as the ϕ-isotypic component of $H^{k-1}_{\rm et}(\bar{\mathcal{E}}_k,\mathcal{O}_\phi)(k-1)$, where $\mathcal{E}_k$ is the $(k-1)$-dimensional Kuga-Sato variety fibered over X_s. In particular, when ϕ has weight 2, V_ϕ is a quotient of $V_1(N_\phi p^s)\otimes\mathcal{O}_\phi$ and $\varpi_\phi: V_1(N_\phi p^s)\otimes\mathcal{O}_\phi\to V_\phi$ denotes the canonical projection.

If N is any multiple of N_ϕ, let $V_\phi(Np^s)$ denote the ϕ-isotypical component of $V_1(Np^s)\otimes\mathcal{O}_\phi$, which is isomorphic to a direct sum of several copies of V_ϕ, and write $\varpi_\phi: V_1(Np^s)\otimes\mathcal{O}_\phi\to V_\phi(Np^s)$ for the resulting projection. Define $V_\phi(Np^s)^+$ and $V_\phi(Np^s)^-$ analogously as above, and likewise for $V_1^{\rm ord}(Np^s)^+$ and $V_1^{\rm ord}(Np^s)^-$. It will also be convenient to introduce the representation

$$V_\phi^*:=V_\phi\otimes\chi_p^{-1}\epsilon_{\rm cyc}^{1-k},\qquad V^*_{\phi\,|G_{\mathbb{Q}_p}}:\begin{pmatrix}\psi_\phi^{-1}\chi_0 & *\\ 0 & \psi_\phi\epsilon_{\rm cyc}^{1-k}\chi_p^{-1}\end{pmatrix},\tag{66}$$

whose restriction to $G_{\mathbb{Q}_p}$ contains an étale submodule (instead of having an étale quotient like V_ϕ). If $k=2$, note that $V_\phi^*\simeq V_\phi(-1)(\langle\epsilon_s^{-1}\rangle)\simeq V_\phi(-1)(\chi_p^{-1})$, which is consistent with (57). Geometrically, the Galois representation V_ϕ^* is naturally realised in the *anti-ordinary quotient* of $H^1_{\rm et}(\bar{X}_s^\dagger,\mathbb{Z}_p)$.

1.5.2. Λ-*adic modular forms.* A Dirichlet character ϵ of conductor dividing p^s, regarded either as a character of Γ_s or as a locally constant character of $\Gamma=\varprojlim\Gamma_s\simeq 1+p\mathbb{Z}_p$, gives rise to a natural ring homomorphism, denoted $\epsilon:\mathbb{C}_p[[\Gamma]]\longrightarrow\mathbb{C}_p$, by extending ϵ to a linear homomorphism on the group ring $\mathbb{C}_p[\Gamma_s]$.

Let $\omega:\mathbb{Z}_p^\times\to\mu_{p-1}$ denote the Teichmüller character, so that $z\omega^{-1}(z)\in 1+p\mathbb{Z}_p$ for any $z\in\mathbb{Z}_p^\times$.

Definition 1.11. A Λ-adic modular cuspform of tame level N and character $\chi:(\mathbb{Z}/N\mathbb{Z})^\times\to\mathbb{C}_p^\times$ is a formal q-series

$$\underline{\phi}:=\sum_n a_n(\underline{\phi})q^n\in\Lambda[[q]]$$

with the property that, for all finite order Dirichlet characters $\epsilon:\Gamma\longrightarrow\Gamma_s\longrightarrow\mathbb{C}_p^\times$, the specialisation

$$\epsilon(\underline{\phi}):=\sum_n\epsilon(a_n(\underline{\phi}))q^n\in\mathbb{C}_p[[q]]$$

is the q-expansion of a modular cuspform of weight 2, level Np^s and character $\chi\epsilon\omega^{-1}$.

Write $S_\Lambda(N,\chi)$ for the space of such Λ-adic modular forms, and $S_\Lambda(N)$ for the direct sum of the spaces $S_\Lambda(N,\chi)$ as χ ranges over all characters mod N.

The natural projection of $\underline{\phi}$ to $\mathbb{Z}_p[\Gamma_s][[q]]$, denoted $\underline{\phi}_s$, can be written as

$$\underline{\phi}_s=\sum_{\sigma\in\Gamma_s}\phi_s^\sigma\cdot\sigma^{-1}\in\mathbb{Z}_p[\Gamma_s][[q]]$$

where $\phi_s\in S_2(Np^s)=\Omega^1(X_s)$ is a classical modular form that will be referred to as the *level s specialisation* of $\underline{\phi}$. With these notations, note that $\epsilon(\underline{\phi})=\epsilon(\underline{\phi}_s)=\theta_\epsilon\phi_s$. Of course, the system $\{\phi_s\}_{s\geq 1}$ of classical modular forms determines $\underline{\phi}$ completely, and vice-versa.

While the forms ϕ_s fail to be compatible under pushforward by either of the transition maps ϖ_1 or ϖ_2, they do satisfy

$$\mu_*(\phi_{s+1})=\pi_1^*(\phi_s),\qquad(\varpi_1)_*(\phi_{s+1})=p\phi_s,\qquad(\varpi_2)_*(\phi_{s+1})=U_p\phi_s,$$

where the first relation is merely a restatement of the fact that the forms $\underline{\phi}_s$ are compatible under the natural projections $\mathbb{Z}_p[\Gamma_{s+1}]\longrightarrow\mathbb{Z}_p[\Gamma_s]$, and the second and third follow directly from the first in light of (33). In particular, the systems of modular forms $\{p^{-s}\phi_s\}$ and $\{U_p^{-s}\phi_s\}$ are compatible under the pushforward maps $(\varpi_1)_*$ and $(\varpi_2)_*$, respectively. It is thus natural to define the maps

$$\mho_s:S_2(Np^s)\to H^1_{\rm dR}(X_s)\quad\text{and}\quad\mho_\infty:S_\Lambda(N)\to H^1_{\rm dR}(X_\infty)$$

given by $\mho_s(\phi)=U_p^{-s}\omega_\phi$ and $\mho_\infty(\underline{\phi})=\{U_p^{-s}\omega_{\phi_s}\}_s$.

For any integrally closed subring $\mathcal{O}$ of $\mathcal{O}_{\mathbb{C}_p}$, set

$$H^1_{\rm dR}(X_\infty)_{\mathcal{O}}:=\left\{\underline{\eta}\in H^1_{\rm dR}(X_\infty)\text{ such that }[\eta,\mho_\infty(\underline{\phi})]_\Gamma\in\mathcal{O}[[\Gamma]]\text{ for all }\underline{\phi}\in S_\Lambda(Np)\right\}.$$

Recall the group ring element $\theta_\epsilon \in \mathbb{C}_p[\Gamma_s]$ introduced in (37). The next lemma records the behaviour of the Λ-adic pairing $[\ ,\ ]_\Gamma$ and the inclusion map $\mho_\infty$ under specialisation.

Lemma 1.12. *Let $\underline{\eta} = \{\eta_s\}$ and $\underline{\phi} = \{\phi_s\}$ be elements of $H^1_{\rm dR}(X_\infty)_{\mathcal{O}}$ and $S_{\Lambda_{\mathcal{O}}}(N)$. For all Dirichlet characters ϵ of conductor $p^{s'}$ with $1 \le s' \le s$,*

$$\epsilon([\underline{\eta}, \mho_\infty(\underline{\phi})]_\Gamma) = \langle \eta_s, ww_s\theta_\epsilon(\phi_s)\rangle_s = \frac{1}{d_s}\langle \theta_\epsilon(\eta_s), ww_s\theta_\epsilon(\phi_s)\rangle_s.$$

Proof. The first equality follows directly from the definitions, since

$$\begin{aligned}\epsilon([\underline{\eta}, \mho_\infty(\underline{\phi})]_\Gamma) &= \epsilon\left(\sum_{\sigma\in\Gamma_s}\langle \sigma\eta_s, ww_sU_p^sU_p^{-s}\phi_s\rangle_s\cdot\sigma^{-1}\right) = \epsilon\left(\sum_{\sigma\in\Gamma_s}\langle \eta_s, ww_s\sigma\phi_s\rangle_s\cdot\sigma^{-1}\right)\\ &= \langle \eta_s, ww_s\theta_\epsilon(\phi_s)\rangle_s = \langle w^{-1}w_s^{-1}\eta_s, \theta_\epsilon(\phi_s)\rangle_s.\end{aligned}$$

The second identity is a direct consequence of the first in light of the fact that θ_ϵ and $\theta_{\epsilon^{-1}}$ are adjoint to each other relative to Poincaré duality, that $\theta_{\epsilon^{-1}}ww_s = ww_s\theta_\epsilon$, and that $\theta_\epsilon^2 = d_s\theta_\epsilon$. $\square$

Note that the formula of Lemma 1.12 does not depend on the value of $s \ge s'$, since

$$\begin{aligned}\langle w_{s+1}\eta_{s+1}, \epsilon(\underline{\phi}_{s+1})\rangle_{s+1} &= \langle w_{s+1}\eta_{s+1}, \varpi_1^*\epsilon(\underline{\phi}_s)\rangle_{s+1} = \langle \varpi_{1*}w_{s+1}\eta_{s+1}, \epsilon(\underline{\phi}_s)\rangle_s\\ &= \langle w_s(\varpi_{2*}\eta_{s+1}), \epsilon(\underline{\phi}_s)\rangle_s = \langle w_s(\eta_s), \epsilon(\underline{\phi}_s)\rangle_s.\end{aligned}$$

1.5.3. *Hida families.* Let $\mathcal{O}$ be a finite extension of $\mathbb{Z}_p$ containing the values of all Dirichlet characters of conductor Np. For any $\mathbb{Z}_p$-module M, write $M_{\mathcal{O}} = M\otimes_{\mathbb{Z}_p}\mathcal{O}$. Assuming that p does not divide $\varphi(N) = |(\mathbb{Z}/N\mathbb{Z})^\times|$, the semi-local ring $\tilde\Lambda_{\mathcal{O}}^{(N)}$ decomposes as a sum over the characters $\psi : (\mathbb{Z}/Np\mathbb{Z})^\times \longrightarrow \mathcal{O}^\times$:

$$\tilde\Lambda_{\mathcal{O}}^{(N)} = \oplus_\psi \Lambda_{\mathcal{O}}^\psi, \qquad \Lambda_{\mathcal{O}}^\psi := e_\psi(\tilde\Lambda_{\mathcal{O}}) \simeq \Lambda_{\mathcal{O}},$$

where e_ψ is the $\mathcal{O}$-algebra homomorphism defined on group-like elements by

$$e_\psi(\langle a;b\rangle) = \psi(\langle a;b\rangle)\langle 1; b\omega^{-1}(b)\rangle \in \Lambda_{\mathcal{O}}.$$

The *weight space* Ω is defined to be the rigid analytic space underlying the formal spectrum of Λ, so that for any complete $\mathbb{Z}_p$-algebra R, the set of R-rational points of Ω is

$$\Omega(R) = \mathrm{Hom}(\Lambda, R) = \mathrm{Hom}_{\rm cts}(\Gamma, R^\times).$$

An element of Ω is said to be *classical* if it corresponds to a character of Γ, denoted $\nu_{k,\epsilon}$, of the form

$$\nu_{k,\epsilon}(n) = n^{k-1}\epsilon(n), \qquad \text{for all } n \in 1 + p\mathbb{Z}_p,$$

for some integer k and some continuous character ϵ of finite order (hence, conductor p^s for some $s \ge 1$, factoring through a primitive character of $\Gamma/\Gamma^{p^{s-1}} \simeq \mathbb{Z}/p^{s-1}\mathbb{Z} \subset (\mathbb{Z}/p^s\mathbb{Z})^\times$).

More generally, if $\tilde\Lambda$ is a finite flat extension of Λ, let $\tilde\Omega := \mathrm{Hom}(\tilde\Lambda, \mathbb{C}_p)$ denote the space of continuous algebra homomorphisms and write $\mathrm{w} : \tilde\Omega \to \Omega$ for the natural "weight map" induced by pull-back from the structure map $\Lambda \hookrightarrow \tilde\Lambda$. A point in $\tilde\Omega$ is said to be classical if its image under w is classical.

Generalising Definition 1.11 slightly, a *Λ-adic ordinary modular form of tame level N and character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}^\times$ with coefficients in $\tilde\Lambda$* is a pair $(\tilde\Omega_{\rm cl}, \underline{\phi})$, where

(i) $\tilde\Omega_{\rm cl}$ is a dense subset of the classical points in $\tilde\Omega$ for the rigid analytic topology;
(ii) $\underline{\phi} := \sum \underline{a}_n q^n \in \tilde\Lambda[[q]]$ is an element of $S_{\tilde\Lambda}(N,\chi) := S_\Lambda(N,\chi)\otimes_\Lambda \tilde\Lambda$, such that for all $x \in \tilde\Omega_{\rm cl}$ of weight $\mathrm{w}(x) = \nu_{k,\epsilon}$, with $k \ge 2$ and character ϵ of conductor p^s, the power series

$$\phi_x := \sum_{n=1}^\infty \underline{a}_n(x)q^n \in \mathbb{C}_p[[q]], \quad \underline{a}_n(x) := x(\underline{a}_n),$$

is the q-expansion of a classical ordinary cusp form in $S_k(Np^s, \chi\epsilon\omega^{1-k}; \mathbb{C}_p)$.

The $\tilde\Lambda$-module of Λ-adic ordinary modular forms of tame level N and character χ is denoted $S^{\rm ord}_{\tilde\Lambda}(N,\chi)$.

Definition 1.13. An element $\underline{\phi} \in S^{\rm ord}_{\tilde\Lambda}(N,\chi)$ whose classical specializations ϕ_x are eigenforms (that is to say, simultaneous eigenvector for the good Hecke operators T_ℓ with $\ell \nmid Np$) is called a *Λ-adic eigenform* or a *Hida family*.

If the subset $\tilde{\Omega}_{\rm cl} \subset \tilde{\Omega}$ can be clearly inferred from the context, such a form will simply be denoted as $\underline{\phi}$, suppressing the datum of $\tilde{\Omega}_{\rm cl}$ from the notation.

If in addition the specialisation ϕ_x of a Λ-adic eigenform $\underline{\phi}$ does not share the same eigenvalues with any eigenform of level N_0p^r with $N_0 \mid N$, $N_0 < N$ and $r \leq s$, then $\underline{\phi}$ is called a Λ-*adic newform.* Note that a form $\underline{\phi}$ as in Definition 1.13 above is in particular a Λ-adic modular form in the weaker sense of Definition 1.11, and hence the material of §1.5.2 applies to it.

Recall the Hecke algebras $\mathbb{T}_1(N)$ and $\mathbb{T}_1(Np^s)$ that were introduced in Section 1.2, and let

$$\mathbb{T}_1(Np^\infty) := e_{\rm ord}\left(\varprojlim_s (\mathbb{T}_1(Np^s)\otimes \mathbb{Z}_p)\right).$$

It follows from a theorem of Hida (cf. [Hi86], [Em99]), that the algebra $\mathbb{T}_1(Np^\infty)$ is locally free of finite rank as a $\tilde{\Lambda}^{(N)}$-module, and the algebra

$$\mathbb{T}_1^\chi(Np^\infty) = e_{\chi\omega^{-1}}\mathbb{T}_1(Np^\infty)_{\mathcal{O}} = \mathbb{T}_1(Np^\infty)\otimes_{e_{\chi\omega^{-1}}} \Lambda_{\mathcal{O}}$$

is therefore free of finite rank over the local ring $\Lambda_{\mathcal{O}}$. Let $\mathcal{L}$ denote the fraction field of $\Lambda_{\mathcal{O}}$, and write

$$\mathbb{T}_1^\chi(Np^\infty)_{\mathcal{L}} := \mathbb{T}_1^\chi(Np^\infty)\otimes_{\Lambda_{\mathcal{O}}} \mathcal{L}.$$

It is a finite-dimensional étale $\mathcal{L}$-algebra, hence decomposes as a product of finite extensions $\mathcal{L}_j$ of $\mathcal{L}$:

$$\mathbb{T}_1^\chi(Np^\infty)_{\mathcal{L}} = \oplus_{j=1}^t \mathcal{L}_j.$$

Furthermore, for each $1 \leq j \leq t$, the integral closure of the image of $\mathbb{T}_1^\chi(Np^\infty)$ in $\mathcal{L}_j$, denoted Λ_j, is a finite flat extension of $\Lambda_{\mathcal{O}}$. The Λ-adic newforms $\underline{\phi}$ of character χ (of conductor dividing N) and primitive tame level N_ϕ (with N_ϕ ranging over the divisors of N that are multiples of the conductor of χ) are in bijection with the resulting homomorphisms

$$\eta_j : \mathbb{T}_1^\chi(Np^\infty) \longrightarrow \Lambda_j.$$

If $\underline{\phi}$ is one such Λ-adic eigenform, let Λ_ϕ denote the corresponding finite flat extension of Λ and corresponding homomorphism $\eta_\phi : \mathbb{T}_1(N_0p^\infty)_{\mathcal{O}} \to \mathbb{T}_1^\chi(Np^\infty) \to \Lambda_\phi$. Let

$$\underline{\epsilon}_{\rm cyc} : G_{\mathbb{Q}} \longrightarrow \Lambda^\times \tag{67}$$

be the Λ-adic cyclotomic character characterised by $\nu_{k,\epsilon}\circ \underline{\epsilon}_{\rm cyc} = \epsilon\cdot\epsilon_{\rm cyc}^{k-1}\cdot\omega^{1-k}$.

Theorem 1.14 (Hida)**.** *Let $\underline{\phi}$ be a Λ-adic newform of tame level N_ϕ and tame character χ. The modules*

$$\mathbb{V}_{\underline{\phi}} := \mathbb{V}_1^{\rm ord}(N_\phi p^\infty)\otimes_{\mathbb{T}_1(N_\phi p^\infty)_{\mathcal{O}},\eta_\phi}\Lambda_\phi, \qquad \mathbb{V}_{\underline{\phi}}^* := \mathbb{V}_{\underline{\phi}}\otimes\underline{\epsilon}_{\rm cyc}^{-1} \tag{68}$$

are both locally free of rank two over Λ_ϕ.

The representations $\mathbb{V}_{\underline{\phi}}$ and $\mathbb{V}_{\underline{\phi}}^*$ will be referred to respectively as the *ordinary* and *anti-ordinary* Λ-*adic representation* of $G_{\mathbb{Q}}$ attached to the Hida family $\underline{\phi}$. Their restrictions to a decomposition group at p are reducible and can be written in upper triangular form as

$$(\mathbb{V}_{\underline{\phi}})_{|G_{\mathbb{Q}_p}} : \begin{pmatrix} \Psi_{\underline{\phi}}^{-1}\chi\underline{\epsilon}_{\rm cyc} & * \\ 0 & \Psi_{\underline{\phi}}\end{pmatrix}, \qquad (\mathbb{V}^*_{\underline{\phi}})_{|G_{\mathbb{Q}_p}} : \begin{pmatrix} \Psi_{\underline{\phi}}^{-1}\chi & * \\ 0 & \Psi_{\underline{\phi}}\underline{\epsilon}_{\rm cyc}^{-1}\end{pmatrix}, \tag{69}$$

where $\Psi_{\underline{\phi}} : G_{\mathbb{Q}_p} \longrightarrow \Lambda_\phi^\times$ is the unramified character defined by $\Psi_{\underline{\phi}}(\mathrm{Frob}_p) = a_p(\underline{\phi}) \in \Lambda_\phi^\times$. As in Definition 1.10, let $\mathbb{V}_{\underline{\phi}}^+$ denote the one-dimensional $G_{\mathbb{Q}_p}$-submodule of $\mathbb{V}_{\underline{\phi}}$ on which $G_{\mathbb{Q}_p}$ acts via $\Psi_{\underline{\phi}}^{-1}\chi\underline{\epsilon}_{\rm cyc}$, and write $\mathbb{V}_{\underline{\phi}}^-$ for the unramified quotient on which $G_{\mathbb{Q}_p}$ acts via $\Psi_{\underline{\phi}}$.

More generally, if N is any multiple of N_ϕ, set

$$\mathbb{V}_{\underline{\phi}}(N) := \mathbb{V}_1^{\rm ord}(Np^\infty)\otimes_{\mathbb{T}_1(Np^\infty)_{\mathcal{O}},\eta_\phi}\Lambda_\phi, \qquad \mathbb{V}^*_{\underline{\phi}}(N) := \mathbb{V}^*_{\underline{\phi}}(N)\otimes\underline{\epsilon}_{\rm cyc}^{-1},$$

and write $\varpi_{\underline{\phi}}$ and $\varpi^*_{\underline{\phi}}$ for the canonical projections

$$\varpi_{\underline{\phi}} : \mathbb{V}_1^{\rm ord}(Np^\infty) \longrightarrow \mathbb{V}_{\underline{\phi}}(N), \qquad \varpi^*_{\underline{\phi}} : \mathbb{V}_1^{\rm ord}(Np^\infty)^\dagger \longrightarrow \mathbb{V}^*_{\underline{\phi}}(N).$$

The quotient $\mathbb{V}_{\underline{\phi}}(N)$ is (non-canonically) isomorphic to a finite direct sum of the Λ-adic representations $\mathbb{V}_{\underline{\phi}}$, and likewise for $\mathbb{V}^*_{\underline{\phi}}(N)$. If $x \in \tilde{\Omega}_{\rm cl}$ is a classical point of weight $\mathrm{w}(x) = \nu_{k,\epsilon}$, with $k \geq 2$ and character ϵ of conductor p^s, it gives rise to $G_{\mathbb{Q}}$-equivariant specialization homomorphisms (which are all denoted by x)

$$x : \mathbb{V}_{\underline{\phi}} \longrightarrow V_{\phi_x}, \qquad x : \mathbb{V}^*_{\underline{\phi}} \longrightarrow V^*_{\phi_x}, \qquad x : \mathbb{V}_{\underline{\phi}}(N) \longrightarrow V_{\phi_x}(Np^s), \qquad x : \mathbb{V}^*_{\underline{\phi}}(N) \longrightarrow V^*_{\phi_x}(Np^s).$$

where V_{ϕ_x} and $V^*_{\phi_x}$ are the Galois representations associated in (66) to the eigenform ϕ_x.

1.6. **The Λ-adic class attached to two Hida families.** Let $f \in S_2(N_f)$ be the newform associated to the elliptic curve E in the introduction and let $\underline{g}$ and $\underline{h}$ be two Λ-adic newforms of tame level N_g and N_h and tame character χ and χ^{-1}, respectively. The prime p arising as the residual characteristic of Λ is taken here not to divide neither $N = \mathrm{lcm}(N_f, N_g, N_h)$ nor $\varphi(N)$. Moreover, as we assumed in (47), we also take p to be a non-Eisenstein prime for f.

Recall the Λ-adic representation $\mathbb{V}^{\mathrm{ord}}_\infty := V_0(Np) \otimes \left(\mathbb{V}^{\mathrm{ord}}_1(Np^\infty) \otimes_{\tilde{\Lambda}} \mathbb{V}^{\mathrm{ord}}_1(Np^\infty)^\dagger\right)$, and define

$$\mathbb{V}_{f\underline{gh}} := V_f \otimes (\mathbb{V}_{\underline{g}} \otimes_\Lambda \mathbb{V}^*_{\underline{h}}), \qquad \mathbb{V}_{f\underline{gh}}(N) := V_f(Np) \otimes (\mathbb{V}_{\underline{g}}(N) \otimes_\Lambda \mathbb{V}^*_{\underline{h}}(N)),$$

which are modules over the ring

$$\Lambda_{f\underline{gh}} := \mathcal{O}_f \otimes (\Lambda_{\underline{g}} \otimes_\Lambda \Lambda_{\underline{h}}),$$

a finite flat extension of the Iwasawa algebra Λ. The rigid analytic space $\Omega_{f\underline{gh}}$ attached to the formal spectrum of $\Lambda_{f\underline{gh}}$ is a finite covering of the weight space Ω. The "classical points" in $\Omega_{f\underline{gh}}$ correspond to pairs of classical points of the form $(y, z) \in \Omega_{\underline{g}} \times \Omega_{\underline{h}}$ satisfying $\mathrm{w}(y) = \mathrm{w}(z)$, that is to say, having the same weight and the same nebentypus at p.

Let g_y and h_z denote as in the previous section the specialisations of $\underline{g}$ and $\underline{h}$ at the points y and z, respectively. Following similar notations as above, let $K = K_{fg_yh_z}$ denote the finite extension of $\mathbb{Q}_p$ generated by K_f, K_{g_y} and K_{h_z} and let $\mathcal{O} = \mathcal{O}_{fg_yh_z}$ denote the valuation ring of K. The specialisation $V_{fg_yh_z}$ of $\mathbb{V}_{f\underline{gh}}$ at the classical point (y, z) of weight ℓ and nebentype character ϵ is a Galois representation of rank 8 over $\mathcal{O}$, and there are natural identifications

$$V_{fg_yh_z} = V_f \otimes V_{g_y} \otimes V^*_{h_z}, \qquad V_{fg_yh_z}(N) := V_f(Np) \otimes V_{g_y}(Np^s) \otimes V^*_{h_z}(Np^s). \tag{70}$$

The reader will note that this Galois representation is pure of weight -1, since a geometric Frobenius element at ℓ acting on V_f has eigenvalues of complex absolute value $\ell^{-1/2}$, while its eigenvalues on $V_{g_y} \otimes V^*_{h_z}$ have complex absolute value 1. Note that $V_{f\underline{gh}}$ interpolates only the *Kummer self-dual* Tate twists of the tensor product of the representations V_f, V_{g_y} and V_{h_z}.

The canonical projections ϖ_f, $\varpi_{\underline{g}}$, and $\varpi^*_{\underline{h}}$ associated to f, $\underline{g}$ and $\underline{h}$ give rise to a surjective Λ-module homomorphism

$$\varpi_{f,\underline{g},\underline{h}} : \mathbb{V}^{\mathrm{ord}}_\infty \longrightarrow \mathbb{V}_{f\underline{gh}}(N). \tag{71}$$

Definition 1.15. The one-variable Λ-adic cohomology class attached to the triple $(f, \underline{g}, \underline{h})$ is the class

$$\kappa(f, \underline{gh}) := \varpi_{f,\underline{g},\underline{h}}(\kappa_\infty) \in H^1(\mathbb{Q}, \mathbb{V}_{f\underline{gh}}(N)).$$

This class takes values in the Galois representation $\mathbb{V}_{f\underline{gh}}(N)$, which is (non-canonically) isomorphic to a finite sum of copies of the representation $\mathbb{V}_{f\underline{gh}}$.

2. Local Λ-adic cohomology classes

2.1. **Local Galois representations.** Let $f \in S_2(N_f)$ be a newform and let $p \nmid N_f$ be a prime at which f is ordinary. Let $\underline{g}$ and $\underline{h}$ be two Λ-adic newforms of tame level N_g and N_h and character χ and χ^{-1}, respectively. Set $N = \mathrm{lcm}(N_f, N_g, N_h)$ as usual. This chapter analyses the restrictions to $G_{\mathbb{Q}_p}$ of the Galois representation $\mathbb{V}_{f\underline{gh}}$ and of the Λ-adic cohomology class $\kappa(f, \underline{gh})$.

Lemma 2.1. *The Galois representation $\mathbb{V}_{f\underline{gh}}$ is endowed with a four-step filtration*

$$0 \subset \mathbb{V}^{++}_{f\underline{gh}} \subset \mathbb{V}^{+}_{f\underline{gh}} \subset \mathbb{V}^{-}_{f\underline{gh}} \subset \mathbb{V}_{f\underline{gh}}$$

by $G_{\mathbb{Q}_p}$-stable $\Lambda_{f\underline{gh}}$-submodules of ranks 0, 1, 4, 7 and 8 respectively. The Galois group $G_{\mathbb{Q}_p}$ acts on the successive quotients for this filtration as a direct sum of $\Lambda_{f\underline{gh}}$-adic characters. More precisely:

$$\begin{array}{rcccl} \mathbb{V}_{f\underline{gh}}^{++} & \simeq & \Lambda_{f\underline{gh}}(\underline{\Upsilon}^+), & & \\ \mathbb{V}_{f\underline{gh}}^{+}/\mathbb{V}_{f\underline{gh}}^{++} & = & \mathbb{V}_{f\underline{gh}}^{f}\oplus\mathbb{V}_{f\underline{gh}}^{g}\oplus\mathbb{V}_{f\underline{gh}}^{h} & \simeq & \Lambda_{f\underline{gh}}(\underline{\Upsilon}_f^+)\oplus\Lambda_{f\underline{gh}}(\underline{\Upsilon}_{\underline{g}}^+)\oplus\Lambda_{f\underline{gh}}(\underline{\Upsilon}_{\underline{h}}^+), \\ \mathbb{V}_{f\underline{gh}}^{-}/\mathbb{V}_{f\underline{gh}}^{+} & = & \mathbb{V}_{f\underline{gh}}^{f}\oplus\mathbb{V}_{f\underline{gh}}^{g}\oplus\mathbb{V}_{f\underline{gh}}^{h} & \simeq & \Lambda_{f\underline{gh}}(\underline{\Upsilon}_f^-)\oplus\Lambda_{f\underline{gh}}(\underline{\Upsilon}_{\underline{g}}^-)\oplus\Lambda_{f\underline{gh}}(\underline{\Upsilon}_{\underline{h}}^-), \\ \mathbb{V}_{f\underline{gh}}/\mathbb{V}_{f\underline{gh}}^{-} & \simeq & \Lambda_{f\underline{gh}}(\underline{\Upsilon}^-), & & \end{array}$$

where the characters arising on the right are described in the following table:

Subquotient	Rank	Galois action
$\mathbb{V}_{f\underline{gh}}^{++}$	1	$\underline{\Upsilon}^+ = (\psi_f\Psi_{\underline{g}}\Psi_{\underline{h}})^{-1}\times\epsilon_{\rm cyc}\underline{\epsilon}_{\rm cyc}$
$\mathbb{V}_{f\underline{gh}}^{+}/\mathbb{V}_{f\underline{gh}}^{++}$	3	$\underline{\Upsilon}_f^+ = \psi_f\Psi_{\underline{g}}^{-1}\Psi_{\underline{h}}^{-1}\times\underline{\epsilon}_{\rm cyc}$ $\underline{\Upsilon}_{\underline{g}}^+ = \psi_f^{-1}\Psi_{\underline{g}}\Psi_{\underline{h}}^{-1}\times\chi^{-1}\times\epsilon_{\rm cyc}$ $\underline{\Upsilon}_{\underline{h}}^+ := \psi_f^{-1}\Psi_{\underline{g}}^{-1}\Psi_{\underline{h}}\times\chi\times\epsilon_{\rm cyc}$
$\mathbb{V}_{f\underline{gh}}^{-}/\mathbb{V}_{f\underline{gh}}^{+}$	3	$\underline{\Upsilon}_{\underline{h}}^- = \psi_f\Psi_{\underline{g}}\Psi_{\underline{h}}^{-1}\times\chi^{-1}$ $\underline{\Upsilon}_{\underline{g}}^- = \psi_f\Psi_{\underline{g}}^{-1}\Psi_{\underline{h}}\times\chi$ $\underline{\Upsilon}_f^- = \psi_f^{-1}\Psi_{\underline{g}}\Psi_{\underline{h}}\times\epsilon_{\rm cyc}\underline{\epsilon}_{\rm cyc}^{-1}$
$\mathbb{V}_{f\underline{gh}}/\mathbb{V}_{f\underline{gh}}^{-}$	1	$\underline{\Upsilon}^- = \psi_f\Psi_{\underline{g}}\Psi_{\underline{h}}\times\underline{\epsilon}_{\rm cyc}^{-1}$.

Proof. The lemma follows directly from (65), (68) and (69). □

Since $\mathbb{V}_{f\underline{gh}}(N)$ is a finite sum of copies of the representation $\mathbb{V}_{f\underline{gh}}$, in view of the above lemma it also admits a natural filtration

$$0\subset\mathbb{V}_{f\underline{gh}}(N)^{++}\subset\mathbb{V}_{f\underline{gh}}(N)^{+}\subset\mathbb{V}_{f\underline{gh}}(N)^{-}\subset\mathbb{V}_{f\underline{gh}}(N)$$

of $\Lambda_{f\underline{gh}}[G_{\mathbb{Q}_p}]$-modules, and

$$\mathbb{V}_{f\underline{gh}}^{+}(N)/\mathbb{V}_{f\underline{gh}}^{++}(N) \quad =: \quad \mathbb{V}_{f\underline{gh}}^{f}(N)\oplus\mathbb{V}_{f\underline{gh}}^{g}(N)\oplus\mathbb{V}_{f\underline{gh}}^{h}(N). \tag{72}$$

For each $\phi=f,g,h$, the subquotient $\mathbb{V}_{f\underline{gh}}^{\phi}(N)$ is a direct sum of finitely many copies of $\mathbb{V}_{f\underline{gh}}^{\phi}$, on which $G_{\mathbb{Q}_p}$ acts via the character $\underline{\Upsilon}_{\phi}^{+}$.

Let

$$\kappa_p(f,\underline{gh}) := {\rm res}_p(\kappa(f,\underline{gh}))\in H^1(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}) \tag{73}$$

denote the natural image of the global class $\kappa(f,\underline{gh})$ in the local cohomology at p, and set

$$\xi_{f\underline{gh}} = (1-\alpha_f a_p(\underline{g})a_p(\underline{h})^{-1}\chi^{-1}(p))(1-\alpha_f a_p(\underline{g})^{-1}a_p(\underline{h})\chi(p)).$$

Proposition 2.2. *The class $\xi_{f\underline{gh}}\cdot\kappa_p(f,\underline{gh})$ belongs to the natural image of $H^1(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}^{+}(N))$ in $H^1(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}(N))$ under the map induced from the inclusion $\mathbb{V}_{f\underline{gh}}^{+}(N)\hookrightarrow\mathbb{V}_{f\underline{gh}}(N)$.*

Proof. Recall the natural projection $\Lambda\longrightarrow\Lambda_s$ to the finite group ring of level p^s. The Galois representation $\mathbb{V}_{f\underline{gh}}(N)\otimes_\Lambda\Lambda_s$ is realised in the middle p-adic étale cohomology of the quotient W_s of $X_0(Np)\times X_s\times X_s^\dagger$, and the natural image of $\kappa_p(f,\underline{gh})$ in $H^1(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}(N)\otimes_\Lambda\Lambda_s)$ is proportional to the image of the cycle Δ_s under the p-adic étale Abel-Jacobi map. The three-fold W_s is smooth and proper over $\mathbb{Q}_p$, and the purity conjecture for the monodromy filtration is known to hold by the work of Saito (cf. [Sa97], [Ne2, (3.2)]). By Theorem 3.1 of [Sa97], it follows that the natural image of $\kappa_p(f,\underline{gh})$ belongs to $H^1_{\rm fin}(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}(N)\otimes_\Lambda\Lambda_s)$, where

$$H^1_{\rm fin}(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}(N)\otimes_\Lambda\Lambda_s)\subset H^1(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}(N)\otimes_\Lambda\Lambda_s) \tag{74}$$

denotes the *finite* submodule in the sense of Bloch-Kato, as defined in [BK93, §3] or [Ne2, (2.2)]. Lemma 1.15 implies that $\mathbb{V}_{f\underline{gh}}(N)\otimes_\Lambda\Lambda_s$ is an ordinary p-adic representation in the sense of Greenberg (cf. [Gr89, (4)], [Fl90]). Hence the finite subspace may be described concretely as

$$H^1_{\rm fin}(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}(N)\otimes_\Lambda\Lambda_s) = \ker\left(H^1(\mathbb{Q}_p,\mathbb{V}_{f\underline{gh}}(N)\otimes_\Lambda\Lambda_s)\longrightarrow H^1(I_p,(\mathbb{V}_{f\underline{gh}}(N)/\mathbb{V}_{f\underline{gh}}^{+}(N))\otimes_\Lambda\Lambda_s)\right),$$

where I_p denotes the inertia group at p. (See [Fl90, Lemma 2, p. 125] for a proof of this useful fact. Note that $H^1_{\rm fin}(\mathbb{Q}_p, \mathbb{V}_{f\underline{gh}}(N)) = H^1_{\rm g}(\mathbb{Q}_p, V_{f\underline{gh}}(N))$ in this case, by [Ne2, (3.1) and (3.2)].) By passing to the inverse limit and using the fact that $\varprojlim_s \mathbb{V}_{f\underline{gh}}(N) \otimes_\Lambda \Lambda_s = \mathbb{V}_{f\underline{gh}}(N)$, one concludes that

$$\kappa_p(f, \underline{gh}) \in \ker\left(H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{gh}}(N)) \longrightarrow H^1(I_p, \mathbb{V}_{f\underline{gh}}(N)/\mathbb{V}^+_{f\underline{gh}}(N))\right).$$

The kernel of the restriction map $H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{gh}}(N)/\mathbb{V}^+_{f\underline{gh}}(N)) \longrightarrow H^1(I_p, \mathbb{V}_{f\underline{gh}}(N)/\mathbb{V}^+_{f\underline{gh}}(N))$ is naturally a quotient of $H^1(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p, (\mathbb{V}_{f\underline{gh}}(N)/\mathbb{V}^+_{f\underline{gh}}(N))^{I_p})$.

The local Galois representation $(\mathbb{V}_{f\underline{gh}}/\mathbb{V}^+_{f\underline{gh}})$ lies in the middle of an exact sequence

$$0 \longrightarrow \mathbb{V}^{f-}_{f\underline{gh}} \oplus \mathbb{V}^{g-}_{f\underline{gh}} \oplus \mathbb{V}^{h-}_{f\underline{gh}} \longrightarrow (\mathbb{V}_{f\underline{gh}}/\mathbb{V}^+_{f\underline{gh}}) \longrightarrow (\mathbb{V}_{f\underline{gh}}/\mathbb{V}^-_{f\underline{gh}}) \longrightarrow 0,$$

which induces an isomorphism $\mathbb{V}^{g-}_{f\underline{gh}} \oplus \mathbb{V}^{h-}_{f\underline{gh}} \longrightarrow (\mathbb{V}_{f\underline{gh}}/\mathbb{V}^+_{f\underline{gh}})^{I_p}$ by Lemma 2.1. The term

$$H^1(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p, \mathbb{V}^{g-}_{f\underline{gh}} \oplus \mathbb{V}^{h-}_{f\underline{gh}}) = (\Lambda_{f\underline{gh}}(\underline{\Upsilon}_g^-) \oplus \Lambda_{f\underline{gh}}(\underline{\Upsilon}_h^-))/(\mathrm{Frob}_p - 1),$$

is annihilated by $\xi_{f\underline{gh}}$, as can be read off from the table in Lemma 2.1, and the same is therefore true of $H^1(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p, (\mathbb{V}_{f\underline{gh}}/\mathbb{V}^-_{f\underline{gh}})^{I_p})$ as well as of $H^1(\mathbb{Q}_p^{\rm ur}/\mathbb{Q}_p, (\mathbb{V}_{f\underline{gh}}(N)/\mathbb{V}^-_{f\underline{gh}}(N))^{I_p})$. It follows that $\xi_{f\underline{gh}} \cdot \kappa_p(f, \underline{gh})$ belongs to the kernel of the natural map to $H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{gh}}(N)/\mathbb{V}^+_{f\underline{gh}}(N))$. □

It will be convenient, for the discussion that follows *and in the remainder of this article* to replace the ring $\Lambda_{f\underline{gh}}$ and all modules over it (such as $\mathbb{V}_{f\underline{gh}}$ and its various subquotients) with their localisations by the multiplicative set generated by $\xi_{f\underline{gh}}$. Since the element $\xi_{f\underline{gh}}$ is non-zero at all classical points of weight $\ell \geq 1$, the passage to this localisation is harmless and still makes it permissible to specialise the Λ-adic class $\kappa(f, \underline{gh})$ at these points. With this convention in place, the following is immediate:

Corollary 2.3. *The class $\kappa_p(f, \underline{gh})$ belongs to the image of $H^1(\mathbb{Q}_p, \mathbb{V}^+_{f\underline{gh}}(N))$ in $H^1(\mathbb{Q}_p, \mathbb{V}_{f\underline{gh}}(N))$ under the map induced from the inclusion $\mathbb{V}^+_{f\underline{gh}}(N) \hookrightarrow \mathbb{V}_{f\underline{gh}}(N)$.*

The decomposition (72) induces a corresponding decomposition of the local cohomology group $H^1(\mathbb{Q}_p, \mathbb{V}^+_{f\underline{gh}}(N)/\mathbb{V}^{++}_{f\underline{gh}}(N))$ into a direct sum of three contributions. Write

$$\kappa_p^f(f, \underline{gh}) \in H^1(\mathbb{Q}_p, \mathbb{V}^f_{f\underline{gh}}(N)), \quad \kappa_p^g(f, \underline{gh}) \in H^1(\mathbb{Q}_p, \mathbb{V}^g_{f\underline{gh}}(N)), \quad \kappa_p^h(f, \underline{gh}) \in H^1(\mathbb{Q}_p, \mathbb{V}^h_{f\underline{gh}}(N)) \tag{75}$$

for the projections of the local class $\kappa_p(f, \underline{gh})$ in each direct summand.

The reader will note that the $\Lambda_{f\underline{gh}}$-adic characters $\underline{\Upsilon}_g^+$ and $\underline{\Upsilon}_h^+$ agree with the cyclotomic character when restricted to the inertia group at p, and thus are not "truly varying", while the character $\underline{\Upsilon}_f^+$ is somewhat more interesting, since it involves the Λ-adic cyclotomic character $\underline{\epsilon}_{\rm cyc}$.

2.2. **Classical specializations.**

Let $(y, z) \in \Omega_g \times \Omega_h$ be a classical point of weight ℓ and character ϵ. The goal of this section is to analyze the specialization

$$\kappa(f, g_y, h_z) := \kappa(f, \underline{gh})_{(y,z)} \in H^1(\mathbb{Q}, V_{fg_yh_z}(N))$$

of the global Λ-adic class $\kappa(f, \underline{gh})$ at (y, z), with special emphasis on its local behaviour at p.

A one-dimensional character $\Upsilon : G_{\mathbb{Q}_p} \longrightarrow \mathbb{C}_p^\times$ is said to be *of Hodge-Tate weight $-j$* if it is equal to a finite order character times the j-th power of the cyclotomic character. The following is an immediate corollary of Lemma 2.1 of the previous chapter.

Corollary 2.4. *The Galois representation $V_{fg_yh_z}$ is endowed with a four-step $G_{\mathbb{Q}_p}$-stable filtration*

$$0 \subset V^{++}_{fg_yh_z} \subset V^+_{fg_yh_z} \subset V^-_{fg_yh_z} \subset V_{fg_yh_z},$$

and the action of $G_{\mathbb{Q}_p}$ on its successive quotients is via the one-dimensional representations described in the following table:

Subquotient	Galois action	Hodge-Tate weights	$\ell = 2$	$\ell = 1$
$V^{++}_{fg_yh_z}$	$\Upsilon^+(y,z) := (\psi_f\psi_{g_y}\psi_{h_z})^{-1} \times \epsilon \times \omega^{1-\ell} \times \epsilon_{\mathrm{cyc}}^{\ell}$	$-\ell$	-2	-1
$V^{+}_{fg_yh_z}/V^{++}_{fg_yh_z}$	$\Upsilon^+_f(y,z) := \psi_f\psi_{g_y}^{-1}\psi_{h_z}^{-1} \times \epsilon \times \omega^{1-\ell} \times \epsilon_{\mathrm{cyc}}^{\ell-1}$	$1-\ell$	-1	0
	$\Upsilon^+_{g_y}(y,z) := \psi_f^{-1}\psi_{g_y}\psi_{h_z}^{-1} \times \chi^{-1} \times \epsilon_{\mathrm{cyc}}$	-1	-1	-1
	$\Upsilon^+_{h_z}(y,z) := \psi_f^{-1}\psi_{g_y}^{-1}\psi_{h_z} \times \chi \times \epsilon_{\mathrm{cyc}}$	-1	-1	-1
$V^{-}_{fg_yh_z}/V^{+}_{fg_yh_z}$	$\Upsilon^-_{h_z}(y,z) := \psi_f\psi_{g_y}\psi_{h_z}^{-1} \times \chi^{-1}$	0	0	0
	$\Upsilon^-_{g_y}(y,z) := \psi_f\psi_{g_y}^{-1}\psi_{h_z} \times \chi$	0	0	0
	$\Upsilon^-_f(y,z) := \psi_f^{-1}\psi_{g_y}\psi_{h_z} \times \epsilon^{-1} \times \omega^{\ell-1} \times \epsilon_{\mathrm{cyc}}^{2-\ell}$	$\ell-2$	0	-1
$V_{fg_yh_z}/V^{-}_{fg_yh_z}$	$\Upsilon^-(y,z) := \psi_f\psi_{g_y}\psi_{h_z} \times \epsilon^{-1} \times \omega^{\ell-1} \times \epsilon_{\mathrm{cyc}}^{1-\ell}$	$\ell-1$	1	0

2.2.1. *Weight two specializations.* When $\ell = 2$, the classes $\kappa(f, g_y, h_z)$ are then, by their very construction, directly related to the étale Abel-Jacobi images of twisted diagonal cycles. Recall that α_{g_y} denotes the eigenvalue of U_p acting on the ordinary modular form g_y.

Proposition 2.5. *Assume that the classical points (y,z) have weight $\ell = 2$ and character ϵ of conductor p^s. Then*

$$\kappa(f, g_y, h_z) = \alpha_{g_y}^{-s}\varpi_{fg_yh_z}(\mathrm{AJ}_{\mathrm{et}}(\Delta_s)) \in H^1(\mathbb{Q}, V_{fg_yh_z}(N)).$$

Proof. By Definition 1.15 and equation (63), $\kappa(f, g_y, h_z) = \varpi_{fg_yh_z} \circ \varpi_{f\underline{gh}}(\kappa_\infty) = \varpi_{fg_yh_z}(\kappa_s)$. But since $\varpi_{fg_yh_z}(1 \otimes U_p \otimes 1) = \alpha_{g_y}\varpi_{fg_yh_z}$ and the projection $\varpi_{fg_yh_z}$ factors through the ordinary projection $e_{\mathrm{ord}} \otimes e_{\mathrm{ord}}$, equation (60) implies that $\kappa(f, g_y, h_z) = \alpha_{g_y}^{-s}\varpi_{fg_yh_z}(\kappa_s^{(3)})$. The projection $\varpi_{fg_yh_z}$ also factors through the Künneth projection pr_{111} of equation (59), and thus $\kappa(f, g_y, h_z) = \alpha_{g_y}^{-s}\varpi_{fg_yh_z}(\kappa_s^{(2)})$. Now, using equation (56) and the fact that $\varpi_{fg_yh_z}\varepsilon_{s,s} = \varpi_{fg_yh_z}$, it can be seen that $\kappa(f, g_y, h_z) = \alpha_{g_y}^{-s}\varpi_{fg_yh_z}(\kappa_s^{(1)})$. Proposition 2.5 now follows from the definition of $\kappa_s^{(1)}$ given in (54). ☐

Corollary 2.6. *For all classical $(y,z) \in \Omega_g \times \Omega_h$ of weight two and character ϵ of conductor p^s, the class $\kappa_p(f, g_y, h_z)$ belongs to the image of $H^1(\mathbb{Q}_p(\zeta_s), V^+_{fg_yh_z}(N))$ in $H^1(\mathbb{Q}_p(\zeta_s), V_{fg_yh_z}(N))$ under the map induced from the inclusion $V^+_{fg_yh_z}(N) \hookrightarrow V_{fg_yh_z}(N)$.*

Proof. This follows by specialising Corollary 2.3 to the point (y,z). (It could also be deduced from Proposition 2.5 in light of the known properties of étale Abel-Jacobi images of algebraic cycles.) ☐

2.2.2. *Weight one specializations.* The richest arithmetic phenomena arise when studying the specialisation of $\kappa(f, \underline{gh})$ at a point (y,z) of weight $\ell = 1$ and trivial nebentype character at p, which will be assumed in this section. The specializations of $\underline{g}$ and $\underline{h}$ at y and z need not be classical in general; let us assume however that they are: the departure point of this section is the assumption that there exist newforms $g \in S_1(N_g, \chi)$ and $h \in S_1(N_h, \chi^{-1})$ such that

$$g_y = g_\alpha \in S_1(N_g p, \chi), \quad h_z = h_\alpha \in S_1(N_h p, \chi^{-1}).$$

Recall from the introduction that associated to g and h there exist Artin representations ϱ_g and ϱ_h with coefficients in a finite extension $L/\mathbb{Q}$. Setting $\varrho = \varrho_g \otimes \varrho_h$, we have $V_{gh} := V_g \otimes V_h \simeq V_\varrho \otimes_L L_p$.

After labelling the two (distinct, by the hypothesis imposed in the introduction) roots of the p-th Hecke polynomial $x^2 - a_p(g) + \chi(p)$ as α_g and β_g, one may define the global cohomology class

$$\kappa(f, g_\alpha, h_\alpha) := \kappa(f, \underline{gh})_{y,z} \in H^1(\mathbb{Q}, V_{fgh}(N))$$

with values in the Galois representation $V_{fgh}(N) = V_f(Np) \otimes V_g(Np) \otimes V_h(Np) \simeq (V_f \otimes V_{gh})^a$ for some $a \geq 1$. Let also $\kappa_p^f(f, g_\alpha, h_\alpha) \in H^1(\mathbb{Q}_p, V^f_{fgh}(N))$ denote the associated local cohomology class arising likewise from (75).

The weight one setting distinguishes itself in (at least) two further respects. Firstly, recall the running assumption that $\alpha_g \neq \beta_g$, $\alpha_h \neq \beta_h$. Let us denote $\underline{g}_\alpha$, $\underline{g}_\beta$, $\underline{h}_\alpha$, $\underline{h}_\beta$ the Hida families passing through g_α, g_β, h_α, h_β, respectively. One thus obtains four (a priori distinct) global classes

(76) $$\kappa(f, g_\alpha, h_\alpha), \quad \kappa(f, g_\alpha, h_\beta), \quad \kappa(f, g_\beta, h_\alpha), \quad \kappa(f, g_\beta, h_\beta) \quad \in H^1(\mathbb{Q}, V_{fgh}(N)).$$

Note that for primes $\ell \neq p$, it follows from [Ne2, (2.5) and (3.2)] that $H^1(\mathbb{Q}_\ell, V_{fgh}(N)) = 0$ and hence a fortiori the restriction to $G_{\mathbb{Q}_\ell}$ of the above cohomology classes are all trivial. This is in general not true for $\ell = p$, as we shall see later in Theorem 6.4.

Since V_{gh} is an Artin representation which is unramified at p, and for which both V_g and V_h are regular, the local cohomology group $H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}(N))$ decomposes into a direct sum of four vector spaces over $L_p = K_{fgh}$:

$$\begin{aligned} H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}(N)) \quad = \quad & H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}^{\alpha\alpha}(N)) \oplus H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}^{\alpha\beta}(N)) \\ & \oplus H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}^{\beta\alpha}(N)) \oplus H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}^{\beta\beta}(N)), \end{aligned}$$

where $V_{gh}^{\alpha\alpha}(N) := V_g(N)^{\alpha_g} \otimes V_h(N)^{\alpha_h}$ denotes the $G_{\mathbb{Q}_p}$-stable subspace in $V_{gh}(N)$ formed from the tensor product of the two eigenspaces in $V_g(N)$ and $V_h(N)$ on which Frob_p acts with eigenvalue α_g and α_h, respectively, and likewise for the other spaces.

Proposition 2.7. *The local class $\kappa_p(f, g_\alpha, h_\alpha)$ belongs to the kernel of the natural homomorphism*

$$H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}(N)) \longrightarrow H^1(\mathbb{Q}_p, V_f(Np) \otimes V_{gh}^{\alpha\alpha}(N)).$$

Proof. This follows from Corollary 2.3 after observing that the specialisation of $\mathbb{V}^+_{f\underline{gh}}$ at (y,z) is contained in

$$V_f(Np) \otimes \left(V_{gh}^{\alpha\beta}(N) \oplus V_{gh}^{\beta\alpha}(N) \oplus V_{gh}^{\beta\beta}(N) \right).$$

□

Although the Galois representation V_{fgh}, which is independent of the choice of stabilisations of g and h, is crystalline under our running assumptions, the same need not be true of the extension classes in (76). This can be seen by observing that the Hodge-Tate weights attached to the specializations of the characters $\underline{\Upsilon}_f^+$ and $\underline{\Upsilon}_f^-$, which in weight $\ell \geq 2$ are < 0 and ≥ 0 respectively, acquire Hodge-Tate weights that are ≥ 0 and < 0 respectively when $\ell = 1$, as summarized in the table of Corollary 2.4. The local condition $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fgh}(N))$ at p defining the Selmer group of $V_{fgh}(N)$ is given by

$$H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fgh}(N)) = \ker \left(H^1(\mathbb{Q}_p, V_{fgh}(N)) \longrightarrow H^1(\mathbb{Q}_p, V_f(Np)^- \otimes V_{gh}(N)) \right), \tag{77}$$

where $V_f(Np)^-$ is the unramified quotient of $V_f(Np)$ of Definition 1.10 and $V_{gh}(N) = V_g(N) \otimes V_h(N)$. Let

$$\partial_p : H^1(\mathbb{Q}, V_{fgh}(N)) \longrightarrow H^1(\mathbb{Q}_p, V_f(Np)^- \otimes V_{gh}(N))$$

be the natural projection to the "singular quotient" of the cohomology group at p, whose kernel is $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_{fgh}(N))$.

Proposition 2.8. *The class $\partial_p(\kappa(f, g_\alpha, h_\alpha))$ belongs to $H^1(\mathbb{Q}_p, V_f(Np)^- \otimes V_{gh}^{\beta\beta}(N))$, and is non-zero if and only $\kappa_p^f(f, g_\alpha, h_\alpha) \neq 0$.*

Proof. The second row in the table in Corollary 2.4 shows that the specialisation of $\underline{\Upsilon}_f^+$ at (y,z) is the unramified character that sends Frob_p to $\alpha_f(\alpha_g\alpha_h)^{-1} = \alpha_f\beta_g\beta_h$ and hence the local class $\kappa_p^f(f, g_\alpha, h_\alpha)$ belongs to $H^1(\mathbb{Q}_p, V_f(N)^- \otimes V_{gh}^{\beta\beta}(N))$. Since

$$\mathbb{V}^+_{f\underline{gh}} / (\mathbb{V}^+_{f\underline{gh}} \cap (V_f^+ \otimes (\mathbb{V}_{\underline{g}} \otimes_\Lambda \mathbb{V}^*_{\underline{h}}))) = \mathcal{O}(\underline{\Upsilon}_f^+),$$

it follows from (72) that $\partial_p(\kappa(f, g_\alpha, h_\alpha)) = \kappa_p^f(f, g_\alpha, h_\alpha) \in H^1(\mathbb{Q}_p, V_f(N)^- \otimes V_{gh}^{\beta\beta}(N))$. □

Note that analogous statements hold for the three other classes attached in (76) to the other p-stabilisations of g and h. In particular, if the four associated singular parts are non-zero, they are necessarily linearly independent over L_p, and therefore generate the target of ∂_p.

2.3. **Dieudonné modules and the Bloch-Kato logarithm.** Let $\mathbf{B}_{\mathrm{dR}}$ denote Fontaine's ring of de Rham p-adic periods. Given a representation V of $G_{\mathbb{Q}_p}$ with coefficients in a finite extension $K/\mathbb{Q}_p$, the *de Rham Dieudonné module* of V is defined as

$$D_{\mathrm{dR}}(V) := (V \otimes \mathbf{B}_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}.$$

It is a finite-dimensional K-vector space equipped with a descending exhaustive filtration $\mathrm{Fil}^j D_{\mathrm{dR}}(V)$ by K-vector subspaces. Falting's comparison theorem (cf. e.g.[Fa97]) asserts that there is an isomorphism of filtered modules

$$(78) \qquad D_{\mathrm{dR}}(V_0(Np)) \simeq H^1_{\mathrm{dR}}(X_0(Np)/\mathbb{Q}_p)(1), \qquad D_{\mathrm{dR}}(V_1(Np^s)) \simeq H^1_{\mathrm{dR}}(X_s/\mathbb{Q}_p)(1).$$

For $V = V_0(Np)$ and $V_1(Np^s)$, the filtration on $D_{\mathrm{dR}}(V)$ is given by

$$\mathrm{Fil}^{-1} D_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V) \supsetneq \mathrm{Fil}^0 D_{\mathrm{dR}}(V) \supsetneq \mathrm{Fil}^1 D_{\mathrm{dR}}(V) = 0$$

and (78) identifies $\mathrm{Fil}^0 D_{\mathrm{dR}}(V)$ with $\Omega^1(X_0(Np)_{/\mathbb{Q}_p})$ and $\Omega^1(X_{s/\mathbb{Q}_p})$ respectively.

Let $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{dR}}$ denote Fontaine's ring of crystalline p-adic periods. The *crystalline Dieudonné modules* associated to the Galois representations V_f and $V_f(Np)$ are

$$D_f := (\mathbf{B}_{\mathrm{cris}} \otimes V_f)^{G_{\mathbb{Q}_p}}, \quad D_f(Np) := (\mathbf{B}_{\mathrm{cris}} \otimes V_f(Np))^{G_{\mathbb{Q}_p}}.$$

These modules are vector spaces over $\mathbb{Q}_p$ equipped with a Hodge filtration and a $\mathbb{Q}_p$-linear action of a Frobenius automorphism Φ, with eigenvalues α_f and β_f. As filtered modules, forgetting the operator Φ, there is canonical isomorphism $D_f(Np) \simeq D_{\mathrm{dR}}(V_f(Np))$ and (78) identifies $D_f(Np)$ with $H^1_{\mathrm{dR}}(X_0(Np)/\mathbb{Q}_p)(1)[f]$. Via this identification, the regular differential form $\omega_{\breve{f}}$ associated to a test vector $\breve{f} \in S_2(Np)[f]$ gives rise to an element of $\mathrm{Fil}^0 D_f(Np)$. Likewise, given a dual test vector $\breve{f}^* \in S_2(Np)^\vee[f]$, let $\eta_{\breve{f}^*} \in D_f(Np)$ denote the unique class such that $\Phi(\eta_{\breve{f}^*}) = \alpha_f \cdot \eta_{\breve{f}^*}$ and

$$(79) \qquad \langle \eta_{\breve{f}^*}, w_1\omega\rangle_{X_0(Np)} = \breve{f}^*(\omega) \qquad \text{for all } \omega \in e_{\mathrm{ord}}\Omega^1(X_0(Np)).$$

As in (1.5.1), let $\phi \in S_2(Np^s, \chi_\phi)$ be an eigenform of weight two, level Np^s and nebentype $\chi_\phi = \chi_0\chi_p$. Assume that ϕ is *primitive at* p in the sense of [MW84, §3], i.e., the p-part of the primitive level of ϕ and the primitive conductor of χ_p are both p^s. The Galois representation V_ϕ (as well as $V_\phi(Np^s)$, of course) is potentially cristalline; more precisely, it becomes cristalline when restricted to $G_{\mathbb{Q}_p(\zeta_s)}$. The crystalline Dieudonné modules

$$D_\phi := (\mathbf{B}_{\mathrm{cris}} \otimes V_\phi)^{G_{\mathbb{Q}_p(\zeta_s)}}, \qquad D_\phi(Np^s) := (\mathbf{B}_{\mathrm{cris}} \otimes V_\phi(Np^s))^{G_{\mathbb{Q}_p(\zeta_s)}}$$

are vector spaces over $K = K_\phi$ equipped with

- a Hodge filtration by $K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_\phi)$-submodules of $D_\phi \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_\phi)$, such that (78) identifies $D_\phi(Np^s)$ with $H^1_{\mathrm{dR}}(X_s/\mathbb{Q}_p(\zeta_s)) \otimes K(1)[\phi]$;
- a K-linear action of a Frobenius operator Φ, with eigenvalues $\alpha_\phi = a_p(\phi)$ and $\beta_\phi = \chi_0(p)\overline{a_p(\phi)}$.

Given a dual test vector $\phi^* \in S_2^{\mathrm{ord}}(Np^s)^\vee[\phi]$, let us denote by $\eta_{\phi^*} \in D_\phi$ the unique class satisfying $\Phi(\eta_{\phi^*}) = \alpha_\phi \cdot \eta_{\phi^*}$ and

$$(80) \qquad \langle \eta_{\phi^*}, w_s\omega\rangle_s = \phi^*(\omega) \qquad \text{for all } \omega \in \Omega^1(X_s)^{\mathrm{ord}}.$$

If $\underline{\phi}^* := \{\phi_s^*\}_{s\geq 1}$ is a collection of elements $\phi_s^* \in S_2^{\mathrm{ord}}(Np^s)^\vee$ which are compatible under the maps $(\varpi_1^*)^\vee$ dual to the pullback by ϖ_1, then the corresponding elements $\eta_{\phi_s^*}$ in $H^1_{\mathrm{dR}}(X_{s/\mathbb{Q}_p(\zeta_s)})$ are compatible under the transition maps ϖ_{2*}, and hence describe an element of $H^1_{\mathrm{dR}}(X_\infty)$.

The weight two specialisations of the Λ-adic test vectors $\underline{\breve{g}} \in S^{\mathrm{ord}}_{\Lambda_g}(N,\chi)[\underline{g}]$ and $\underline{\breve{h}} \in S^{\mathrm{ord}}_{\Lambda_h}(N,\chi^{-1})[\underline{h}]$ in the introduction provide natural examples of eigenforms ϕ that are primitive at p. Namely, let ϵ be a Dirichlet character of conductor p^s and let $(y,z) \in \Omega_g \times_\Omega \Omega_h$ be a classical point of weight $(2,\epsilon)$. Assuming that $\epsilon \neq \omega$, the eigenforms g_y, h_z and $\breve{g}_y \in S_2(Np^s, \chi\epsilon\omega^{-1})$ and $\breve{h}_z \in S_2(Np^s, \chi^{-1}\epsilon\omega^{-1})$ are primitive at p. Let

$$D_{fg_yh_z}(N) := (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_{fg_yh_z}(N))^{G_{\mathbb{Q}_p(\zeta_s)}} \simeq D_f(Np) \otimes D_{g_y}(Np^s) \otimes D_{h_z}(Np^s)(-1)$$

denote the Dieudonné module associated to the Galois representation $V_{fg_yh_z}(N)$ introduced in (70).

Let $H^1_{\exp}(\mathbb{Q}_p(\zeta_s), V_{fg_yh_z}(N)) \subseteq H^1_{\mathrm{fin}}(\mathbb{Q}_p(\zeta_s), V_{fg_yh_z}(N)) \subseteq H^1(\mathbb{Q}_p(\zeta_s), V_{fg_yh_z}(N))$ be the subspaces introduced in [BK93] by Bloch and Kato and let

$$\log_p : H^1_{\exp}(\mathbb{Q}_p(\zeta_s), V_{fg_yh_z}(N)) \longrightarrow D_{fg_yh_z}(N)/(\mathrm{Fil}^0(D_{fg_yh_z}(N)) + D_{fg_yh_z}(N)^{\Phi=1})$$

denote the Bloch-Kato logarithm map of loc. cit.

Since the eigenvalues of the Frobenius operator Φ acting on $D_{fg_yh_z}(N)$ are $\alpha_f\alpha_{g_y}\alpha_{h_z},\dots,\beta_f\beta_{g_y}\beta_{h_z}$, they all are algebraic numbers of complex absolute value $p^{3/2}$, and it follows from [BK93, Corollary 3.8.4] that $D_{fg_yh_z}(N)^{\Phi=1}=0$ and $H^1_{\exp}(\mathbb{Q}_p(\zeta_s),V_{fg_yh_z}(N))=H^1_{\mathrm{fin}}(\mathbb{Q}_p(\zeta_s),V_{fg_yh_z}(N))$. Bloch-Kato's logarithm map may therefore be recast as a map

$$\log_p : H^1_{\mathrm{fin}}(\mathbb{Q}_p(\zeta_s),V_{fg_yh_z}(N)) \longrightarrow D_{fg_yh_z}(N)/\mathrm{Fil}^0(D_{fg_yh_z}(N)) \simeq \mathrm{Fil}^0(D_{fg_yh_z}(N))^\vee, \tag{81}$$

where the latter isomorphism is induced by Poincaré duality.

The p-adic syntomic Abel-Jacobi map is then defined as the composition

$$\mathrm{AJ}_p : \mathrm{CH}^2(W_s)_0(\mathbb{Q}_p(\zeta_s)) \xrightarrow{\varpi_{fg_yh_z}\circ\mathrm{AJ}_{\mathrm{et}}} H^1_{\mathrm{fin}}(\mathbb{Q}_p(\zeta_s),V_{fg_yh_z}(N)) \xrightarrow{\log_p} \mathrm{Fil}^0(D_{fg_yh_z}(N))^\vee. \tag{82}$$

2.4. **Ohta's periods and Λ-adic test vectors.** Let $\hat{\mathbb{Z}}_p^{\mathrm{ur}}$ denote the completion of the ring of integers in the maximal unramified extension $\hat{\mathbb{Q}}_p^{\mathrm{ur}}$ of $\mathbb{Q}_p$. Recall the convention whereby the symbol $\hat{\otimes}$ denotes the completed tensor product, so that, for example, $\Lambda\hat{\otimes}\hat{\mathbb{Z}}_p^{\mathrm{ur}}$ is isomorphic to the power series ring $\hat{\mathbb{Z}}_p^{\mathrm{ur}}[[T]]$.

Define $D(V_1^{\mathrm{ord}}(Np^s)^-):=(V_1^{\mathrm{ord}}(Np^s)^-\otimes\hat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}$. Since the modules $V_1^{\mathrm{ord}}(Np^s)^-$ are unramified, they are admissible for the period ring $\hat{\mathbb{Z}}_p^{\mathrm{ur}}$ and hence $D_{\mathrm{dR}}(V_1^{\mathrm{ord}}(Np^s)^-)=D(V_1^{\mathrm{ord}}(Np^s)^-)$. Define also

$$\mathbb{D}(\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^-) := (\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^-\hat{\otimes}\hat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}} = \varprojlim_s D(V_1^{\mathrm{ord}}(Np^s)^-).$$

Recall that the comparison theorem of (78) leads to a natural inclusion

$$S_2^{\mathrm{ord}}(Np^s)=e_{\mathrm{ord}}\Omega^1(X_s)\longrightarrow e_{\mathrm{ord}}H^1_{\mathrm{dR}}(X_s)=D_{\mathrm{dR}}(V_1^{\mathrm{ord}}(Np^s)).$$

Write

$$\mathrm{cmp}_s : S_2^{\mathrm{ord}}(Np^s)\longrightarrow D(V_1^{\mathrm{ord}}(Np^s)^-)$$

for the natural map obtained by composing this inclusion with the natural projection

$$D_{\mathrm{dR}}(V_1^{\mathrm{ord}}(Np^s))\longrightarrow D(V_1^{\mathrm{ord}}(Np^s)^-)$$

arising from the $G_{\mathbb{Q}_p}$-invariant filtration on $V_1^{\mathrm{ord}}(Np^s)$.

We begin by recalling a theorem of Ohta [Oh95], as refined subsequently by Wake [Wa] which provides a p-adic interpolation of the maps cmp_s, relating the space $S_\Lambda^{\mathrm{ord}}(N)$ of ordinary Λ-adic modular forms to the Galois representation $\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)$.

Proposition 2.9 (Ohta). *There is a canonical isomorphism*

$$\mathrm{cmp}_\infty : S_\Lambda^{\mathrm{ord}}(N)\longrightarrow \mathbb{D}(\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^-)$$

characterised by the commutativity of the diagram

$$\begin{tikzcd} S_\Lambda^{\mathrm{ord}}(N) \arrow[r, "\mathrm{cmp}_\infty"] \arrow[d, "\mho_s"'] & \mathbb{D}(\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^-) \arrow[d] \\ S_2^{\mathrm{ord}}(Np^s) \arrow[r, "\mathrm{cmp}_s"] & D(V_1^{\mathrm{ord}}(Np^s)^-) \end{tikzcd}$$

where the left vertical arrow sends the ordinary Λ-adic form $\underline{g}=\{g_s\}$ to $\mho_s(g_s)=U_p^{-s}g_s$, and the right vertical arrow denotes the natural specialisation map of level p^s.

The reader will recall that the system $\{\mho_s(g_s)\}$ of ordinary modular forms on X_s are compatible under the pushforward maps ϖ_{2*}, in harmony with the fact that the inverse limit defining $\mathbb{D}(\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^-))$ is also taken relative to these maps. The statement differs slightly from the one in Ohta's work, which focusses on the anti-ordinary part of the étale cohomology of the tower X_s, in which the inverse limits are taken relative to the maps ϖ_{1*}. Working with the ordinary part obviates the application of the automorphism w_s to the specialisation at level s which occurs in Ohta's result.

Since the U_p operator (and hence, the ordinary projector, as well) are self-adjoint relative to the twisted Λ-adic Weil pairing $[\ ,\ ]_\Gamma$ of (40), this pairing induces a pairing on the ordinary parts, which shall be denoted by the same symbol:

$$[\,,]_\Gamma : \mathbb{V}_1^{\mathrm{ord}}(Np^\infty) \times \mathbb{V}_1^{\mathrm{ord}}(Np^\infty) \longrightarrow \Lambda(\underline{\epsilon}_{\mathrm{cyc}}). \tag{83}$$

The submodule $\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^+$ is isotropic for this pairing, which therefore induces a natural $\Lambda(\underline{\epsilon}_{\mathrm{cyc}})$-valued duality between $\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^+$ and the quotient $\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^-$. (This duality is the one described, for instance, in Theorem B of [Oh95].) Combining $[\ ,\ ]_\Gamma$ with Ohta's map cmp_∞ yields canonical $G_{\mathbb{Q}_p}$-equivariant Λ-linear maps, as described in [Wa, Theorem 1.3]:

$$\begin{aligned} \underline{c}^+ :& S_\Lambda^{\mathrm{ord}}(N) \longrightarrow \mathrm{Hom}_\Lambda(\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^+, \Lambda(\underline{\epsilon}_{\mathrm{cyc}}) \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}}), \\ \underline{c}^- :& S_\Lambda^{\mathrm{ord}}(N)^\vee \longrightarrow \mathrm{Hom}_\Lambda(\mathbb{V}_1^{\mathrm{ord}}(Np^\infty)^-, \Lambda \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}}), \end{aligned} \tag{84}$$

which are given by the explicit formulae

$$\begin{aligned} \underline{c}^+(\underline{\phi})(\underline{\xi}) &= [\mho_\infty(\underline{\phi}), \underline{\xi}]_\Gamma = \left\{ \langle U_p^{-s}\phi_s, ww_s U_p^s \xi_s \rangle_{\Gamma_s} \right\}_{s\geq 1} \\ &= \{\langle \phi_s, ww_s\xi_s\rangle_{\Gamma_s}\}_{s\geq 1} = \left\{ -\langle \xi_s, (ww_s)^{-1}\phi_s\rangle_{\Gamma_s}\right\}_{s\geq 1}. \end{aligned} \tag{85}$$

$$\underline{c}^-(\underline{\phi}^*)(\underline{\xi}) = [\underline{\phi}^*, \underline{\xi}]_\Gamma = \left\{ -\langle \xi_s, (ww_s)^{-1} U_p^s \eta_{\phi_s^*}\rangle_{\Gamma_s}\right\}_{s\geq 1}. \tag{86}$$

After tensoring over Λ with Λ_g and taking $\underline{g}$-isotypic parts in (84), one deduces Λ_g-linear $G_{\mathbb{Q}_p}$-equivariant homomorphisms

$$\underline{c}_g^+ : S_{\Lambda_g}(N)[\underline{g}] \longrightarrow \mathrm{Hom}_{\Lambda_g}(\mathbb{V}_g^+(N), \Lambda_g(\underline{\epsilon}_{\mathrm{cyc}}) \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}}), \tag{87}$$

$$\underline{c}_g^- : S_{\Lambda_g}(N)^\vee[\underline{g}] \longrightarrow \mathrm{Hom}_{\Lambda_g}(\mathbb{V}_g^-(N), \Lambda_g \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}}), \tag{88}$$

which are described by the rules, deduced from (85) and (86),

$$\underline{c}_g^+(\underline{\breve{g}})(\underline{\xi}) = \left\{ -\langle \xi_s, (ww_s)^{-1}\breve{g}_s\rangle_{\Gamma_s}\right\}_{s\geq 1}, \tag{89}$$

$$\underline{c}_g^-(\underline{\breve{g}}^*)(\underline{\xi}) = \left\{ \alpha_{\underline{g}_s}^s \langle \xi_s, (ww_s)^{-1}\eta_{\breve{g}_s^*}\rangle_{\Gamma_s}\right\}_{s\geq 1}. \tag{90}$$

If $\tilde{\Lambda}$ is any complete Noetherian $\mathbb{Z}_p$-algebra and $\Psi : G_{\mathbb{Q}_p} \longrightarrow \tilde{\Lambda}^\times$ is an unramified character, it is convenient to define

$$\tilde{\Lambda}\{\Psi\} := (\tilde{\Lambda} \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}})^\Psi$$

to be the $\tilde{\Lambda}$-submodule of $\tilde{\Lambda} \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}}$ on which $G_{\mathbb{Q}_p}$ acts via the character Ψ. As explained in [LZ12, Prop. 3.3], $\tilde{\Lambda}\{\Psi\}$ is non-canonically isomorphic to $\tilde{\Lambda}(\Psi)$ as a $\tilde{\Lambda}[G_{\mathbb{Q}_p}]$-module. The advantage of working with $\tilde{\Lambda}\{\Psi\}$ is that one has a *canonical* identification

$$\mathbb{D}(\tilde{\Lambda}\{\Psi\}) = (((\tilde{\Lambda} \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}})^\Psi) \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}} = \tilde{\Lambda}, \tag{91}$$

the last equality being simply induced from the multiplication on the coefficients.

The $G_{\mathbb{Q}_p}$- equivariance of the maps (87) and (88) above shows that $\underline{c}_g^+(\underline{\breve{g}})$ and $\underline{c}_g^-(\underline{\breve{g}}^*)$ give rise to $G_{\mathbb{Q}_p}$-equivariant Λ_g-linear maps

$$\underline{c}_g^+(\underline{\breve{g}}) : \mathbb{V}_g(N)^+ \longrightarrow \Lambda\{\Psi_{\underline{g}}^{-1}\}(\underline{\epsilon}_{\mathrm{cyc}}), \qquad \underline{c}_g^-(\underline{\breve{g}}^*) : \mathbb{V}_g(N)^- \longrightarrow \Lambda\{\Psi_{\underline{g}}\},$$

with similar remarks applying, of course, to the elements $\underline{c}_h^+(\underline{\breve{h}})$ and $\underline{c}_h^-(\underline{\breve{h}}^*)$. The maps $\underline{c}_g^+(\underline{\breve{g}})$ and $\underline{c}_g^-(\underline{\breve{g}}^*)$ also admit simpler variants associated to the fixed forms $\breve{f}$ and $\breve{f}^*$,

$$c_f^+(\breve{f}) : V_f(Np)^+ \longrightarrow \mathbb{Q}_p\{\psi_f^{-1}\}(\epsilon_{\mathrm{cyc}}), \qquad \underline{c}_f^-(\breve{f}^*) : V_f(Np)^- \longrightarrow \mathbb{Q}_p\{\psi_f\}.$$

Recall from Lemma 2.1 that there are canonical isomorphisms of $\Lambda_{f\underline{gh}}[G_{\mathbb{Q}_p}]$-modules

$$\begin{aligned} \mathbb{V}_{f\underline{gh}}^f(N) &\simeq V_f(Np)^- \otimes \mathbb{V}_{\underline{g}}(N)^+ \otimes_\Lambda \mathbb{V}_{\underline{h}}(N)^+(\underline{\epsilon}_{\mathrm{cyc}}^{-1}), \\ \mathbb{V}_{f\underline{gh}}^g(N) &\simeq V_f(Np)^+ \otimes \mathbb{V}_{\underline{g}}(N)^- \otimes_\Lambda \mathbb{V}_{\underline{h}}(N)^+(\underline{\epsilon}_{\mathrm{cyc}}^{-1}), \\ \mathbb{V}_{f\underline{gh}}^h(N) &\simeq V_f(Np)^+ \otimes \mathbb{V}_{\underline{g}}(N)^+ \otimes_\Lambda \mathbb{V}_{\underline{h}}(N)^-(\underline{\epsilon}_{\mathrm{cyc}}^{-1}). \end{aligned}$$

Choose a triple of Λ-adic test vectors

$$\breve{f} \in S_2(Np)[f], \qquad \underline{\breve{g}} \in S_{\Lambda_g}^{\mathrm{ord}}(N,\chi)[\underline{g}], \qquad \underline{\breve{h}} \in S_{\Lambda_h}^{\mathrm{ord}}(N,\chi^{-1})[\underline{h}], \tag{92}$$

and a triple of dual test vectors

$$\breve{f}^* \in S_2(Np)^\vee[f], \qquad \underline{\breve{g}}^* \in S^{\mathrm{ord}}_{\Lambda_g}(N,\chi^{-1})^\vee[\underline{g}], \qquad \underline{\breve{h}}^* \in S^{\mathrm{ord}}_{\Lambda_h}(N,\chi)^\vee[\underline{h}] \tag{93}$$

as in the introduction.

We can then define the local Λ-adic cohomology classes

$$\begin{aligned}
\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}}) &:= c_f^-(\breve{f}^*) \otimes \underline{c}_g^+(\underline{\breve{g}}) \otimes \underline{c}_h^+(\underline{\breve{h}})(\kappa_p^f(f, \underline{gh})) &\in\ & H^1(\mathbb{Q}_p, \Lambda_{f\underline{gh}}\{\psi_f \Psi_{\underline{g}}^{-1}\Psi_{\underline{h}}^{-1}\}(\underline{\epsilon}_{\mathrm{cyc}})),\\
\kappa_p^g(\breve{f}, \underline{\breve{g}^*\breve{h}}) &:= c_f^+(\breve{f}) \otimes \underline{c}_g^-(\underline{\breve{g}}^*) \otimes \underline{c}_h^+(\underline{\breve{h}})(\kappa_p^g(f, \underline{gh})) &\in\ & H^1(\mathbb{Q}_p, \Lambda_{f\underline{gh}}\{\psi_f^{-1} \Psi_{\underline{g}}\Psi_{\underline{h}}^{-1}\chi^{-1}\}(\epsilon_{\mathrm{cyc}})),\\
\kappa_p^h(\breve{f}, \underline{\breve{g}\breve{h}^*}) &:= c_f^+(\breve{f}) \otimes \underline{c}_g^+(\underline{\breve{g}}) \otimes \underline{c}_h^-(\underline{\breve{h}}^*)(\kappa_p^h(f, \underline{gh})) &\in\ & H^1(\mathbb{Q}_p, \Lambda_{f\underline{gh}}\{\psi_f^{-1} \Psi_{\underline{g}}^{-1}\Psi_{\underline{h}}\chi\}(\epsilon_{\mathrm{cyc}})).
\end{aligned} \tag{94}$$

The modules in which the above classes take values are non-canonically isomorphic to $\Lambda_{f\underline{gh}}(\underline{\Upsilon}_{\underline{f}}^+)$, $\Lambda_{f\underline{gh}}(\underline{\Upsilon}_{\underline{g}}^+)$ and $\Lambda_{f\underline{gh}}(\underline{\Upsilon}_{\underline{h}}^+)$, respectively, but elements of the former incorporate a choice of period (in the style of [LZ12]) for the relevant unramified Galois characters.

As a piece of notation, we let

$$\begin{aligned}
\kappa_p^f(\breve{f}^*, \breve{g}_y, \breve{h}_z) &\in H^1(\mathbb{Q}_p, K_x\{\psi_f \Psi_{g_y}^{-1}\Psi_{h_z}^{-1}\}(\epsilon_{\mathrm{cyc}}^{\ell-1}\epsilon\omega^{1-\ell})),\\
\kappa_p^g(\breve{f}, \breve{g}_y^*, \breve{h}_z) &\in H^1(\mathbb{Q}_p, K_x\{\psi_f^{-1} \Psi_{g_y}\Psi_{h_z}^{-1} \times \chi^{-1}\}(\epsilon_{\mathrm{cyc}})),
\end{aligned} \tag{95}$$

denote the local cohomology classes obtained by specialising the Λ-adic classes of (94) at a classical point $x := (y,z) \in \Omega_g \times_\Omega \Omega_h$ of weight (ℓ, ϵ) for some positive integer $\ell \geq 1$ and character ϵ of conductor p^s, corresponding to a ring homomorphism $x : \Lambda_{f\underline{gh}} \longrightarrow K_x$ to a finite extension K_x of $\mathbb{Q}_p$.

Our most immediate goal is now to relate the images of the classes $\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}})$ and $\kappa_p^g(\breve{f}, \underline{\breve{g}^*\breve{h}})$ under Perrin-Riou's Λ-adic logarithm map to the restriction to $\Omega_{f\underline{gh}}$ of the Garrett-Hida p-adic L-functions $\mathscr{L}_p^{f_\alpha}(\underline{\breve{f}}^*, \underline{\breve{g}}, \underline{\breve{h}})$ and $\mathscr{L}_p^{g_\alpha}(\underline{\breve{f}}, \underline{\breve{g}}^*, \underline{\breve{h}})$, respectively. (A similar relation exists between $\mathscr{L}_p^{h_\alpha}(\underline{\breve{f}}, \underline{\breve{g}}, \underline{\breve{h}}^*)$ and $\kappa_p^h(\breve{f}, \underline{\breve{g}\breve{h}^*})$, since g and h play symmetric roles in all the constructions; the details have been omitted because the latter relation plays no role in the proof of the main theorems of the article.)

The family $\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}})$ and the p-adic L-function $\mathscr{L}_p^{f_\alpha}(\underline{\breve{f}}^*, \underline{\breve{g}}, \underline{\breve{h}})$ play a crucial role in the proofs of Theorems A and C, while $\kappa_p^g(\breve{f}, \underline{\breve{g}^*\breve{h}})$ and $\mathscr{L}_p^{g_\alpha}(\underline{\breve{f}}, \underline{\breve{g}}^*, \underline{\breve{h}})$ are the main ingredient in the proofs of Theorems B and D. It is worth remarking that the family of cohomology classes $\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}})$, whose underlying representation space involves the Λ-adic cyclotomic character, is of greater arithmetic complexity than the family $\kappa_p^g(\breve{f}, \underline{\breve{g}^*\breve{h}})$, whose restriction to the inertia group at p is equal to ϵ_{cyc}, and whose specialisations are therefore independent of (y,z) when restricted to inertia.

2.5. **The logarithm of weight two specialisations.** Let $\epsilon \neq \omega$ be a Dirichlet character of conductor p^s and let $x := (y,z) \in \Omega_g \times_\Omega \Omega_h$ be a classical point of weight $(2,\epsilon)$. Put $K = K_x = K_{fg_yh_z}$ as above and let $\kappa_p^f(\breve{f}^*, \breve{g}_y, \breve{h}_z)$ and $\kappa_p^g(\breve{f}, \breve{g}_y^*, \breve{h}_z)$ be the local cohomology classes introduced in (95). Since the Hodge-Tate weights of the target Galois representations are all -1, the Bloch-Kato p-adic logarithm maps induce isomorphisms

$$\begin{aligned}
\log_p &: H^1(\mathbb{Q}_p, K_x\{\psi_f \Psi_{g_y}^{-1}\Psi_{h_z}^{-1}\}(\epsilon_{\mathrm{cyc}}\epsilon\omega^{-1})) \longrightarrow K\\
\log_p &: H^1(\mathbb{Q}_p, K_x\{\psi_f^{-1} \Psi_{g_y}\Psi_{h_z}^{-1} \times \chi^{-1}\}(\epsilon_{\mathrm{cyc}})) \longrightarrow K.
\end{aligned} \tag{96}$$

We refer to (142) in §5.1 below for a more detailed discussion of this fact in a more general setting and a discussion of the periods involved in the above isomorphisms.

The goal of this section is to give an explicit formula for the Bloch-Kato p-adic logarithms of the classes $\kappa_p^f(\breve{f}^*, \breve{g}_y, \breve{h}_z)$ and $\kappa_p^g(\breve{f}, \breve{g}_y^*, \breve{h}_z)$ in terms of the image of the twisted diagonal cycles introduced in the previous chapter under the p-adic syntomic Abel-Jacobi map (82).

Recall the de Rham cohomology classes of (79) and (80), and let

$$\begin{aligned}
\omega^\circ_{\breve{f}} &:= (ww_s)^{-1}\omega_{\breve{f}}, &\qquad \omega^\circ_{\breve{g}_y} &:= (ww_s)^{-1}\omega_{\breve{g}_y}, &\qquad \omega^\circ_{\breve{h}_z} &= (ww_s)^{-1}\eta_{\breve{h}_z},\\
\eta^\circ_{\breve{f}^*} &:= (ww_s)^{-1}\eta_{\breve{f}^*}, &\qquad \eta^\circ_{\breve{g}_y^*} &:= (ww_s)^{-1}\eta_{\breve{g}_y^*}, &\qquad \eta^\circ_{\breve{h}_z^*} &= (ww_s)^{-1}\eta_{\breve{h}_z^*},
\end{aligned}$$

viewed as elements of the anti-ordinary part of $H^1_{\mathrm{dR}}(X_0(Np)_{/\mathbb{Q}_p})$ and $H^1_{\mathrm{dR}}(X_{s/\mathbb{Q}_p(\zeta_s)})$, respectively. The triple tensor classes

$$\eta^\circ_{\breve f^*}\otimes\omega^\circ_{\breve g_y}\otimes\omega^\circ_{\breve h_z},\qquad \omega^\circ_{\breve f}\otimes\eta^\circ_{\breve g_y^*}\otimes\omega^\circ_{\breve h_z},\qquad \omega^\circ_{\breve f}\otimes\omega^\circ_{\breve g_y}\otimes\eta^\circ_{\breve h_z^*},$$

are thus elements in the anti-ordinary part of $H^3_{\mathrm{dR}}(W^\dagger_{s,s/\mathbb{Q}_p(\zeta_s)})=H^3_{\mathrm{dR}}(W_{s,s/\mathbb{Q}_p(\zeta_s)})$. Since the N-part (resp. the p-part) of the nebentype characters of g_y and h_z are inverse one of another (resp. equal), these classes are invariant under the action of the group D_s of diamond operators acting *diagonally at* N and *anti-diagonally at* p on $W_{s,s}$, hence they descend to classes in $H^3_{\mathrm{dR}}(W_{s/\mathbb{Q}_p(\zeta_s)})$, denoted

$$\eta^\circ_{\breve f^*}\otimes\omega^\circ_{\breve g_y}\omega^\circ_{\breve h_z},\quad \omega^\circ_{\breve f}\otimes\eta^\circ_{\breve g_y^*}\omega^\circ_{\breve h_z},\quad \omega^\circ_{\breve f}\otimes\omega^\circ_{\breve g_y}\eta^\circ_{\breve h_z^*}\in H^3_{\mathrm{dR}}(W_{s/\mathbb{Q}_p(\zeta_s)}),$$

which are characterized by the relation

$$\mathrm{pr}_s^*(\eta^\circ_{\breve f^*}\otimes\omega^\circ_{\breve g_y}\omega^\circ_{\breve h_z})=\eta^\circ_{\breve f^*}\otimes\omega^\circ_{\breve g_y}\otimes\omega^\circ_{\breve h_z},\tag{97}$$

and likewise for the other two. Faltings' comparison isomorphism (78) together with Künneth's formula (55) allows to regard them as elements of $\mathrm{Fil}^0(D_{fg_yh_z}(N))$ and it thus makes sense to evaluate the functional $\mathrm{AJ}_p(\Delta_s)$ at these classes.

Proposition 2.10. *For all arithmetic points $x=(y,z)$ of weight $(2,\epsilon)$ as above, the specialisations*

$$\kappa_p^f(\breve f^*,\breve g_y,\breve h_z):=\kappa_p^f(\breve f^*,\underline{\breve g\breve h})(y,z),\qquad \kappa_p^g(\breve f,\breve g_y^*,\breve h_z):=\kappa_p^g(\breve f,\underline{\breve g^*\breve h})(y,z)$$

satisfy the formulae

$$\begin{aligned}\log_p(\kappa_p^f(\breve f^*,\breve g_y,\breve h_z)) &= \alpha_g^{-s}\mathrm{AJ}_p(\Delta_s)(\eta^\circ_{\breve f^*}\otimes\omega^\circ_{\breve g_y}\omega^\circ_{\breve h_z}),\\ \log_p(\kappa_p^g(\breve f,\breve g_y^*,\breve h_z)) &= \mathrm{AJ}_p(\Delta_s)(\omega^\circ_{\breve f}\otimes\eta^\circ_{\breve g_y^*}\omega^\circ_{\breve h_z}).\end{aligned}$$

Proof. Let $\langle\ ,\ \rangle_{W_s}$ denote the Poincaré pairing (in both its étale and de Rham incarnations) on the $(1,1,1)$-Künneth component of the middle cohomology of W_s, which is related to the Poincaré duality on the curves $X_0(Np)$ and X_s by the rules

$$\langle\eta_1\otimes\eta_2\eta_3,\omega_1\otimes\omega_2\omega_3\rangle_{W_s}=\langle\eta_1,\omega_1\rangle_{X_0(Np)}\langle\eta_2,\omega_2\rangle_s\langle\eta_3,\omega_3\rangle_s.$$

Let $\langle\ ,\ \rangle_{W_s,\Gamma_s}$ denote the group-ring valued pairing constructed from the above by setting

$$\langle\eta_1\otimes\eta_2\eta_3,\omega_1\otimes\omega_2\omega_3\rangle_{W_s,\Gamma_s}=\langle\eta_1,\omega_1\rangle_{X_0(Np)}\langle\eta_2,\omega_2\rangle_{\Gamma_s}\langle\eta_3,\omega_3\rangle_{\Gamma_s}.$$

The formulae for $\underline{c}_g^+(\underline{\breve g})$ and $\underline{c}_h^+(\underline{\breve h})$ described in equation (89) lead to the following equality in the local cohomology group $H^1(\mathbb{Q}_p,K_x(\Upsilon_f^+(y,z)))$, where $\kappa_p(f,\underline{gh})_s$ denotes the natural projection of the ordinary Λ-adic class $\kappa_p(f,\underline{gh})$ to the s-th component in the inverse limit:

$$\kappa_p^f(\breve f^*,\breve g_y,\breve h_z):=\nu_{(y,z)}(\kappa_p^f(\breve f^*,\underline{\breve g},\underline{\breve h}))=\nu_{(y,z)}(\langle\kappa_p(f,\underline{gh})_s,\eta^\circ_{f^*}\otimes\omega^\circ_{\breve g_s}\omega^\circ_{\breve h_s}\rangle_{W_s,\Gamma_s}).\tag{98}$$

It then follows from Lemma 1.12 that

$$\kappa_p^f(\breve f^*,\breve g_y,\breve h_z)=\langle\kappa_p(f,g_y,h_z)_s,\eta^\circ_{f^*}\otimes\omega^\circ_{\breve g_y}\omega^\circ_{\breve h_z}\rangle_{W_s}.\tag{99}$$

Proposition 2.5 thus yields $\kappa_p^f(\breve f^*,\breve g_y,\breve h_z)=\alpha_{g_y}^{-s}\langle\mathrm{AJ}_{\mathrm{et}}(\Delta_s),\eta^\circ_{\breve f^*}\otimes\omega^\circ_{\breve g_y}\omega^\circ_{\breve h_z}\rangle_{W_s}$ and the first assertion of the proposition follows by appying the Bloch-Kato logarithm to both sides. The second is proved by the same calculation, but replacing (98) by the analogous formula arising from both (89) and (90), namely $\kappa_p^g(\breve f,\breve g_y^*,\breve h_z)=\alpha_{g_y}^s\nu_{(y,z)}(\langle\kappa(f,\underline{gh}),\eta^\circ_{\breve f}\otimes\eta^\circ_{\breve g_s^*}\omega^\circ_{\breve h_s}\rangle_{W_s,\Gamma_s})$. □

For the later calculations, it will be convenient to dispose of a simpler formula for $\log_p(\kappa_p^f(\breve f^*,\breve g_y,\breve h_z))$ and $\log_p(\kappa_p^g(\breve f,\breve g_y^*,\breve h_z))$ involving the product variety $W_{s,s}$ rather than its quotient W_s, and the classes $\eta_{\breve f^*}\otimes\omega_{\breve g_y}\omega_{\breve h_z}$ and $\omega_{\breve f}\otimes\eta_{\breve g_y^*}\omega_{\breve h_z}$ instead of their anti-ordinary counterparts. To do this, define the cycle

$$\Delta^\circ_{s,s}:=((ww_1)^{-1},(ww_s)^{-1},(ww_s)^{-1})_*\Delta_{s,s}\in\mathrm{CH}^2(W_{s,s})_0(\mathbb{Q}_p(\zeta_N\zeta_s)).$$

Equations (41) and (49) show that $\Delta^\circ_{s,s}$ is given by the simple equation

$$\begin{aligned}\Delta^\circ_{s,s}&=((ww_1)^{-1},(ww_s)^{-1},(ww_s)^{-1})_*\varepsilon_{s,s*}(j_1\circ\varpi_2^{s-1}\circ w_s,\mathrm{Id},w_s)_*\delta_*(X_s)\\ &=\varepsilon_{s,s*}(j_1\circ\varpi_1^{s-1},w_s^{-1},\mathrm{Id})_*(w^{-1},w^{-1},w^{-1})_*\delta_*(X_s)=\varepsilon_{s,s*}(j_1\circ\varpi_1^{s-1},w_s^{-1},\mathrm{Id})_*\delta_*(X_s).\end{aligned}\tag{100}$$

Corollary 2.11. *For all arithmetic points $x = (y, z)$ of weight $(2, \epsilon)$,*

$$\begin{aligned}\log_p(\kappa_p^f(\breve{f}^*, \breve{g}_y, \breve{h}_z)) &= \alpha_g^{-s} d_s^{-1} \mathrm{AJ}_p(\Delta_{s,s}^\circ)(\eta_{\breve{f}^*} \otimes \omega_{\breve{g}_y}\omega_{\breve{h}_z}),\\ \log_p(\kappa_p^g(\breve{f}, \breve{g}_y^*, \breve{h}_z)) &= d_s^{-1} \mathrm{AJ}_p(\Delta_{s,s}^\circ)(\omega_{\breve{f}} \otimes \eta_{\breve{g}_y^*}\omega_{\breve{h}_z}).\end{aligned}$$

Proof. This follows directly from Proposition 2.10 in light of the functorial properties of Abel-Jacobi maps with respect to automorphisms. □

Corollary 2.11 motivates the study of the image of the twisted diagonal cycle $\Delta_{s,s}^\circ$ under the p-adic syntomic Abel-Jacobi map, which is taken up in the next two chapters.

3. The syntomic Abel-Jacobi map on products of semistable curves

3.1. **The cohomology of semistable curves.** This section recalls the description (following [CI99], and [CI10]) of the de Rham cohomology of a semistable curve and the attendant structures with which it is equipped. Let F be a finite extension of $\mathbb{Q}_p$, let $\mathcal{O}_F$ denote its ring of integers, fix an uniformizer π_F and let $k_F = \mathcal{O}_F/\pi_F$ denote the residue field. Let F_0 denote the maximal unramified subfield of F, which is isomorphic to the fraction field of the ring of Witt vectors of k_F. Write $\sigma_0 \in \mathrm{Gal}\,(F_0/\mathbb{Q}_p)$ for the Frobenius element generating the Galois group of $F_0/\mathbb{Q}_p$. Let X be a smooth, connected, proper curve over F and assume it possesses a regular semistable model $\mathcal{X}$ over $\mathcal{O}_F$, meaning that locally at every point, $\mathcal{X}$ is either smooth over $\mathrm{Spec}(\mathcal{O}_F)$ or étale over $\mathrm{Spec}(\mathcal{O}_F[X, Y]/(XY - \pi_F))$.

Let $\tilde{\mathcal{X}}$ denote the special fiber of $\mathcal{X}$. It is assumed that the irreducible components of $\tilde{\mathcal{X}}$ are smooth, geometrically irreducible, and defined over k_F, that there are at least two of them, and that distinct components intersect only in ordinary double points defined over k_F. Let $\mathcal{G}$ denote the dual graph of the special fiber, whose set $\mathcal{V}(\mathcal{G})$ of vertices is in bijection with the irreducible components of $\tilde{\mathcal{X}}$ and whose set $\mathcal{E}(\mathcal{G})$ of oriented edges is in bijection with the singular points of the special fiber, together with an ordering of the two components which intersect at that point. Given $v \in \mathcal{V}(\mathcal{G})$, let $\tilde{\mathcal{X}}_v$ denote the associated component, and let $\tilde{\mathcal{X}}_v^{\mathrm{sm}}$ denote the smooth locus of $\tilde{\mathcal{X}}_v$, that is to say, the complement in $\tilde{\mathcal{X}}_v$ of the singular points of $\tilde{\mathcal{X}}$. If $e \in \mathcal{E}(\mathcal{G})$ is an edge of $\mathcal{G}$, write $s(e)$ and $t(e)$ for its source and target respectively, and x_e for the associated ordinary double point, satisfying $\tilde{\mathcal{X}}_{s(e)} \cap \tilde{\mathcal{X}}_{t(e)} = \{x_e\}$. Write also $\bar{e}$ for the reversed edge, satisfying $s(\bar{e}) = t(e)$ and $t(\bar{e}) = s(e)$; note that $x_{\bar{e}} = x_e$.

Let

$$\mathrm{red} : X(\mathbb{C}_p) \longrightarrow \tilde{\mathcal{X}}(\bar{\mathbb{F}}_p)$$

denote the reduction map and, for each $v \in \mathcal{V}(\mathcal{G})$, let

$$\mathcal{W}_v := \mathrm{red}^{-1}(\tilde{\mathcal{X}}_v(\bar{\mathbb{F}}_p)), \qquad \mathcal{A}_v := \mathrm{red}^{-1}(\tilde{\mathcal{X}}_v^{\mathrm{sm}}(\bar{\mathbb{F}}_p)) \subset \mathcal{W}_v$$

denote the wide open subspace and underlying affinoid associated to the component $\tilde{\mathcal{X}}_v$. If e is an edge of $\mathcal{G}$, the wide opens $\mathcal{W}_{s(e)}$ and $\mathcal{W}_{t(e)}$ intersect in the wide open annulus

$$\mathcal{W}_e := \mathcal{W}_{s(e)} \cap \mathcal{W}_{t(e)} = \mathrm{red}^{-1}(\{x_e\}).$$

The collection $\{\mathcal{W}_v\}_{v\in\mathcal{V}(\mathcal{G})}$ gives rise to an *admissible cover* of $X(\mathbb{C}_p)$ by wide open subsets defined over F, whose nerve is encoded in the graph $\mathcal{G}$. The de Rham cohomology group $H^1_{\mathrm{dR}}(X)$ can be described rigid analytically as the set of hypercocycles

$$\left\{ (\{\omega_v\}_v, \{f_e\}_e) \in \prod_{v\in\mathcal{V}(\mathcal{G})} \Omega^1(\mathcal{W}_v) \times \prod_{e\in\mathcal{E}(\mathcal{G})} \mathcal{O}_{\mathcal{W}_e} \text{ such that } df_e = \omega_{t(e)} - \omega_{s(e)}, f_{\bar{e}} = -f_e \text{ for all } e \in \mathcal{E}(\mathcal{G}) \right\},$$

taken modulo the hyper-coboundaries, which are hypercocycles of the form $(df_v, f_{t(e)} - f_{s(e)})$ attached to a collection $\{f_v\}_v$ of elements of $\mathcal{O}_{\mathcal{W}_v}$ indexed by $v \in \mathcal{V}(\mathcal{G})$.

The non-trivial rigid cohomology groups of $\mathcal{W} = \mathcal{W}_v$ or $\mathcal{W}_e$ are

$$H^0_{\mathrm{rig}}(\mathcal{W}) = F \quad \text{and} \quad H^1_{\mathrm{rig}}(\mathcal{W}) = \Omega^1(\mathcal{W})/d\mathcal{O}_{\mathcal{W}}.$$

Following [CI99, (1.1)], the Mayer-Vietoris sequence attached to this admissible covering places $H^1_{\mathrm{dR}}(X)$ in the middle of a short exact sequence

$$0 \longrightarrow \frac{\oplus_e H^0_{\mathrm{rig}}(\mathcal{W}_e)^-}{\delta(\oplus_v H^0_{\mathrm{rig}}(\mathcal{W}_v))} \xrightarrow{i} H^1_{\mathrm{dR}}(X) \xrightarrow{r} \ker\left(\oplus_v H^1_{\mathrm{rig}}(\mathcal{W}_v) \xrightarrow{\delta} \oplus_e H^1_{\mathrm{rig}}(\mathcal{W}_e)^-\right) \longrightarrow 0, \tag{101}$$

where

- $\oplus_e H^i_{\mathrm{rig}}(\mathcal{W}_e)^-$ is the (-1)-eigen-subspace of $\oplus_e H^i_{\mathrm{rig}}(\mathcal{W}_e)$ which respect to the involution $e \mapsto \bar{e}$;
- the indices e and v run over $\mathcal{E}(\mathcal{G})$ and $\mathcal{V}(\mathcal{G})$ respectively;
- the map δ sends $\{\kappa_v\}_v$ to the class of $\{\kappa_{t(e)} - \kappa_{s(e)}\}_e$;
- the map i sends a collection $\{\lambda_e\}_e$ to the class of the hypercocycle $(\{0\}_v, \{\lambda_e\}_e)$;
- the map r is the natural restriction map sending the class of $(\{\omega_v\}_v, \{f_e\}_e)$ to $\{\omega_v\}_v$.

The cohomology group $H^1_{\mathrm{dR}}(X)$ is an F-vector space equipped with the following further structures:

(i) A decreasing Hodge filtration of F-vector spaces given by $\mathrm{Fil}^i = H^1_{\mathrm{dR}}(X)$ for $i \le 0$, $\mathrm{Fil}^1 = \Omega^1(X)$ and $\mathrm{Fil}^i = \{0\}$ for $i > 1$.

(ii) A canonical F_0-structure $H^1_{\log-\mathrm{cris}}(\tilde{\mathcal{X}})$ afforded by the log-crystalline cohomology of $\tilde{\mathcal{X}}$ with coefficients in F_0. This is a F_0-vector space for which there is a canonical comparison isomorphism

$$H^1_{\log-\mathrm{cris}}(\tilde{\mathcal{X}}) \otimes_{F_0} F \simeq H^1_{\mathrm{dR}}(X). \tag{102}$$

(iii) A σ_0-semilinear Frobenius operator φ on $H^1_{\log-\mathrm{cris}}(\tilde{\mathcal{X}})$ arising from the Frobenius on k_F.

(iv) A F_0-linear *monodromy map* $N : H^1_{\log-\mathrm{cris}}(\tilde{\mathcal{X}}) \longrightarrow H^1_{\log-\mathrm{cris}}(\tilde{\mathcal{X}})$ which via (102) extends to a F-linear map

$$N : H^1_{\mathrm{dR}}(X) \longrightarrow H^1_{\mathrm{dR}}(X), \tag{103}$$

denoted with the same letter. It sends the class of $(\{\omega_v\}_v, \{f_e\}_e)$ to the class of $(\{0\}_v, \{\lambda_e\}_e)$ with $\lambda_e = \partial_{\mathcal{W}_e}\omega_{s(e)}$. Here, $\partial_{\mathcal{W}_e} : H^1_{\mathrm{rig}}(\mathcal{W}_e) \longrightarrow H^0_{\mathrm{rig}}(\mathcal{W}_e)$ denotes the p-adic annular residue on $\mathcal{W}_e$, taken relative to an orientation on $\mathcal{W}_e$ which is determined by e.

(v) A section

$$s : H^1_{\mathrm{dR}}(X) \longrightarrow \oplus_e H^0_{\mathrm{rig}}(\mathcal{W}_e)^- / \delta\left(\oplus_v H^0_{\mathrm{rig}}(\mathcal{W}_v)\right) \tag{104}$$

of the map i in the exact sequence (101), defined by the rule

$$s(\{\omega_v\}_v, \{f_e\}_e) := \{\lambda_e\}_e, \qquad \lambda_e := f_e + F_{\omega_{s(e)}} - F_{\omega_{t(e)}},$$

where F_{ω_v} denotes the *Coleman primitive* of the differential ω_v. This primitive is a canonically defined up to addition of a single constant of integration. It is analytic on each residue disk in $\mathcal{A}_v$ and its restriction to the annulus $\mathcal{W}_e$ (for e any edge having v as either source or target) is generated by a single logarithmic function over the ring of rigid analytic functions on $\mathcal{W}_e$.

Set $d = [F_0 : \mathbb{Q}_p]$. Since $\Phi := \varphi^d$ is F_0-linear, it extends via the comparison isomorphism (102) to a F-linear operator on $H^1_{\mathrm{dR}}(X)$ that is still written as Φ.

The action of Φ on $H^1_{\mathrm{dR}}(X)$ admits a direct description in terms of the exact sequence (101), in which the action of Φ on the left and right-hand terms is realized concretely by choosing a system Φ_v of characteristic 0 lifts of Frobenius to a system $\mathcal{A}_v \subset \mathcal{W}_v[\epsilon] \subset \mathcal{W}_v$ of Frobenius neighbourhoods of the underlying affinoids $\mathcal{A}_v$ in $\mathcal{W}_v$, indexed by a real parameter $0 < \epsilon < 1$. While Φ_v is only defined on $\mathcal{W}_v[\epsilon]$ for suitable $\epsilon > 0$ and does not preserve this domain, the map $\omega_v \mapsto \Phi^*\omega_v$ still induces a well-defined transformation

$$\Phi : \frac{\Omega^1(\mathcal{W}_v)}{d\mathcal{O}_{\mathcal{W}_v}} \xrightarrow{\mathrm{res}_{\mathcal{W}_v[\epsilon]}} \frac{\Omega^1(\mathcal{W}_v[\epsilon])}{d\mathcal{O}_{\mathcal{W}_v[\epsilon]}} \xrightarrow{\Phi^*} \frac{\Omega^1(\mathcal{W}_v[\epsilon'])}{d\mathcal{O}_{\mathcal{W}_v[\epsilon']}} \xrightarrow{\mathrm{res}^{-1}_{\mathcal{W}_v[\epsilon']}} \frac{\Omega^1(\mathcal{W}_v)}{d\mathcal{O}_{\mathcal{W}_v}} \tag{105}$$

where $\mathrm{res}_{\mathcal{W}_v[\epsilon]}$ is the isomorphism induced by restriction to the wide open $\mathcal{W}_v[\epsilon]$. This action of Frobenius on the cohomology does not depend on the choice of characteristic zero liftings $\{\Phi_v\}$. Following the definitions in [CI99], the action of Φ then extends to the full $H^1_{\mathrm{dR}}(X)$ via the requirement that Φ be compatible with the section s of (104).

Definition 3.1. A class in $H^1_{\mathrm{dR}}(X)$ is said to be *pure* if it is in the kernel of both N and s.

The set of pure classes, denoted $H^1_{\mathrm{dR}}(X)_{\mathrm{pure}}$, is a subgroup of $H^1_{\mathrm{dR}}(X)$ which is preserved by the action of Frobenius and is identified via the map r of (101) with

$$H^1_{\mathrm{dR}}(X)_{\mathrm{pure}} = \bigoplus_v \frac{\Omega^1_{\mathrm{sk}}(\mathcal{W}_v)}{d\mathcal{O}_{\mathcal{W}_v}}, \tag{106}$$

where $\Omega^1_{\mathrm{sk}}(\mathcal{W}_v) \subset \Omega^1(\mathcal{W}_v)$ denotes the space of rigid differentials on $\mathcal{W}_v$ *of the second kind*, i.e., have vanishing annular residues. This identification is compatible with the action of Φ on both groups.

Proposition 3.2. *The eigenvalues of Φ on $H^1_{\mathrm{dR}}(X)_{\mathrm{pure}}$ have complex absolute value $\sqrt{p}$.*

Proof. Equation (106) shows that $H^1_{\mathrm{dR}}(X)_{\mathrm{pure}}$ is isomorphic as a Frobenius module to the crystalline cohomology of the disjoint union of the components $\tilde{\mathcal{X}}_v$ of the special fiber $\tilde{\mathcal{X}}$. More precisely, there is an exact sequence

$$0 \to H^1_{\mathrm{cris}}(\tilde{\mathcal{X}}_v) \to H^1_{\mathrm{cris}}(\tilde{\mathcal{X}}_v^{\mathrm{sm}}) \to \oplus_e F(-1) \xrightarrow{\Sigma} F(-1) \to 0 \tag{107}$$

where the comparison theorems allow us to identify

$$H^1_{\mathrm{cris}}(\tilde{\mathcal{X}}_v) \otimes_{F_0} F \simeq \frac{\Omega^1_{\mathrm{sk}}(\mathcal{W}_v)}{d\mathcal{O}_{\mathcal{W}_v}} \quad \text{and} \quad H^1_{\mathrm{cris}}(\tilde{\mathcal{X}}_v^{\mathrm{sm}}) \otimes_{F_0} F \simeq \frac{\Omega^1(\mathcal{W}_v)}{d\mathcal{O}_{\mathcal{W}_v}} = H^1_{\mathrm{rig}}(\mathcal{W}_v).$$

Since the components $\tilde{\mathcal{X}}_v$ are smooth and projective, the theorem follows from the Riemann hypothesis for curves. $\square$

In light of Proposition 3.2, it would have been more appropriate to use the expression *pure of weight* 1 to designate the classes in $H^1_{\mathrm{dR}}(X)_{\mathrm{pure}}$. The vaguer but more concise terminology adopted in Definition 3.1 was preferred because the qualifier *weight* 1 is also used in the setting of modular forms with a different meaning. It follows from Proposition 3.2 that the exact sequence (107) has a natural Frobenius-equivariant splitting and hence the injective map $r : H^1_{\mathrm{dR}}(X)_{\mathrm{pure}} \longrightarrow \oplus_v H^1_{\mathrm{rig}}(\mathcal{W}_v)$ admits a natural Frobenius-equivariant section

$$r^\iota : \oplus_v H^1_{\mathrm{rig}}(\mathcal{W}_v) \longrightarrow H^1_{\mathrm{dR}}(X)_{\mathrm{pure}}. \tag{108}$$

The Poincaré duality on $H^1_{\mathrm{dR}}(X)$ descends to a perfect pairing on $H^1_{\mathrm{dR}}(X)_{\mathrm{pure}}$, which is described by the formula

$$\langle \{\omega_v\}, \{\eta_v\} \rangle_X = \sum_{e \in \mathcal{E}(\mathcal{G})} \mathrm{res}_{\mathcal{W}_e}(F_e \eta_{s(e)}), \tag{109}$$

where the sum is taken over all the oriented edges of $\mathcal{G}$ and F_e is an analytic primitive of the restriction to the annulus $\mathcal{W}_e$ of $\omega_{s(e)}$, which exists because ω_v has vanishing annular residues for all $v \in \mathcal{V}(\mathcal{G})$, and is well-defined up to a constant. (The choice of constant in the local primitive F_e does not affect the term on the right in (109), since η_v also has vanishing residues for all v.)

3.2. **An analytic recipe for the syntomic Abel-Jacobi map.** The goal of this section is to give an analytic expression for the syntomic Abel-Jacobi map on the Chow group of codimension two cycles on the triple product of semistable curves, in terms of Coleman's theory of p-adic integration of rigid differential forms. It will be convenient to place oneself in the abstract setting given by conditions (a)-(d) below. While ostensibly somewhat special, this setting will nonetheless suffice for the applications in this paper.

(a) The ambient three-fold is a product $W = X_1 \times X_2 \times X_3$ of three semistable curves over F.

For $i = 1, 2, 3$, write $\mathcal{G}_i$ for the dual graph of the special fiber of $\mathcal{X}_i$, and let $\Phi_i = \{\Phi_{i,v}\}_{v \in \mathcal{V}(\mathcal{G}_i)}$ denote a system of liftings of Frobenius to each of the basic wide open spaces in the admissible covering of $X_i(\mathbb{C}_p)$ described in the previous section. Denote by $\Phi_{23} := \Phi_2 \otimes \Phi_3$ the corresponding Frobenius endomorphism on $H^1_{\mathrm{dR}}(X_2)_{\mathrm{pure}} \otimes H^1_{\mathrm{dR}}(X_3)_{\mathrm{pure}} \subset H^2_{\mathrm{dR}}(X_2 \times X_3)$. Our second assumption, concerning the cycle Δ itself, is:

(b) The cycle $\Delta \in \mathrm{CH}^2(W)$ is obtained from the data of a semistable curve X, of three semistable morphisms $\pi_i : X \longrightarrow X_i$ and of three correspondences $\varepsilon_i \in \mathrm{Corr}(X_i)$, by setting

$$\Delta := (\varepsilon_1, \varepsilon_2, \varepsilon_3)_*(\pi_1, \pi_2, \pi_3)_*(X).$$

(c) The correspondences ε_i annihilate the H^0 and H^2 terms in the cohomology of X_i. By the same reasoning as in the proof of Proposition 1.4, this implies that Δ is null-homologous, i.e., that it belongs to $\mathrm{CH}^2(W)_0$.

Let

$$\mathrm{AJ}_p : \mathrm{CH}^2(W)_0 \longrightarrow (\mathrm{Fil}^2 H^3_{\mathrm{dR}}(W))^\vee \tag{110}$$

denote the p-adic syntomic Abel-Jacobi map, as discussed e.g. in [Be00, Introduction and §5], [BLZ, §3] and [Ne1, (3.7)]. It follows from [NeNi, Theorem B] that (110) is canonically identified with the composition of the étale Abel-Jacobi map with Bloch-Kato's logarithm, as in (82).

Even under assumptions (a), (b), and (c), we fall short of giving a formula for the full image $\mathrm{AJ}_p(\Delta)$ but only for its values on cohomology classes of the form

$$\eta_1 \otimes \omega_2 \otimes \omega_3 \in H^1_{\mathrm{dR}}(X_1) \otimes \Omega^1(X_2) \otimes \Omega^1(X_3) \subset \mathrm{Fil}^2(H^3_{\mathrm{dR}}(W)),$$

assuming that:

(d) η_1, ω_2 and ω_3 belong to the pure subspaces of $H^1_{\mathrm{dR}}(X_1)$, $H^1_{\mathrm{dR}}(X_2)$, and $H^1_{\mathrm{dR}}(X_3)$ respectively.

Our recipe for describing $\mathrm{AJ}_p(\Delta)(\eta_1 \otimes \omega_2 \otimes \omega_3)$ rests on the following lemma.

Lemma 3.3. *There exists a polynomial $P \in \mathbb{Q}[X]$ satisfying:*

(1) *$P(\Phi_{23})$ annihilates the class of $\omega_2 \otimes \omega_3$ in $H^2_{\mathrm{dR}}(X_2 \times X_3)$;*
(2) *$P(\Phi_X)$ acts invertibly on $\ker N \subset H^1_{\mathrm{dR}}(X)$.*

Proof. By Proposition 3.2, there is a polynomial P satisfying the first condition, all of whose roots have complex absolute value p. But any such P automatically satisfies the second condition, since Frobenius acts on the kernel of N with eigenvalues of complex absolute value either 1 or $\sqrt{p}$. □

Fix a polynomial P as in the above lemma. Since $P(\Phi_{23})$ annihilates the class of $\omega_2 \otimes \omega_3$ in $H^2_{\mathrm{dR}}(X_2 \times X_3)$, there is a system of rigid 1-forms

$$\rho := \rho_P^{v,w}(\omega_2, \omega_3) \in \Omega^1(\mathcal{W}_v \times \mathcal{W}_w), \qquad v \in \mathcal{V}(\mathcal{G}_2), \quad w \in \mathcal{V}(\mathcal{G}_3) \tag{111}$$

satisfying

$$d\rho_P^{v,w}(\omega_2, \omega_3) = P(\Phi_{23})\left((\omega_2 \wedge \omega_3)_{|\mathcal{W}_v \times \mathcal{W}_w}\right).$$

The system $\rho_P^{v,w}(\omega_2, \omega_3)$ of rigid 1-forms is well-defined, up to a system of closed 1-forms on each product of basic wide opens in $X_2 \times X_3$. One next defines a system $\{\tilde{\xi}_P^v(\omega_2, \omega_3)\}_{v \in \mathcal{V}(\mathcal{G}_X)}$ of rigid 1-forms on each basic wide open of X by pulling back the differentials $\rho_P^{v,w}(\omega_2, \omega_3)$ via the maps $(\pi_2, \pi_3) : X \longrightarrow X_2 \times X_3$ and $(\varepsilon_2, \varepsilon_3) X_2 \times X_3 \longrightarrow X_2 \times X_3$:

$$\{\tilde{\xi}_P^v(\omega_2, \omega_3)\}_{v \in \mathcal{V}(\mathcal{G}_X)} = (\pi_2, \pi_3)^* (\varepsilon_2, \varepsilon_3)^* \left(\{\rho_P^{v,w}(\omega_2, \omega_3)\}\right).$$

Condition (c) combined with the Künneth formula for $H^1_{\mathrm{rig}}(X_2 \times X_3)$ implies that for each $v \in \mathcal{V}(\mathcal{G}_X)$, the class of the differential $\tilde{\xi}_P^v(\omega_2, \omega_3)$ in $H^1_{\mathrm{rig}}(\mathcal{W}_v)$ does not depend on the choice of system $\rho_P^{v,w}(\omega_2, \omega_3)$ satisfying (111). Hence the class of $\tilde{\xi}_P(\omega_2, \omega_3) \in \oplus_{v \in \mathcal{V}(\mathcal{G}_X)} H^1_{\mathrm{rig}}(\mathcal{W}_v)$ is independent of this choice. Let

$$\xi_P(\omega_2, \omega_3) := r^\iota(\tilde{\xi}_P(\omega_2, \omega_3)) \in H^1_{\mathrm{dR}}(X)_{\mathrm{pure}},$$

where r^ι is the Frobenius-equivariant section introduced in (108).

Applying the second condition in Lemma 3.3 shows that there is a unique class

$$\xi(\omega_2, \omega_3) \in H^1_{\mathrm{dR}}(X)_{\mathrm{pure}} \quad \text{satisfying} \quad P(\Phi_C)\xi(\omega_2, \omega_3) = \xi_P(\omega_2, \omega_3).$$

Theorem 3.4. *For all $\Delta \in \mathrm{CH}^2(W)_0$ and classes η_1, ω_2 and ω_3 satisfying conditions $(a) - (d)$,*

$$\mathrm{AJ}_p(\Delta)(\eta_1 \otimes \omega_2 \otimes \omega_3) = \langle \pi_1^* \varepsilon_1^*(\eta_1), \xi(\omega_2, \omega_3) \rangle_X.$$

Proof. The proof of this formula closely follows the one given in [DR13, §3.3] for the Gross-Kudla-Schoen diagonal cycle on the cube of a modular curve with good reduction at p, which exploits Besser's finite polynomial cohomology for smooth varietes over $\mathbb{Z}_p$ introduced in [Be00]. This theory has been extended recently to the setting of semistable varieties by Besser, Loeffler and Zerbes in [BLZ], building crucially on prior work of Nekovar and Niziol [NeNi]. Namely, [BLZ, Proposition 3.3] asserts in this case that

$$\mathrm{AJ}_p(\Delta)(\eta_1 \otimes \omega_2 \otimes \omega_3) = \mathrm{tr}_{\mathcal{X}}\left((\pi_1, \pi_2, \pi_3)^* (\varepsilon_1, \varepsilon_2, \varepsilon_3)^* (\tilde{\eta}_1 \otimes \tilde{\omega}_2 \otimes \tilde{\omega}_3)\right) \qquad \text{where}$$

- $\tilde{\eta}_1 \in H^1_{\mathrm{fp}}(\mathcal{X}_1, 0)$, $\tilde{\omega}_2 \in H^1_{\mathrm{fp}}(\mathcal{X}_2, 1)$, $\tilde{\omega}_3 \in H^1_{\mathrm{fp}}(\mathcal{X}_3, 1)$ are respectively lifts of η_1, ω_2, ω_3 to P-syntomic cohomology in the sense of [BLZ, §2.2]. In order to lighten the notations, the polynomials implicit in the construction shall be omitted.

- $(\pi_1,\pi_2,\pi_3)^*(\varepsilon_1,\varepsilon_2,\varepsilon_3)^*(\tilde\eta_1\otimes\tilde\omega_2\otimes\tilde\omega_3)\in H^3_{\rm fp}(\mathcal{X},2)$ stands for the pull-back to $\mathcal{X}$ of the tensor product of these classes in syntomic cohomology.
- $\mathrm{tr}_{\mathcal{X}}: H^3_{\rm fp}(\mathcal{X},2)\xrightarrow{\sim} F$ is the trace isomorphism of [BLZ, Definition 3.1].

Let $\langle\,,\rangle_{\rm fp}: H^1_{\rm fp}(\mathcal{X},0)\times H^2_{\rm fp}(\mathcal{X},2)\longrightarrow F$ denote the cup-product described in [BLZ, §1.3, §2.4]. By the compatibility between the trace and cup-product in syntomic cohomology afforded by [BLZ, Theorem 2.20], the above quantity may be recast as

$$\mathrm{tr}_{\mathcal{X}}\big((\pi_1,\pi_2,\pi_3)^*(\varepsilon_1,\varepsilon_2,\varepsilon_3)^*(\tilde\eta_1\otimes\tilde\omega_2\otimes\tilde\omega_3)\big)=\langle\pi_1^*\varepsilon_1^*(\tilde\eta_1),(\pi_2,\pi_3)^*(\varepsilon_2,\varepsilon_3)^*(\tilde\omega_2\otimes\tilde\omega_3)\rangle_{\rm fp}.$$

According to the explicit description of finite polynomial cohomology provided in [BLZ, Definition 2.4], the class $\tilde\omega_2\otimes\tilde\omega_3$ in $H^2_{\rm fp}(\mathcal{X}^2,2)$ may be represented by the pair $(\rho,\omega_2\otimes\omega_3)$ where ρ is as in (111), and hence the class $(\pi_2,\pi_3)^*(\varepsilon_2,\varepsilon_3)^*(\tilde\omega_2\otimes\tilde\omega_3)$ in $H^2_{\rm fp}(\mathcal{X},2)$ is represented by the image of $\xi=\xi(\omega_2,\omega_3)$ under the natural morphism $\mathrm{i}: H^1_{\rm dR}(X)\longrightarrow H^2_{\rm fp}(\mathcal{X},2)$, $\mathrm{i}(\xi)=P(\Phi)(\xi)$. It follows that

$$\langle\pi_1^*\varepsilon_1^*(\tilde\eta_1),(\pi_2,\pi_3)^*(\varepsilon_2,\varepsilon_3)^*(\tilde\omega_2\otimes\tilde\omega_3)\rangle_{\rm fp}=\langle\pi_1^*\varepsilon_1^*(\tilde\eta_1),\mathrm{i}(\xi(\omega_2,\omega_3))\rangle_{\rm fp}=\langle\pi_1^*\varepsilon_1^*(\eta_1),\xi(\omega_2,\omega_3)\rangle_X,$$

where the second equality follows as in [Be00, (14)]. □

4. Bloch-Kato logarithms of weight two specializations

As was explained at the end of §2.4, the aim of this chapter is to show that the one-variable p-adic L-functions

$$\mathscr{L}_p^{f_\alpha}(\breve f^*,\underline{\breve g\breve h}):=\mathscr{L}_p^{f_\alpha}(\underline{\breve f}^*,\underline{\breve g},\underline{\breve h})_{|\Omega_{f\underline{gh}}},\quad \mathscr{L}_p^{g_\alpha}(\breve f,\underline{\breve g^*\breve h}):=\mathscr{L}_p^{g_\alpha}(\underline{\breve f},\underline{\breve g}^*,\underline{\breve h})_{|\Omega_{f\underline{gh}}} \tag{112}$$

arising as the restriction to $\Omega_{f\underline{gh}}$ of the Garrett-Hida p-adic L-functions discussed in the introduction can be recovered as the image of the Λ-adic classes $\kappa_p^f(\breve f^*,\underline{\breve g\breve h})$ and $\kappa_p^g(\breve f,\underline{\breve g^*\breve h})$ and under Perrin-Riou's Λ-adic logarithm map. We shall do so by comparing their values at classical points (y,z) of weight 2. Corollary 2.11 asserts that the Bloch-Kato logarithm of the specialisation of these classes at such points may be computed as the value of the functional $\mathrm{AJ}_p(\Delta^\circ_{s,s})$ at the classes $\eta_{\breve f^*}\otimes\omega_{\breve g_y}\otimes\omega_{\breve h_z}$ and $\omega_{\breve f}\otimes\eta_{\breve g_y^*}\otimes\omega_{\breve h_z}$, respectively.

Armed with Theorem 3.4, we can now proceed with this program, starting with an analysis of the above forms in the rigid analytic cohomology of the modular curves $X_0(Np)$ and X_s.

4.1. **The rigid geometry of modular curves.** Let $\mathcal{X}_1(Np^s)$ be the proper, flat, regular model of X_s over $\mathbb{Z}_p[\zeta_s]$ considered in [KM85, §12]. Its special fiber is the union of a finite number of reduced Igusa curves Ig_j over $\mathbb{F}_p$, meeting (not always transversally) at their supersingular points. Among these irreducible components there are two which are isomorphic to the Igusa curve $\mathrm{Ig}(Np^s)$ representing the moduli problem $(\Gamma_1(N),\mathrm{Ig}(p^s))$ over $\mathbb{F}_p$ specified in [KM85, §12]. These are called the *good components*, and are labelled Ig_∞ and Ig_0. More precisely, Ig_∞ classifies ordinary elliptic curves with $\Gamma_1(N)$-level structure and a chosen section of order p^s in its canonical subgroup of order p^s.

Let $F/\mathbb{Q}_p(\zeta_s)$ be a finite extension over which X_s acquires semistable reduction, with valuation ring $\mathcal{O}_F$ and residue field k_F. Let $\pi:\mathcal{X}_s\to\mathcal{X}_1(Np^s)\times\mathcal{O}_F$ be a semistable model of X_s over $\mathcal{O}_F$, obtained by applying sufficiently many blow-ups to the base change of $\mathcal{X}_1(Np^s)$ to $\mathcal{O}_F$. The map π is birational and induces an isomorphism between the generic fibers and also between two of the components of the closed fibre of $\mathcal{X}_s$ with $\mathrm{Ig}_\infty\times k_F$ and $\mathrm{Ig}_0\times k_F$, respectively. By an abuse of notation, continue to denote Ig_∞ and Ig_0 the two components of the closed fibre of $\mathcal{X}_s$ corresponding to the two latter. Let

$$\mathrm{red}:\mathcal{X}_s(F)\longrightarrow\tilde{\mathcal{X}}_s(k_F)$$

denote the "mod p" reduction map. The inverse images of the irreducible components of the special fiber of $\mathcal{X}_s$ give a finite collection of wide open subsets of $\mathcal{X}_s(F)$ which form an admissible cover of this rigid analytic space, as in the discussion in §3.1. Let $WO(\mathcal{X}_s)$ denote the collection of wide open spaces in this admissible cover.

Denote by $\mathcal{W}_\infty(p^s)$ and $\mathcal{W}_0(p^s)$ the wide open spaces in $WO(\mathcal{X}_s)$ arising as the inverse image of Ig_∞ and Ig_0, respectively, under the reduction map. The inverse image of the smooth locus of Ig_∞, denoted $\mathcal{A}_\infty(p^s)$, is the underlying affinoid of $\mathcal{W}_\infty(p^s)$ and an analogous definition is made for $\mathcal{A}_0(p^s)$.

The F-valued points of the rigid analytic space $\mathcal{A}_\infty(p^s)$ are in bijection with triples (A,i_N,i_p) where A is an ordinary (generalised) elliptic curve over $\mathcal{O}_F$, $i_N:\mu_N\longrightarrow A[N]$ is an injective map of group schemes over $\mathcal{O}_F$, and $i_p : \mu_{p^s} \longrightarrow A\{p^s\}$ is an isomorphism between μ_{p^s} and the canonical subgroup $A\{p^s\}$ of A of order p^s. As explained in [Co97] and [BrEm10, Lemme 4.4.1], the F-vector spaces

$$H^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^s)) = \frac{\Omega^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^s))}{d\mathcal{O}_{\mathcal{W}_\infty(p^s)}}, \qquad H^1_{\mathrm{rig}}(\mathcal{W}_0(p^s)) = \frac{\Omega^1_{\mathrm{rig}}(\mathcal{W}_0(p^s))}{d\mathcal{O}_{\mathcal{W}_0(p^s)}}$$

are equipped with a natural action of Hecke operators T_ℓ for $\ell \nmid Np$. As already explained in (105), they are also equipped with natural F-linear Frobenius endomorphisms, which are defined by choosing characteristic zero lifts Φ_∞ and Φ_0 of the Frobenius endomorphism in characteristic p to a system of wide open neighbourhoods of the affinoids $\mathcal{A}_\infty(p^s)$ and $\mathcal{A}_0(p^s)$ in $\mathcal{W}_\infty(p^s)$ and $\mathcal{W}_0(p^s)$, respectively. In the setting at hand, a natural choice of Φ_∞ is given by the Deligne-Tate mapping defined on triples (A, i_N, i_p) as above by the rule

$$\Phi_\infty(A, i_N, i_p) = (A/A\{p\}, \pi_{\{p\}} \circ i_N, \tilde{i}_p),$$

where

1) $A\{p\}$ is the canonical subgroup of A of order p;
2) $\pi_{\{p\}}$ is the natural isogeny $A \longrightarrow A/A\{p\}$;
3) $\tilde{i}_p : \mu_{p^s} \longrightarrow A$ sends ζ_s to $\pi_{\{p\}}(P_{s+1})$ where $P_{s+1} \in A\{p^{s+1}\}$ satisfies $pP_{s+1} = i_p(\zeta_s)$.

Let $(\mathbb{G}_m/q^{\mathbb{Z}}, i_N, i_p)$ denote the Tate curve, where i_N and i_p are induced by the natural embeddings $\mu_N, \mu_{p^s} \hookrightarrow \mathbb{G}_m$. Then Φ_∞ acts on it by the rule $\Phi_\infty(\mathbb{G}_m/q^{\mathbb{Z}}, i_N, i_p) = (\mathbb{G}_m/q^{p\mathbb{Z}}, i_N^p, i_p)$, and therefore

$$\Phi_\infty = p\langle p; 1\rangle V \tag{113}$$

where V is the operator on p-adic modular forms whose action is given by $V\left(\sum_{n=1}^\infty a_n q^n\right) = \sum_n a_n q^{np}$ on q-expansions. To define Φ_0, recall the Atkin-Lehner automorphism w_s attached to the choice of primitive p^s-th root of unity ζ_s. It interchanges the wide opens $\mathcal{W}_\infty(p^s)$ and $\mathcal{W}_0(p^s)$, and one defines

$$\Phi_0 := w_s^{-1}\Phi_\infty w_s = \langle p^{-1}; 1\rangle w_s \Phi_\infty w_s. \tag{114}$$

For each $\mathcal{W} \in WO(\mathcal{X}_s)$, recall that

$$\mathrm{res}_{\mathcal{W}} : H^1_{\mathrm{dR}}(X_1(Np^s)) \longrightarrow H^1_{\mathrm{rig}}(\mathcal{W})$$

is the the map induced by restriction to $\mathcal{W}$, and write res_0 and res_∞ for $\mathrm{res}_{\mathcal{W}_0(p^s)}$ and $\mathrm{res}_{\mathcal{W}_\infty(p^s)}$.

Let $H^1_{\mathrm{dR}}(X_s/F)^{\mathrm{prim}}$ denote the subspace of the de Rham cohomology of X_s associated to the *primitive subspace* $S_2(Np^s)_F^{\mathrm{prim}}$ in the sense of [MW84, §3].

Theorem 4.1. [Co97], [BrEm10] *The restriction maps* res_∞ *and* res_0 *induce an isomorphism*

$$H^1_{\mathrm{dR}}(X_{s/F})^{\mathrm{prim}} \simeq H^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^s))^{\mathrm{pure}} \oplus H^1_{\mathrm{rig}}(\mathcal{W}_0(p^s))^{\mathrm{pure}},$$

where $H^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^s))^{\mathrm{pure}}$ *denotes the subspace of the de Rham cohomology which is pure of weight one, i.e., the space generated by classes of rigid differentials with vanishing annular residues. The restriction morphism is equivariant with respect to*

- *the Hecke operators* T_ℓ *for* $\ell \nmid Np$ *acting on both sides;*
- *the Frobenius endomorphism* Φ *acting on* $H^1_{\mathrm{dR}}(X_{s/F})$*;*
- *the Frobenius endomorphism* (Φ_∞, Φ_0) *acting on* $H^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^s)) \oplus H^1_{\mathrm{rig}}(\mathcal{W}_0(p^s))$*.*

A similar but simpler analysis also holds for the modular curve $X_0(Np)$, since it admits a proper regular model over $\mathbb{Z}_p$, whose special fiber is the union of two irreducible components, each isomorphic to the special fiber of the smooth integral model of $X_0(N)$. Write $\{\mathcal{W}_\infty, \mathcal{W}_0\}$ for the standard admissible covering of $X_0(Np)$ by wide open neighbourhoods of the Hasse domain of $X_0(N)$. As above, the group $H^1_{\mathrm{dR}}(X_0(Np))$ is again endowed with the action of a Frobenius map Φ and the Hecke operator U_p, which mutually commute (by functoriality of log-crystalline cohomology with respect to correspondences).

The *ordinary unit root subspace*

$$H^1_{\mathrm{dR}}(X_0(Np)_{/\mathbb{Q}_p})^{\mathrm{ord,ur}} \subset e_{\mathrm{ord}} H^1_{\mathrm{dR}}(X_0(Np)_{/\mathbb{Q}_p})$$

is defined to be the subspace spanned by the eigenvectors of Φ whose eigenvalue is a p-adic unit.

Lemma 4.2. *The map* $H^1_{\mathrm{dR}}(X_0(Np)_{/F})^{\mathrm{ord,ur}} \longrightarrow H^1_{\mathrm{rig}}(\mathcal{W}_\infty)$ *induced by restriction is the zero map, and for any eigenform* ϕ *of weight* 2 *and level dividing* N *the map*

$$H^1_{\mathrm{dR}}(X_0(Np)_{/F})^{\mathrm{ord,ur}}[\phi] \longrightarrow H^1_{\mathrm{rig}}(\mathcal{W}_0)^{\mathrm{ur}}[f] \simeq H^1_{\mathrm{dR}}(X_0(N)_{/F})^{\mathrm{ur}}[\phi] \tag{115}$$

is an isomorphism.

Proof. This follows from noting that the action of $\Phi = \Phi_\infty$ and U on $H^1_{\mathrm{rig}}(\mathcal{W}_\infty)$ are related by the rule $\Phi_\infty = \langle p\rangle pV = \langle p\rangle pU^{-1}$. □

4.2. **Test vectors and their associated de Rham cohomology classes.** As was explained in the introduction and recalled above, the Λ-adic cohomology classes and the Garrett-Hida p-adic L-functions considered in this work depend on a choice of test vectors $\breve{f}$, $\underline{\breve{g}}$, $\underline{\breve{h}}$ and dual test vectors $\breve{f}^*$, $\underline{\breve{g}}^*$, $\underline{\breve{h}}^*$ that were fixed in (92) and (93). We focus first on the former, and analyze the regular differential forms associated to $\breve{f}$ and the weight two specialisations of $\underline{\breve{g}}$ and $\underline{\breve{h}}$.

Assume that $\breve{f} \in S_2(Np)[f]$ arises as the ordinary p-stabilisation of some test vector in $S_2(N)[f]$, and recall that $\omega_{\breve{f}} \in \Omega^1(X_0(Np))$ denotes the regular differential form asociated to $\breve{f}$. Recall also the maps $X_s \xrightarrow{\varpi_1^{s-1}} X_1 \xrightarrow{j_1} X_0(Np)$ that arise in the equation (100) describing the cycle $\Delta^\circ_{s,s}$, and define

$$\omega_{\breve{f},s} := (\varpi_1^{s-1})^* \circ j_1^*(\omega_{\breve{f}}), \tag{116}$$

to be the pull-back of $\omega_{\breve{f}}$ to X_s via these projections.

Let $(y,z) \in \Omega_g \times_\Omega \Omega_h$ be a classical point of weight 2 and character $\epsilon \neq \omega$ of primitive conductor p^s. In order to lighten notations, set $g = g_y$, $h = h_z$ and

$$\breve{g} = \breve{g}_y \in S_2(Np^s, \chi\epsilon\omega^{-1}), \qquad \breve{h} = \breve{h}_z \in S_2(Np^s, \chi^{-1}\epsilon\omega^{-1}).$$

These notations are in force only in Chapter 4, and g and h will revert to denoting modular forms of weight 1 in subsequent chapters, as was done in the introduction. Let $h^\iota \in S_2(N_h p^s, \chi^{-1}\omega\epsilon^{-1})$ be the newform whose q-expansion coefficients are determined by

$$a_m(h^\iota) = \begin{cases} \omega\epsilon^{-1}(m)\cdot a_m(h) & \text{if } p \nmid m; \\ \chi^{-1}(p)\overline{a}_p(h)^r & \text{if } m = p^r. \end{cases}$$

If $\breve{h} = \sum_{d|N/N_h} \lambda_d \cdot h(q^d)$, set also

$$\breve{h}^\iota := \sum_{d|N/N_h} \lambda_d h^\iota(q^d) \in S_2(Np^s, \chi^{-1}\omega\epsilon^{-1}),$$

so that for all n with $p \nmid n$, the fourier coefficient $a_n(\breve{h}^\iota)$ is given by $a_n(\breve{h}^\iota) := (\omega\epsilon^{-1})(n)\cdot a_n(\underline{\breve{h}})$.

For any Dirichlet character α, let $\mathfrak{g}(\alpha) \in \bar{\mathbb{Q}}_p$ denote the Gauss sum attached to α, on which $G_{\mathbb{Q}_p}$ acts through this character.

Proposition 4.3. *The identity* $w_s(\breve{h}) = \mathfrak{g}(\omega^{-1}\epsilon)\alpha_h^{-s}\breve{h}^\iota$ *holds in* $S_2(N_h p^s, \chi^{-1}\omega\epsilon^{-1})$.

Proof. This follows from [AL78, Theorem 2.1] (with $k = 2$ and $Q = p^s$), taking into account that w_s commutes with the level-raising operators at the divisors of N/N_h, which are prime to p. □

For any regular differential form $\omega \in \Omega^1(X_{s/F})$, let ω_∞ and ω_0 denote its restrictions to $\mathcal{W}_\infty(p^s)$ and $\mathcal{W}_0(p^s)$ respectively, and write $[\omega_\infty]$, $[\omega_0]$ for their classes in cohomology.

Proposition 4.4. *For* $\omega = \omega_{\breve{f},s}$, $\omega_{\breve{g}}$ *and* $\omega_{\breve{h}}$,

$$\Phi_\infty[\omega_\infty] = \beta\cdot[\omega_\infty], \qquad \Phi_0[\omega_0] = \alpha\cdot[\omega_0], \tag{117}$$

where $(\alpha,\beta) = (\alpha_f,\beta_f)$, $(a_p(g), \chi(p)\overline{a_p(g)})$ *and* $(a_p(h), \chi^{-1}(p)\overline{a_p(h)})$ *respectively.*

Proof. Let us prove the claim for $\omega = \omega_{\breve{g}}$, as the other cases follow similarly. Together with V, let $U = U_p$ be the usual operator on p-adic modular forms defined on q-expansions as in (29). A direct calculation shows that

$$U(\breve{g}) = a_p(g)\breve{g}, \qquad UV = VU = 1 \text{ on } H^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^s)),$$

hence $V([\omega_{\breve{g},\infty}]) = a_p(g)^{-1}\cdot[\omega_{\breve{g},\infty}]$. The first equality in (117) now follows from the fact that the canonical lift Φ_∞ of Frobenius agrees with $p\langle p;1\rangle V$, as stated in (113), and that $a_p(g)$ is a Weil number of weight one, i.e., a complex number of absolute value $\sqrt{p}$. To study the action of Φ_0 on $\omega_{\breve{g},0}$, note that

$$\Phi_0(\omega_{\breve{g},0}) = w_s^{-1}\Phi_\infty w_s(\omega_{\breve{g},0}) = w_s^{-1}\beta_{g^\iota} w_s\omega_{\breve{g},0},$$

where the first equality follows from (114) and the second from Proposition 4.3. The second equality in (117) now follows from the fact that $\beta_{g^\iota} = a_p(g) = \alpha_g$. $\square$

If $\phi(q)$ is any overconvergent p-adic modular form, its *p-depletion* is defined to be the overconvergent modular form whose q-expansion is given by

$$\phi^{[p]}(q) := (1-VU)\phi(q) = \sum_{p\nmid n} a_n(\phi)q^n,$$

and $\omega_\phi^{[p]}$ denotes the associated rigid differential on a wide open neighbourhood of $\mathcal{A}_\infty(p^s)$ in $\mathcal{W}_\infty(p^s)$.

Corollary 4.5. (1) *The class of the rigid differential* $\omega^{[p]}_{\breve{f},\infty} = (1-\beta_f^{-1}\Phi_\infty)(\omega_{\breve{f},\infty})$ *is trivial in* $H^1_{\mathrm{rig}}(\mathcal{W}_\infty)$, *i.e.,* $\omega^{[p]}_{\breve{f},\infty} = d\breve{F}$ *for some* $\breve{F}\in\mathcal{O}_{\mathcal{W}_\infty}$. *The q-expansion of* $\breve{F}$ *is*

$$\breve{F}(q) = \sum_{p\nmid n}\frac{a_n(\breve{f})}{n}q^n.$$

(2) *The class of the rigid differential* $\omega^{[p]}_{\breve{g},\infty} = (1-\beta_g^{-1}\Phi_\infty)(\omega_{\breve{g},\infty})$ *is trivial in* $H^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^s))$, *i.e.,* $\omega^{[p]}_{\breve{g},\infty} = d\breve{G}$ *for some* $\breve{G}_\infty\in\mathcal{O}_{\mathcal{W}_\infty(p^s)}$. *The q-expansion of G is given by*

$$\breve{G}(q) = \sum_{p\nmid n}\frac{a_n(\breve{g})}{n}q^n.$$

(3) *The class of the rigid differential* $\omega^{[p]}_{\breve{h},0} = (1-\alpha_h^{-1}\Phi_0)(\omega_{\breve{h},0})$ *is trivial in* $H^1_{\mathrm{rig}}(\mathcal{W}_0(p^s))$, *i.e.,* $\omega^{[p]}_{\breve{h},0} = d\breve{H}^\circ$ *for some* $\breve{H}\circ\in\mathcal{O}_{\mathcal{W}_0(p^s)}$. *The q-expansion of* $w_s\breve{H}^\circ$ *is given by*

$$(w_s\breve{H}^\circ)(q) = \mathfrak{g}(\omega^{-1}\epsilon)\alpha_h^{-s}H^\iota(q), \qquad H^\iota(q) = \sum_{p\nmid n}\frac{a_n(\breve{h}^\iota)}{n}q^n, \qquad a_n(\breve{h}^\iota) = (\omega\epsilon^{-1})(n)a_n(\breve{h}).$$

Proof. The triviality statements follow from Prop. 4.4 and the q-expansion of the primitives in (1) and (2) are obtained by a direct calculation. A similar reasoning, using Prop. 4.3, implies (3). $\square$

We turn now to analyze the dual test vectors and their associated de Rham cohomology classes. Given a dual test vector $\breve{f}^*\in S_2^{\mathrm{ord}}(\Gamma_0(Np))^\vee[f]$, recall that $\eta_{\breve{f}^*}\in H^1_{\mathrm{dR}}(X_0(Np))^{\mathrm{ord,ur}}[f]$ denotes the unique class satisfying

$$\Phi(\eta_{\breve{f}^*}) = \alpha_f\cdot\eta_{\breve{f}^*}, \quad \text{and} \quad \langle\eta_{\breve{f}^*}, w_1\omega\rangle_{X_0(Np)} = \breve{f}^*(\omega) \quad \forall\omega\in S_2^{\mathrm{ord}}(\Gamma_0(Np)). \tag{118}$$

As in equation (116), define

$$\eta_{\breve{f}^*,s} := (\varpi_1^{s-1})^*\circ j_1^*(\eta_{\breve{f}^*}). \tag{119}$$

Likewise, set

$$\breve{g}^* = \breve{g}_y^*\in S_2(Np^s,\chi^{-1}\epsilon\omega^{-1})^\vee, \qquad \breve{h}^* = \breve{h}_z^*\in S_2(Np^s,\chi\epsilon\omega^{-1})^\vee,$$

and let $\eta_{\breve{g}^*}$ and $\eta_{\breve{h}^*}\in H^1_{\mathrm{dR}}(X_{s/F})$ be the de Rham cohomology classes associated to them as in (80). Note that

$$\Phi(\eta) = \alpha\cdot\eta \text{ and } U_p(\eta) = \alpha\cdot\eta \tag{120}$$

for $\eta = \eta_{\breve{f}^*,s}$, $\eta_{\breve{g}^*}$ and $\eta_{\breve{h}^*}$, where $\alpha = \alpha_f$, α_g and α_h, respectively.

Lemma 4.6. *The classes* $\eta_{\breve{g}^*}$ *and* $\eta_{\breve{h}^*}$ *are supported on* $\mathcal{W}_0(p^s)$.

Proof. Arguing as in Lemma 4.2 and taking (120) into account, it follows that the restriction of $\eta_{\breve{g}^*}$ and $\eta_{\breve{h}^*}$ to $\mathcal{W}_\infty(p^s)$ vanish, and hence they are supported on $\mathcal{W}_0(p^s)$ by Theorem 4.1. $\square$

Remark 4.7. Lemma 4.6 is consistent with the fact that the value of $\langle \eta_{\breve{g}^*}, w_s\omega\rangle_s$ depends only on the restriction of the ordinary class ω to $\mathcal{W}_\infty(p^s)$, i.e., on the overconvergent modular form attached to ω.

4.3. **Twisted diagonal cycles under the syntomic Abel-Jacobi map.** We are now in a position to evaluate the image of the twisted diagonal cycles under the syntomic Abel-Jacobi map.

4.3.1. *Evaluation of* $\mathrm{AJ}_p(\Delta^\circ_{s,s})$ *at* $\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}$.

Lemma 4.8. *The data* $(\Delta,\eta_1,\omega_2,\omega_3) = (\Delta^\circ_{s,s},\eta_{\breve{f}^*},\omega_{\breve{g}},\omega_{\breve{h}})$ *satisfies conditions (a)-(d) of §3.2.*

Proof. Condition (a) is clear, since $W_{s,s}$ is a triple product of curves with semistable reduction over F. The cycle $\Delta^\circ_{s,s}$ is described as in condition (b) of Section 3.2 by setting

$$X = X_s, \qquad (\pi_1,\pi_2,\pi_3) = (j_1\circ\varpi_1^{s-1}, w_s, \mathrm{Id}), \qquad (\varepsilon_1,\varepsilon_2,\varepsilon_3) = (\varepsilon,\varepsilon_s,\varepsilon_s).$$

Condition (c) is built into the assumptions, while Condition (d) follows from the fact that the form $\eta_{\breve{f}^*}$ belongs to the f-isotypic subspace attached to an eigenform of prime-to-p level, and $\breve{g}$ and $\breve{h}$ are primitive at p. ☐

Let $\xi(\omega_{\breve{g}},\omega_{\breve{h}})\in H^1_{\mathrm{dR}}(X_{s/F})$ be the cohomology class attached to the data $(\Delta^\circ_{s,s},\omega_{\breve{g}},\omega_{\breve{h}})$ following the recipe of §3.2. Then Theorem 3.4 applies and asserts that

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) = \langle \eta_{\breve{f}^*,s}, \xi(\omega_{\breve{g}},\omega_{\breve{h}})\rangle_s. \tag{121}$$

Lemma 4.9. *The class* $\xi(\omega_{\breve{g}},\omega_{\breve{h}})$ *arises as the pull-back of a class in* $H^1_{\mathrm{dR}}(X_0(Np^s)_{/F})$ *via* j_s.

Proof. Let

$$\rho_P^{v,w}(\omega_{\breve{g}},\omega_{\breve{h}})\in\Omega^1(\mathcal{W}_v\times\mathcal{W}_w), \qquad v,w\in\mathcal{V}(\mathcal{G}(\tilde{\mathcal{X}}_s)),$$

be a system of rigid 1-forms satisfying (111). The operators $(\langle a;b\rangle,\langle a;b^{-1}\rangle)$ in D_s leave the system $\{\rho_P^{v,w}\}$ invariant *up to systems of closed* 1*-forms in* $\Omega^1(\mathcal{W}_v\times\mathcal{W}_w)$; this follows because D_s fixes the class $\omega_{\breve{g}}\otimes\omega_{\breve{h}}\in H^2_{\mathrm{dR}}(X_s\times X_s)$. It follows that the class

$$\xi_P(\omega_{\breve{g}},\omega_{\breve{h}})\in H^1_{\mathrm{dR}}(X_{s/F}) \tag{122}$$

introduced in §3.2 is well-defined, and fixed by the diamond operators. Since these operators commute with Frobenius, the same holds for $\xi(\omega_{\breve{g}},\omega_{\breve{h}})$, and the lemma follows. ☐

The degeneracy maps $\pi_1,\pi_2: X_s^\flat\longrightarrow X_s$ that were introduced in (32) induce similar maps from $X_0(Np^{s+1})\longrightarrow X_0(Np^s)$ by passing to the quotient by the action of the group $(\mathbb{Z}/Np^s)^\times$ of diamond operators of level Np^s on both sides. Let us continue to denote these maps as π_1 and π_2, by a slight abuse of notation. The class $\eta_{\breve{f}^*,s}$ likewise arises as the pull-back via j_s of a class on $H^1_{\mathrm{dR}}(X_0(Np^s)_{/F})$, which shall be denoted by the same symbol. (Any ambiguity arising from this double use of notation shall be avoided by consistently specifying the modular curve on which the Poincaré pairing being computed. Furthermore, the object $\eta_{\breve{f}^*,s}$ will play only a provisional role, disappearing after the proof of Lemma 4.12.) With these new notations,

$$\eta_{\breve{f}^*,s} = (\pi_1^{s-1})^*(\eta_{\breve{f}^*}), \tag{123}$$

as follows directly from (119). Since $d_s=\deg(j_s)$, the functoriality of Poincaré duality relative to pull-backs now allows us to rewrite (121) in terms of the Poincaré pairing on $X_0(Np^s)$, namely as

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) = d_s\cdot\langle \eta_{\breve{f}^*,s}, \xi(\omega_{\breve{g}},\omega_{\breve{h}})\rangle_{X_0(Np^s)}. \tag{124}$$

Recall the ordinary projector e_{ord} of (30) that was already exploited in the discussion of Hida theory in §1.5 as well as its anti-ordinary counterpart e^*_{ord}. Both will now be considered as endomorphisms of $H^1_{\mathrm{dR}}(X_0(Np^s)_{/F})$. Just like U_p and U_p^*, these endomorphisms are adjoint to each other relative to Poincaré duality.

Lemma 4.10. *For all* $s\geq 1$,

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) = d_s\langle \eta_{\breve{f}^*,s}, e^*_{\mathrm{ord}}\xi(\omega_{\breve{g}},\omega_{\breve{h}})\rangle_{X_0(Np^s)}.$$

Proof. This follows from (124) in light of the fact that the class $\eta_{\breve{f}^*,s}$ is ordinary and hence equal to its image under the ordinary projector, so that

$$\langle \eta_{\breve{f}^*,s}, \xi(\omega_{\breve{g}}, \omega_{\breve{h}})\rangle_{X_0(Np^s)} = \langle e_{\mathrm{ord}}\eta_{\breve{f}^*,s}, \xi(\omega_{\breve{g}}, \omega_{\breve{h}})\rangle_{X_0(Np^s)} = \langle \eta_{\breve{f}^*,s}, e^*_{\mathrm{ord}}\xi(\omega_{\breve{g}}, \omega_{\breve{h}})\rangle_{X_0(Np^s)}.$$

□

Lemma 4.11. *The pulback $(\pi_1^{s-1})^*$ induces an isomorphism*

$$(\pi_1^{s-1})^* : e_{\mathrm{ord}} H^1_{\mathrm{dR}}(X_0(Np)) \longrightarrow e_{\mathrm{ord}} H^1_{\mathrm{dR}}(X_0(Np^s)).$$

Proof. The second displayed equation in the proof of Thm. 4.1. of [Em99] (after replacing s by 1 and r by s) asserts that the homomorphism

$$e_{\mathrm{ord}}((\Gamma_1(Np)\cap\Gamma_0(p^s))^{\mathrm{ab}}\otimes\mathbb{Z}_p) \longrightarrow e_{\mathrm{ord}}(\Gamma_1(Np)^{\mathrm{ab}}\otimes\mathbb{Z}_p)$$

induced from the natural inclusion $\Gamma_1(Np)\cap\Gamma_0(p^s)\subset\Gamma_1(Np)$ is an isomorphism. The abelianisation of a congruence subgroup (tensored with $\mathbb{Z}_p$) is naturally identified with the homology with $\mathbb{Z}_p$-coefficients of the associated open modular curve. Taking a further quotient by the subgroup generated by the parabolic elements and passing to $\mathbb{Z}_p$-duals, one concludes that the natural map

$$e_{\mathrm{ord}} H^1(X_1(Np), \mathbb{Z}_p) \longrightarrow e_{\mathrm{ord}} H^1(X_1(Np)\times_{X_0(Np)} X_0(Np^s), \mathbb{Z}_p)$$

is an isomorphism. This map agrees with $(\pi_1^{s-1})^*$ (viewed as a map on singular cohomology). Hence Lemma 4.11 follows after taking invariants under the action of the group $(\mathbb{Z}/Np\mathbb{Z})^\times$ of diamond operators acting on both sides, and invoking the comparison isomorphism between singular and de Rham cohomology. □

Applying the ordinary projector to $\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}}) := w_s\xi(\omega_{\breve{g}}, \omega_{\breve{h}})$, it follows from Lemma 4.11 that

$$e_{\mathrm{ord}}\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}}) = (\pi_1^{s-1})^*(\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_1), \tag{125}$$

for a suitable class $\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_1 \in e_{\mathrm{ord}} H^1_{\mathrm{dR}}(X_0(Np)_{/F})$. Let $\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_{\mathcal{W}_\infty}$ denote the restriction of the class $\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_1$ to the Hasse domain $\mathcal{W}_\infty$ viewed as a p-adic modular form of weight two.

Lemma 4.12. *For all $s\geq 1$,*

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) = d_s\alpha_f^{s-1}\breve{f}^*(\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_{\mathcal{W}_\infty}).$$

Proof. Since $w_s = w_s^{-1}$ on $X_0(Np^s)$, one has $e^*_{\mathrm{ord}} = w_s e_{\mathrm{ord}} w_s$. Lemma 4.10 implies that

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) = d_s\langle \eta_{\breve{f}^*,s}, w_s e_{\mathrm{ord}}\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})\rangle_{X_0(Np^s)}.$$

It folllows from (125) that

$$\begin{aligned}\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) &= d_s\langle \eta_{\breve{f}^*,s}, w_s(\pi_1^{s-1})^*\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_1\rangle_{X_0(Np^s)}\\ &= d_s\langle \pi_{2*}^{s-1}\eta_{\breve{f}^*,s}, w_1^*\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_1\rangle_{X_0(Np)},\end{aligned}$$

where the facts that $w_s^*(\pi_1^{s-1})^* = (\pi_2^{s-1})^* w_1^*$ and that the adjoint of π_2^* relative to Poincaré duality is π_{2*} have been used to derive the last equation. By (123) and (33),

$$\pi_{2*}^{s-1}\eta_{\breve{f}^*,s} = \pi_{2*}^{s-1}(\pi_1^{s-1})^*\eta_{\breve{f}^*} = U_p^{s-1}\eta_{\breve{f}^*} = \alpha_f^{s-1}\eta_{\breve{f}^*},$$

and therefore

$$\begin{aligned}\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) &= d_s\alpha_f^{s-1}\langle \eta_{\breve{f}^*}, w_1^*\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_1\rangle_{X_0(Np)}\\ &= d_s\alpha_f^{s-1}\breve{f}^*(\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})_{\mathcal{W}_\infty}),\end{aligned}$$

where the last equality follows from equation (118) defining $\eta_{\breve{f}^*}$. □

Since the pull-back π_1^* induces the identity on q-expansions, Lemma 4.12 implies that

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) = d_s\alpha_f^{s-1}\breve{f}^*(e_{\mathrm{ord}}\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})), \tag{126}$$

in light of (125). We now turn to the calculation of the ordinary p-adic modular form $e_{\mathrm{ord}}\xi^\iota(\omega_{\breve{g}}, \omega_{\breve{h}})$. Let $P\in\mathbb{C}_p[t]$ be a polynomial satisfying the conditions of Lemma 3.3, i.e. such that

(i) $P(\Phi\times\Phi)$ annihilates the class of $\omega_{\breve{g},\infty}\otimes\omega_{\breve{h},0}$ in $H^2_{\mathrm{rig}}(\mathcal{W}_\infty(p^s)\times\mathcal{W}_0(p^s))$.
(ii) None of the roots of $P(t)$ are of complex absolute value $\sqrt{p}$.

In light of Proposition 4.4, one may take $P(x) := 2(1-\beta_g^{-1}\alpha_h^{-1}x) = 2(1-p^{-1}\chi^{-1}(p)\alpha_g\alpha_h^{-1}x)$. The polynomial identity

$$P(xy) = (1-\beta_g^{-1}x)(1+\alpha_h^{-1}y) + (1+\beta_g^{-1}x)(1-\alpha_h^{-1}y)$$

combined with Corollary 4.5 implies that

$$P(\Phi_2\times\Phi_3)(\omega_{\breve{g},\infty}\otimes\omega_{\breve{h},0}) = (d\breve{G})\otimes(1+\alpha_h^{-1}\Phi_0)\omega_{\breve{h},0} + (1+\beta_g^{-1}\Phi_\infty)\omega_{\breve{g},\infty}\otimes(d\breve{H}^\circ).$$

It follows that the rigid analytic differential 1-form $\varrho_P(\omega_{\breve{g}},\omega_{\breve{h}})\in\Omega^1(\mathcal{W}_\infty(p^s)\times\mathcal{W}_0(p^s))$ satisfying

$$d\varrho_P(\omega_{\breve{g}},\omega_{\breve{h}}) = P(\Phi_2\times\Phi_3)(\omega_{\breve{g},\infty}\otimes\omega_{\breve{h},0})$$

can be chosen to be

$$\varrho_P(\omega_{\breve{g}},\omega_{\breve{h}}) = \breve{G}\otimes(1+\alpha_h^{-1}\Phi_0)\omega_{\breve{h},0} - (1+\beta_g^{-1}\Phi_\infty)\omega_{\breve{g},\infty}\otimes\breve{H}^\circ, \tag{127}$$

where $\breve{G}\in\mathcal{O}_{\mathcal{W}_\infty(p^s)}$ and $\breve{H}^\circ\in\mathcal{O}_{\mathcal{W}_0(p^s)}$ are the primitives described in Corollary 4.5.

After setting $\xi_P^\iota(\omega_{\breve{g}},\omega_{\breve{h}}) := w_s\xi_P(\omega_{\breve{g}},\omega_{\breve{h}})$ and observing that the correspondences ε_s fix the differentials $\omega_{\breve{g}}$, $\omega_{\breve{h}}$ as well as their primitives $\breve{G}$ and $\breve{H}^\circ$, it follows that

$$\xi_P(\omega_{\breve{g}},\omega_{\breve{h}}) = \left((w_s^{-1}\breve{G})\times(1+\chi^{-1}(p)\alpha_h^{-1}\Phi_\infty)\omega_{\breve{h},\infty} - (1+\beta_g^{-1}\Phi_\infty)(w_s^{-1}\omega_{\breve{g},\infty})\times\breve{H}^\circ\right), \tag{128}$$

$$\xi_P^\iota(\omega_{\breve{g}},\omega_{\breve{h}}) = \left(\breve{G}\times(1+\chi^{-1}(p)\alpha_h^{-1}\Phi_\infty)(w_s\omega_{\breve{h},\infty}) - (1+\beta_g^{-1}\Phi_\infty)\omega_{\breve{g},\infty}\times(w_s\breve{H}^\circ)\right). \tag{129}$$

By Corollary 4.5, the q-expansion of this p-adic modular form is given by

$$\xi_P^\iota(\omega_{\breve{g}},\omega_{\breve{h}})(q) = \mathfrak{g}(\omega^{-1}\epsilon)\alpha_h^{-s}\left(\breve{G}\times(1+\chi^{-1}(p)\alpha_h^{-1}\Phi_\infty)\breve{h}^\iota - (1+\beta_g^{-1}\Phi_\infty)\breve{g}_\infty\times\breve{H}^\iota\right)(q)\frac{dq}{q}. \tag{130}$$

This is a rigid differential on a wide open neighbourhood of $\mathcal{A}_\infty(p^s)$ in $\mathcal{W}_\infty(p^s)$, and hence corresponds to an overconvergent modular form of weight 2 on $\mathcal{W}_\infty(p^s)$. Since the nebentypus character of $\breve{g}$ and $\breve{G}$ is $\chi\omega^{-1}\epsilon$, and that of $\breve{h}^\iota$ and $\breve{H}^\iota$ is $\chi^{-1}\omega\epsilon^{-1}$, it follows that $\xi_P^\iota(\omega_{\breve{g}},\omega_{\breve{h}})$ arises as a pullback under the projection j_s of a rigid differential on $\mathcal{W}_\infty$.

Proposition 4.13. *The class of* $e_{\mathrm{ord}}(\xi_P^\iota(\omega_{\breve{g}},\omega_{\breve{h}}))$ *in* $H^1_{\mathrm{rig}}(\mathcal{W}_\infty)$ *is equal to the class of the weight two p-adic modular form*

$$-2\mathfrak{g}(\omega^{-1}\epsilon)\times\alpha_h^{-s}\times\breve{g}\breve{H}^\iota\in\Omega^1(\mathcal{W}_\infty).$$

Proof. If f_1 and f_2 are overconvergent modular forms, then $f_1^{[p]}\times(Vf_2)$ lies in the kernel of U and hence a fortiori in the kernel of e_{ord}. Since $\Phi_\infty = \langle p\rangle pV$, after applying e_{ord} to (130) one obtains

$$\begin{aligned} e_{\mathrm{ord}}(\xi_P^\iota(\omega_{\breve{g}},\omega_{\breve{h}})) &= \mathfrak{g}(\omega^{-1}\epsilon)\times\alpha_h^{-s}\times e_{\mathrm{ord}}(\breve{G}\times\breve{h}^\iota_\infty - \breve{g}_\infty\times\breve{H}^\iota) \\ &= \mathfrak{g}(\omega^{-1}\epsilon)\times\alpha_h^{-s}\times e_{\mathrm{ord}}(\breve{G}\times\breve{h}^{\iota[p]}_\infty - \breve{g}^{[p]}_\infty\times\breve{H}^\iota). \end{aligned} \tag{131}$$

Applying e_{ord} to the identity $d(\breve{G}\breve{H}^\iota) = \breve{G}\times\breve{h}^{\iota[p]}_\infty + \breve{g}^{[p]}_\infty\times\breve{H}^\iota$, and using the fact that the image of d on overconvergent forms is in the kernel of e_{ord}, it follows that

$$e_{\mathrm{ord}}(\breve{G}\times\breve{h}^{\iota[p]}_\infty) = -e_{\mathrm{ord}}(\breve{g}^{[p]}_\infty\times\breve{H}^\iota), \tag{132}$$

and the result follows from (132) combined with (131). $\square$

We can now state the main formula concerning the value of $\mathrm{AJ}_p(\Delta^\circ_{s,s})$ at the vector $\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}$:

Theorem 4.14. *Setting* $\mathcal{E}_f(f,g,h) := -2(1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})^{-1}$, *one has*

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) = \mathcal{E}_f(f,g,h)\times\mathfrak{g}(\omega^{-1}\epsilon)\times\alpha_f^{s-1}\alpha_h^{-s}\times d_s\times\breve{f}^*(\breve{g}\breve{H}^\iota).$$

Proof. By combining (126) with Proposition 4.13,

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\eta_{\breve{f}^*}\otimes\omega_{\breve{g}}\otimes\omega_{\breve{h}}) = -2\times\mathfrak{g}(\omega^{-1}\epsilon)\times\alpha_f^{s-1}\alpha_h^{-s}\times d_s\times\breve{f}^*(P(\Phi_\infty)^{-1}(\breve{g}\breve{H}^\iota)).$$

But for all classes $\xi\in H^1_{\mathrm{rig}}(\mathcal{W}_\infty)$, $\breve{f}^*(\Phi\xi) = p\alpha_f^{-1}\breve{f}^*(\xi) = \beta_f\breve{f}^*(\xi)$. More generally, $\breve{f}^*(Q(\Phi)\xi) = Q(\beta_f)\breve{f}^*(\xi)$, for all rational functions Q with no pole at β_f. The result follows. $\square$

4.3.2. *Evaluation of* $\mathrm{AJ}_p(\Delta^\circ_{s,s})$ *at* $\omega_{\breve f}\otimes\eta_{\breve g^*}\otimes\omega_{\breve h}$. Thanks to Lemma 4.8, the data $(\Delta_{s,s},\omega_{\breve f},\eta_{\breve g^*},\omega_{\breve h})$ satisfies conditions (a)-(d) of §3.2. The class $w_s^{-1}\eta_{\breve g^*}$ is anti-ordinary and supported on $\mathcal{W}_\infty(p^s)$ by Lemma 4.6. Theorem 3.4 therefore applies and asserts that

$$\mathrm{AJ}_p(\Delta_{s,s})(\omega_{\breve f}\otimes\eta_{\breve g^*}\otimes\omega_{\breve h})=\langle w_s^{-1}\eta_{\breve g^*},\xi(\omega_{\breve f},\omega_{\breve h})\rangle_s=\langle\eta_{\breve g^*},w_s e_{\mathrm{ord}}(\xi(\omega_{\breve f},\omega_{\breve h}))\rangle_s \tag{133}$$

where $\xi(\omega_{\breve f},\omega_{\breve h})\in H^1_{\mathrm{dR}}(X_{s/F})$ is the class associated to the triple $(\Delta_{s,s},\omega_{\breve f},\omega_{\breve h})$ as in §3.2 and the second equality follows as in Lemma 4.10.

In order to compute the class $\xi(\omega_{\breve f},\omega_{\breve h})$ explicitly, consider the polynomial $P(x):=2(1-\beta_f^{-1}\beta_h^{-1}x)$. It satisfies the conditions of Lemma 3.3, since the endomorphism $P(\Phi_1\times\Phi_3)$ annihilates the class of $\omega_{\breve f,\infty}\otimes\omega_{\breve h,\infty}$ in $H^2_{\mathrm{rig}}(\mathcal{W}_\infty\times\mathcal{W}_\infty(p^s))$ by Proposition 4.4 and (120). The polynomial identity

$$P(xy)=(1-\beta_f^{-1}x)(1+\beta_h^{-1}y)+(1+\beta_f^{-1}x)(1-\beta_h^{-1}y)$$

combined with Corollary 4.5 implies that

$$P(\Phi_1\times\Phi_3)(\omega_{\breve f,\infty}\otimes\omega_{\breve h,\infty})=\omega^{[p]}_{\breve f,\infty}\otimes(1+\beta_h^{-1}\Phi_\infty)\omega_{\breve h,\infty}+(1+\beta_f^{-1}\Phi_\infty)\omega_{\breve f,\infty}\otimes\omega^{[p]}_{\breve h,\infty}.$$

It follows that the form $\varrho_P(\omega_{\breve f},\omega_{\breve h})\in\Omega^1(\mathcal{W}_\infty\times\mathcal{W}_\infty(p^s))$ satisfying (111) can be chosen to be

$$\varrho_P(\omega_{\breve f},\omega_{\breve h})=\breve F\otimes(1+\beta_h^{-1}\Phi_\infty)\omega_{\breve h,\infty}-(1+\beta_f^{-1}\Phi_\infty)\omega_{\breve f,\infty}\otimes\breve H, \tag{134}$$

where $\breve H\in\mathcal{O}_{\mathcal{W}_\infty(p^s)}$ is the rigid analytic primitive of $\omega^{[p]}_{\breve h,\infty}$ following the definitions of Corollary 4.5 (with g and G replaced by h and H). By the analogous calculation that led to (128), the pull-back $\xi_P(\omega_{\breve f},\omega_{\breve h})$ of $\varrho_P(\omega_{\breve f},\omega_{\breve h})$ under the correspondence

$$(\varepsilon,\varepsilon_s)(j_1\circ\varpi_1^{s-1},\mathrm{Id}):X_0(Np)\times X_s\longrightarrow X_0(Np)\times X_s$$

arising in the definition of the cycle $\Delta^\circ_{s,s}$ is given by

$$\xi_P(\omega_{\breve f},\omega_{\breve h})=\breve F\times(1+\chi(p)\alpha_h\Phi_\infty)\omega_{\breve h,\infty}-(1+\beta_f^{-1}\Phi_\infty)\omega_{\breve f,s,\infty}\times\breve H, \tag{135}$$

where the fact that ϖ_1^* and j_1^* induce the identity on q-expansions has been used to show that $\breve F$ pulls back to the same overconvergent modular form (viewed this time as a rigid differential on $\mathcal{O}_{\mathcal{W}_\infty(p^s)}$). In this calculation, copious use has been made of the fact that the correspondence ε was chosen so as to leave $\omega_{\breve f}$ and $\breve F$ invariant, and likewise that the correspondence ε_s fixes $\omega_{\breve h}$ and $\breve H$.

Arguing exactly as in the proof of Proposition 4.13, it follows that

$$e_{\mathrm{ord}}\xi_P(\omega_{\breve f},\omega_{\breve h})_{\mathcal{W}_\infty}=2\breve F\cdot\omega_{\breve h,\infty}\ \text{ and hence }\ e_{\mathrm{ord}}\xi(\omega_{\breve f},\omega_{\breve h})_{\mathcal{W}_\infty}=2P(\Phi_\infty)^{-1}\big(\breve F\cdot\omega_{\breve h,\infty}\big). \tag{136}$$

Theorem 4.15. *Let* $\mathcal{E}_g(f,g,h):=2(1-\chi^{-1}(p)\beta_f^{-1}\beta_g\beta_h^{-1})^{-1}=2(1-\chi(p)p^{-1}\alpha_f\alpha_g^{-1}\alpha_h)^{-1}$. *Then*

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\omega_{\breve f}\otimes\eta_{\breve g^*}\omega_{\breve h})=\mathcal{E}_g(f,g,h)\times\breve g^*(\breve F\times\breve h).$$

Proof. Combining (133) and (136), and taking into account that the adjoint of w_s^{-1} with respect to $\langle\,,\rangle_s$ is w_s, it follows that

$$\mathrm{AJ}_p(\Delta^\circ_{s,s})(\omega_{\breve f}\otimes\eta_{\breve g^*}\omega_{\breve h})=2\langle\eta_{\breve g^*},w_sP(\Phi_\infty)^{-1}\big(\breve F\cdot\omega_{\breve h,\infty}\big)\rangle_s=\mathcal{E}_g(f,g,h)\langle\eta_{\breve g^*},w_s\big(\breve F\cdot\omega_{\breve h,\infty}\big)\rangle_s.$$

The theorem is then a consequence of the definition of $\eta_{\breve g^*}$. □

4.4. **Garrett-Hida p-adic L-functions.** We now come to the main results of this chapter, expressing the Bloch-Kato logarithms of the specialisation of the classes $\kappa_p^f(\breve f^*,\underline{\breve g\breve h})$ and $\kappa_p^g(\breve f,\underline{\breve g^*\breve h})$ at weight 2 points in terms of the values of the one-variable Garret-Hida p-adic L-functions introduced in (112).

Theorem 4.16. *For all arithmetic points* $x=(y,z)\in\Omega_g\times_\Omega\Omega_h$ *of weight* $(2,\epsilon)$,

$$\log_p(\kappa_p^f(\breve f^*,\breve g_y,\breve h_z))=\mathcal{E}_f(f,g_y,h_z)\times\mathfrak{g}(\omega^{-1}\epsilon)\times\alpha_f^{s-1}\alpha_{g_y}^{-s}\alpha_{h_z}^{-s}\times\mathscr{L}_p{}^{f_\alpha}(\breve f^*,\underline{\breve g\breve h})(y,z).$$

Proof. Combining Corollary 2.11 with Theorem 4.14 leads to the equality

$$\log_p(\kappa_p^{f_\alpha}(\breve f^*,\breve g_y,\breve h_z))=\mathcal{E}_f(f,g_y,h_z)\times\mathfrak{g}(\omega^{-1}\epsilon)\times\alpha_f^{s-1}\alpha_{g_y}^{-s}\alpha_{h_z}^{-s}\times\breve f^*(\breve g_y\times d^{-1}\breve h_z^\iota). \tag{137}$$

The formal q-expansion with coefficients in Λ_h which is given by $\underline{\breve H}^\iota(q):=\sum_{p\nmid n}\langle n^{-1}\rangle a_n(\breve h)q^n$, where $\langle n\rangle\in\Lambda$ denotes the group-like element associted to $n\in\mathbb{Z}_p^\times$, satisfies the following properties:

(i) For any arithmetic point $z \in \Omega_h$ of weight-character $(2,\epsilon)$ as above, $\underline{\breve{H}}^{\iota}_z = d^{-1}\breve{h}^{\iota}_z$.

(ii) For any point $z \in \Omega_{\underline{h}}$ of weight $k \geq 2$ with $k \equiv 1 \pmod{p-1}$ and trivial nebentypus character at p, $\underline{\breve{H}}^{\iota}_z = d^{1-k}\breve{h}_z$. In particular, it is an overconvergent modular form of weight $2-k$.

If $\underline{\breve{g}}$ is any Λ-adic modular form of level N and character χ, then the product $\underline{\breve{g}} \times \underline{\breve{H}}^{\iota}$ therefore specialises, at all points (y,z) with common weights (k,ϵ), to a p-adic overconvergent modular form of weight 2 on $\Gamma_0(Np)$. Since ordinary overconvergent modular forms of weight 2 are classical, it follows that $e_{\mathrm{ord}}(\underline{\breve{g}} \times \underline{\breve{H}}^{\iota})$ belongs to the space $S_2(\Gamma_0(Np)) \otimes \Lambda_{gh}$ of classical modular forms tensored with the ring $\Lambda_{gh} = \Lambda_g \otimes_\Lambda \Lambda_h$. Property (ii) together with the interpolation property (6) implies that $\breve{f}^*(\underline{\breve{g}\breve{H}}^{\iota}) = \mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \underline{\breve{g}\breve{h}})$, since these two elements of $\Lambda_{f\underline{gh}}$ coincide on a dense set of points in $\Omega_g \times_\Omega \Omega_h$. In particular, $\breve{f}^*(\breve{g}_y \times d^{-1}\breve{h}^{\iota}_z) = \mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \underline{\breve{g}\breve{h}})(y,z)$ and the theorem follows from (137). $\square$

Theorem 4.17. *For all arithmetic points $x = (y,z) \in \Omega_g \times_\Omega \Omega_h$ of weight $(2,\epsilon)$,*

$$\log_p(\kappa_p^g(\breve{f}^*, \breve{g}_y, \breve{h}_z)) = \mathcal{E}_{g_y}(f, g_y, h_z) \times \mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}^*\breve{h}})(y,z).$$

Proof. Combining the second statement in Corollary 2.11 with Theorem 4.15,

$$\log_p(\kappa_p^g(\breve{f}, \breve{g}_y^*, \breve{h}_z)) = d_s^{-1} \times \mathcal{E}_{g_y}(f, g_y, h_z) \times \breve{g}_y^*(\breve{F} \times \breve{h}_z). \tag{138}$$

By definition,

$$\breve{g}_y^*(\breve{F} \times \breve{h}_z) = \langle \eta_{\breve{g}_y^*}, w_s(\breve{F} \times \breve{h}_z)\rangle_s = d_s \times (y,z)(\langle \eta_{\breve{g}_s^*}, w_s(\breve{F} \times \breve{h}_s)\rangle_{\Gamma_s}) = d_s \times (y,z)(\underline{\breve{g}}^*(\breve{F} \times \underline{\breve{h}})_s), \tag{139}$$

where the penultimate equality follows from Lemma 1.12, and the expression $\underline{\breve{g}}^*(\breve{F} \times \underline{\breve{h}})_s \in \Lambda_{f\underline{gh}} \otimes_\Lambda \Lambda_s$ occuring in the last term denotes the natural projection "to level s" of an element of $\Lambda_{f\underline{gh}}$. The interpolation property (6) of the p-adic L-function $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}}^*, \underline{\breve{h}})$ at points of weight $(2,\ell,\ell)$ and trivial nebentypus character shows that

$$\underline{\breve{g}}^*(\breve{F} \times \underline{\breve{h}}) = \mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}}^*, \underline{\breve{h}}),$$

and hence, since $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}^*\breve{h}})$ is just the restriction of $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}}^*, \underline{\breve{h}})$ to the one-dimensional rigid analytic space $\Omega_{f\underline{gh}}$, that

$$(y,z)(\underline{\breve{g}}^*(\breve{F} \times \underline{\breve{h}})_s) = \mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}^*\breve{h}})(y,z). \tag{140}$$

The theorem follows by combining (138), (139) and (140). $\square$

5. The explicit reciprocity law

5.1. Perrin-Riou's Λ-adic regulator.

Let $\tilde{\Lambda}$ be a finite flat extension of Λ and let $\Psi : G_{\mathbb{Q}_p} \longrightarrow \tilde{\Lambda}^\times$ be an unramified $\tilde{\Lambda}$-valued character. Let

$$\underline{\kappa}^g \in H^1(\mathbb{Q}_p, \tilde{\Lambda}\{\Psi\}(1)), \qquad \underline{\kappa}^f \in H^1(\mathbb{Q}_p, \tilde{\Lambda}\{\Psi\}(\underline{\epsilon}_{\mathrm{cyc}})) \tag{141}$$

be $\tilde{\Lambda}$-adic classes with values in the $\tilde{\Lambda}[G_{\mathbb{Q}_p}]$-modules $\tilde{\Lambda}\{\Psi\}(1) = \tilde{\Lambda}\{\Psi\}(\epsilon_{\mathrm{cyc}})$ and $\tilde{\Lambda}\{\Psi\}(\underline{\epsilon}_{\mathrm{cyc}})$), respectively. The notations used to designate these classes are motivated by (94), which asserts that $\underline{\kappa}^g = \kappa_p^g(\breve{f}, \underline{\breve{g}^*\breve{h}})$ and $\underline{\kappa}^f = \kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}})$ yield particular instances of (141), after taking

$$\Psi = \psi_f^{-1}\Psi_{\underline{g}}\Psi_{\underline{h}}^{-1} \times \chi^{-1} \qquad \text{and} \quad \Psi = \psi_f \Psi_{\underline{g}}^{-1}\Psi_{\underline{h}}^{-1}$$

for $\underline{\kappa}^g$ and $\underline{\kappa}^f$ respectively.

In general, the Λ-adic classes $\underline{\kappa}^g$ and $\underline{\kappa}^f$ gives rise to a collection of classical specialisations

$$\kappa_x^g \in H^1(\mathbb{Q}_p, K_x\{\Psi_x\}(1)), \qquad \kappa_x^f \in H^1(\mathbb{Q}_p, K_x\{\Psi_x\}(\epsilon_x)) \quad \text{where } \epsilon_x = \epsilon_{\mathrm{cyc}}^{\ell-1}\omega^{1-\ell}\epsilon,$$

as $x \in \tilde{\Omega}_{\mathrm{cl}}$ ranges over the set of classical points of $\tilde{\Omega} = \mathrm{Spf}(\tilde{\Lambda})$ satisfying $\mathrm{w}(x) = \nu_{\ell,\epsilon}$ for some $\ell \geq 1$ and a Dirichlet character ϵ of conductor p^s, $s \geq 1$.

One has canonical isomorphisms of Dieudonné modules for any $x \in \tilde{\Omega}_{\mathrm{cl}}$,

$$\begin{aligned} \mathrm{D}_{\mathrm{dR}}(K_x\{\Psi_x\}(1)) &:= \mathrm{D}_{\mathrm{dR}}(K_x\{\Psi_x\}) \otimes \mathrm{D}_{\mathrm{dR}}(K_x(1)) = \mathrm{D}_{\mathrm{dR}}(K_x(1)) \overset{\cdot t}{\simeq} K_x \\ \mathrm{D}_{\mathrm{dR}}(K_x\{\Psi_x\}(\epsilon_x)) &:= \mathrm{D}_{\mathrm{dR}}(K_x\{\Psi_x\}) \otimes \mathrm{D}_{\mathrm{dR}}(K_x(\epsilon_x)) = \mathrm{D}_{\mathrm{dR}}(K_x(\epsilon_x)) \overset{\cdot t^{\ell-1}}{\simeq} K_x, \end{aligned} \tag{142}$$

where (91) has been invoked in deriving the penultimate equalities, and the last isomorphism is given by multiplication by the corresponding power of Fontaine's period $t \in B_{\mathrm{dR}}$, the p-adic analogue of $2\pi i$ on which $G_{\mathbb{Q}_p}$ acts as multiplication by ϵ_{cyc}. (Note that the field K_x is assumed to contain ζ_s, and hence, that it contains the periods of the finite order character $\omega^{1-\ell}\epsilon$.)

Assume that for any $x \in \tilde{\Omega}_{\mathrm{cl}}$ the character Ψ_x is not trivial. Arguing as in [BDR2, Lemma 3.8], it then follows that the Bloch-Kato logarithm and dual exponential map induce isomorphisms

$$
\begin{aligned}
\log_p : \quad & H^1(\mathbb{Q}_p, K_x\{\Psi_x\}(1)) && \longrightarrow K_x && \text{for any } x \in \tilde{\Omega}_{\mathrm{cl}}, \\
\log_p : \quad & H^1(\mathbb{Q}_p, K_x\{\Psi_x\}(\epsilon_x)) && \longrightarrow K_x && \text{for any } x \in \tilde{\Omega}_{\mathrm{cl}} \text{ such that } \ell \geq 2, \\
\exp_p^* : \quad & H^1(\mathbb{Q}_p, K_x\{\Psi_x\}(\epsilon_x)) && \longrightarrow K_x && \text{for any } x \in \tilde{\Omega}_{\mathrm{cl}} \text{ such that } \ell = 1.
\end{aligned}
\tag{143}
$$

Note that the logarithm maps quoted in (96) are particular cases of the above. The two propositions below describe how the images of the classes κ_x^g and κ_x^f under the maps in (143) can be interpolated p-adically. The p-adic interpolation of the logarithms of the classes κ_x^g is treated first in Proposition 5.1 below, inspired by the elegant treatment of the p-adic interpolation of unramified periods given in [LZ12]. The corresponding interpolation problem for the classes κ_x^f, which is taken up in Proposition 5.2, lies considerably deeper and relies crucially on the "explicit reciprocity law" of Perrin-Riou (as refined in [LZ12]).

Proposition 5.1. *There exists an element $\mathcal{L}(\underline{\kappa}^g) \in \tilde{\Lambda}$ such that $\mathcal{L}(\underline{\kappa}^g)(x) = \log_p(\kappa_x^g)$ for all $x \in \tilde{\Omega}_{\mathrm{cl}}$.*

Proof. Let

$$G^{\mathrm{ur}} = \mathrm{Gal}\,(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p) = \hat{\mathbb{Z}}$$

be the Galois group of the maximal unramified extension of $\mathbb{Q}_p$, which is canonically isomorphic to $\hat{\mathbb{Z}}$ as a profinite group via the canonical topological generator given by the Frobenius automorphism $\sigma_p \in G^{\mathrm{ur}}$. Let $\mathbb{Z}_p[[G^{\mathrm{ur}}]]$ be the completed group ring for G^{ur}, and write $\mathbb{Z}_p^{\circ}[[G^{\mathrm{ur}}]]$ for the same completed group ring viewed as a rank one module over itself and equipped with the "tautological" unramified action of $G_{\mathbb{Q}_p}$ in which the Frobenius automorphism acts as multiplication by the group-like element σ_p. If R is a p-adic ring, and $\alpha \in R^{\times}$ is any invertible element of R, the assignment $\sigma_p \mapsto \alpha$ extends uniquely to a continuous group homomorphism $\Psi : G^{\mathrm{ur}} \longrightarrow R^{\times}$ since $R^{\times}$ is a profinite group, and this homomorphism extends by $\mathbb{Z}_p$-linearity to a continuous ring homomorphism

$$\Psi : \mathbb{Z}_p[[G^{\mathrm{ur}}]] \longrightarrow R.$$

Let $R\{\Psi\}$ denote the free R-module of rank one equipped with an action of $G_{\mathbb{Q}_p}$ in which $G_{\mathbb{Q}_p}$ acts on R via the unramified character Ψ. With these notations, the representation $R(\Psi)(1)$ of $G_{\mathbb{Q}_p}$ can be written as

$$R(\Psi)(1) = \mathbb{Z}_p^{\circ}[[G^{\mathrm{ur}}]](1)\hat{\otimes}_{\Psi} R.$$

It follows, after invoking Shapiro's Lemma for the second equality, that

$$H^1(\mathbb{Q}_p, R(\Psi)(1)) = H^1(\mathbb{Q}_p, \mathbb{Z}_p^{\circ}[[G^{\mathrm{ur}}]](1))\hat{\otimes}_{\Psi} R = (\varprojlim_n H^1(\mathbb{Q}_{p^n}, \mathbb{Z}_p(1)))\hat{\otimes}_{\Psi} R,$$

where the inverse limit is taken with respect the corestriction maps and $\mathbb{Q}_{p^n}$ denotes the unramified extension of $\mathbb{Q}_p$ of degree n for each $n \geq 1$. Hilbert's theorem 90 implies that

$$H^1(\mathbb{Q}_p, R(\Psi)(1)) = (\varprojlim_n (\mathbb{Q}_{p^n}^{\times} \otimes \mathbb{Z}_p))\hat{\otimes}_{\Psi} R = (\varprojlim_n (\mathcal{O}_n^{\times} \otimes \mathbb{Z}_p))\hat{\otimes}_{\Psi} R,$$

where the inverse limit is taken with respect to the norm maps, and $\mathcal{O}_n$ denotes the ring of integers of $\mathbb{Q}_{p^n}$. The standard p-adic logarithm $\log_p : \mathcal{O}_n^{\times} \to \mathcal{O}_n$ therefore gives rise to a natural homomorphism

$$\log_p : H^1(\mathbb{Q}_p, R(\Psi)(1)) \longrightarrow (\varprojlim_n \mathcal{O}_n)\hat{\otimes}_{\Psi} R, \tag{144}$$

where the inverse limit is taken relative to the trace maps. The elements of $\varprojlim_n \mathcal{O}_n$ can be interpreted as $\hat{\mathbb{Z}}_p^{\mathrm{ur}}$-valued measures on G^{ur} by associating to an element $a = \{a_n\}$ of this inverse limit the measure μ_a defined by

$$\mu_a(\sigma \cdot \mathrm{Gal}\,(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_{p^n})) := \sigma a_n, \qquad \forall n \geq 1, \quad \sigma \in \mathrm{Gal}\,(\mathbb{Q}_{p^n}/\mathbb{Q}_p).$$

The assignment

$$a \otimes 1 \mapsto \int_{G^{\mathrm{ur}}} \Psi(t) d\mu_a(t)$$

gives rise to a natural homomorphism

$$(\varprojlim_n \mathcal{O}_n)\hat{\otimes}_\Psi R \longrightarrow (R\hat{\otimes}\hat{\mathbb{Z}}_p^{\mathrm{ur}})^{\sigma_p=\alpha^{-1}} = (R(\Psi)\hat{\otimes}\hat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}. \tag{145}$$

Combining (144) and (145) allows one to parlay the logarithm map into a canonical homomorphism

$$H^1(\mathbb{Q}_p, R(\Psi)(1)) \longrightarrow (R(\Psi)\hat{\otimes}\hat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}. \tag{146}$$

Equivalently, after invoking (91), one obtains a canonical logarithm map

$$\mathcal{L} : H^1(\mathbb{Q}_p, R\{\Psi\}(1)) \longrightarrow R. \tag{147}$$

The homomorphism $\mathcal{L}$ is functorial in the sense that for all G^{ur}-equivariant homomorphisms $\varphi : R\{\Psi\} \longrightarrow R'\{\Psi'\}$, the diagram

$$\begin{tikzcd}
H^1(\mathbb{Q}_p, R\{\Psi\}(1)) \arrow[r, "\mathcal{L}"] \arrow[d, "\varphi"'] & R \arrow[d, "\varphi"'] \\
H^1(\mathbb{Q}_p, R'\{\Psi'\}(1)) \arrow[r, "\mathcal{L}"] & R'
\end{tikzcd}$$

commutes. In particular, after setting $R := \tilde{\Lambda}\{\Psi\}$ and letting $\varphi : \tilde{\Lambda}\{\Psi\} \longrightarrow K_x\{\Psi_x\}$ be the homomorphism induced by the evaluation map at x, the element $\mathcal{L}(\underline{\kappa}^g) \in \tilde{\Lambda}$ satisfies the interpolation property claimed in Proposition 5.1, in light of the fact that the map $\mathcal{L} : H^1(\mathbb{Q}_p, K_x\{\Psi_x\}(1)) \longrightarrow K_x$ agrees with the p-adic logarithm map $\log_p$. $\square$

In order to treat the analogous question for the Λ-adic cohomology class $\underline{\kappa}^f$, define

$$\mho_x := (-1)^{1-\ell} \cdot \Gamma^*(2-\ell) \cdot \mathfrak{g}(\epsilon^{-1}\omega^{\ell-1}) \cdot \Psi_x(\mathrm{Frob}_p)^{-s} \cdot \mathcal{E}_x \cdot p^{s(2-\ell)} \in K_x, \tag{148}$$

where

$$\Gamma^*(2-\ell) = \frac{1}{(\ell-2)!} \text{ if } \ell \geq 2; \qquad \Gamma^*(2-\ell) = (1-\ell)! \text{ if } \ell \leq 1,$$

and

$$\mathcal{E}_x = (1 - p^{1-\ell}\Psi_x(\mathrm{Frob}_p)^{-1})(1 - p^{\ell-2}\Psi_x(\mathrm{Frob}_p))^{-1} \text{ if } \epsilon = \omega^{\ell-1}; \qquad \mathcal{E}_x = 1 \text{ otherwise.}$$

Proposition 5.2. *There exists an element $\mathcal{L}(\underline{\kappa}^f) \in \tilde{\Lambda}$ such that for all $x \in \tilde{\Omega}_{\mathrm{cl}}$ with $\mathrm{w}(x) = \nu_{\ell,\epsilon}$,*

$$\mathcal{L}(\underline{\kappa}^f)(x) = \begin{cases} \mho_x \cdot \log_p(\kappa_x^f), & \text{if } \ell \geq 2, \\ \mho_x \cdot \exp_p^*(\kappa_x^f), & \text{if } \ell = 1. \end{cases}$$

Proof. Let K_∞ denote the abelian extension of $\mathbb{Q}_p$ through which the $\tilde{\Lambda}$-adic character $\Psi\underline{\epsilon}_{\mathrm{cyc}}$ factors, let $G = \mathrm{Gal}(K_\infty/\mathbb{Q}_p)$, and let $\Lambda_G := \mathbb{Z}_p[[G]]$ be the completed group ring attached to G, equipped with its tautological action of $G_{\mathbb{Q}_p}$. The decomposition $K_\infty = K_\infty^{\mathrm{ur}}\mathbb{Q}_p(\mu_{p^\infty})$, where K_∞^{ur} is the maximal unramified subfield of K_∞, determines a canonical decomposition $G = U \times \mathbb{Z}_p^\times$. The unramified character $\Psi : U \longrightarrow \tilde{\Lambda}^\times$ extends by linearity to a natural $G_{\mathbb{Q}_p}$-equivariant projection

$$\theta_\Psi : \Lambda_G \longrightarrow \tilde{\Lambda}\{\Psi\}(\underline{\epsilon}_{\mathrm{cyc}}). \tag{149}$$

The p-adic regulator of [LZ12], specialised to the case where V is the trivial one-dimensional representation of $G_{\mathbb{Q}_p}$, yields a "two-variable regulator map"

$$\mathcal{L}^G : H^1(\mathbb{Q}_p, \Lambda_G) \longrightarrow \Lambda_G \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}}.$$

Let $\tilde{\theta}_\Psi : H^1(\mathbb{Q}_p, \Lambda_G) \longrightarrow H^1(\mathbb{Q}_p, \tilde{\Lambda}\{\Psi\}(\underline{\epsilon}_{\mathrm{cyc}}))$ denote the map induced by (149) in cohomology. Note that $\tilde{\theta}_\Psi$ is surjective because the cohomological dimension of $\mathbb{Q}_p$ is 1. Let $\underline{\tilde{\kappa}}^f \in H^1(\mathbb{Q}_p, \Lambda_G)$ be any lift of the class $\underline{\kappa}^f$ under $\tilde{\theta}_\Psi$, and set

$$\mathcal{L}(\underline{\kappa}^f) := \theta_\Psi(\mathcal{L}^G(\underline{\tilde{\kappa}}^f)) \in \tilde{\Lambda}\{\Psi\} \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}}. \tag{150}$$

It is not hard to see (and follows, for instance, from Prop. 4.9. of [LZ12]) that $\mathcal{L}(\underline{\kappa}^f)$ is fixed under the action of Frob_p and hence $\mathcal{L}(\underline{\kappa}^f)$ belongs to $\mathbb{D}(\tilde{\Lambda}\{\Psi\})$, which is canonically isomorphic to $\tilde{\Lambda}$ by (91).

Now [LZ12, Theorem 4.15], with $j = 1-\ell$ and $\Phi^n = \Psi_x(\mathrm{Frob}_p)^{-s}$ applied to the character $\omega = \epsilon_{\mathrm{cyc}}^{\ell-1}$ in the notation of loc.cit. implies that the element $\mathcal{L}(\underline{\kappa}^f)$ has the required interpolation property. $\square$

5.2. **The triple product p-adic L-functions via Λ-adic cohomology classes.** We are now in position to prove the following result, which lends support to Perrin-Riou's vision according to which p-adic L-functions ought to arise as the images of p-adic families of distinguished global elements (referred to loosely as the Iwasawa theoretic incarnations of "Euler systems") under suitable p-adic regulator maps. Realising a p-adic L-function in this way has strong arithmetic consequences, some of which shall be explored in the remainder of this article.

Theorem 5.3. *The following equalities hold:*

$$\mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \underline{\breve{g}\breve{h}}) = \alpha_f/2 \cdot (1-\chi^{-1}(p)\alpha_f^{-1}a_p(\underline{g})a_p(\underline{h})^{-1}) \times \mathcal{L}(\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}})),$$
$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}^*\breve{h}}) = 1/2 \cdot (1-\chi(p)p^{-1}\alpha_f a_p^{-1}(\underline{g})a_p(\underline{h})) \times \mathcal{L}(\kappa_p^g(\breve{f}, \underline{\breve{g}^*\breve{h}})).$$

Proof. By specialising Propositions 5.1 and 5.2 to $\underline{\kappa}^g := \kappa_p^g(\breve{f}, \underline{\breve{g}^*\breve{h}})$ and $\underline{\kappa}^f := \kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}})$ respectively, one obtains from (95) that

$$\log_p(\kappa_p^f(\breve{f}^*, \breve{g}_y, \breve{h}_z)) = \alpha_f^s a_p(g_y)^{-s} a_p(h_z)^{-s} \times \mathcal{L}(\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}}))(y,z),$$
$$\log_p(\kappa_p^g(\breve{f}, \breve{g}_y^*, \breve{h}_z)) = \mathcal{L}(\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}}))(y,z),$$

at all classical points $x = (y,z) \in \Omega_g \times_\Omega \Omega_h$ of weight-character $(2,\epsilon)$ with $\epsilon\omega^{-1}$ of conductor p^s with $s \geq 1$. Comparing this identity with Theorems 4.16 and 4.17 respectively shows that

$$\mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \underline{\breve{g}}, \underline{\breve{h}})(y,z) = \alpha_f/2 \cdot (1-\chi^{-1}(p)\alpha_f^{-1}\alpha_{g_y}\alpha_{h_z}^{-1}) \times \mathcal{L}(\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}}))(y,z),$$
$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}}^*, \underline{\breve{h}})(y,z) = 1/2 \cdot (1-\chi(p)p^{-1}\alpha_f\alpha_{g_y}^{-1}\alpha_{h_z}) \times \mathcal{L}(\kappa_p^g(\breve{f}, \underline{\breve{g}^*\breve{h}}))(y,z),$$

for infinitely many points (y,z) of weight-character of the form $(2,\epsilon)$. But a non-zero element of $\Lambda_{f\underline{gh}}$ can only vanish at finitely many points of $\Omega_g \times_\Omega \Omega_h$. The theorem follows, since $a_p(g_y) = \alpha_{g_y}$ and $a_p(h_z) = \alpha_{h_z}$. $\square$

6. Application to the Birch and Swinnerton-Dyer conjecture

As in the introduction, let

$$f \in S_2(N_f), \qquad g \in S_1(N_g, \chi), \qquad h \in S_1(N_h, \chi^{-1})$$

be three newforms and assume that the level N_f of f is relatively prime to $N_g N_h$. Assume also that the weight two modular form f has rational fourier coefficients, and hence is associated to an elliptic curve E. (This last assumption is only made for notational simplicity and the arguments of this chapter would extend to the setting where E is replaced by any simple abelian variety quotient of $J_0(N)$, at the cost of slight technical complications.) The goal of this chapter is to prove Theorems A, B, C and D and their corollaries stated in the introduction, concerning the arithmetic of the twist of E by the four-dimensional self-dual Artin representation attached to

$$\varrho_{gh} := \varrho_g \otimes \varrho_h : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_4(L),$$

where ϱ_g and ϱ_h are the odd, irreducible, two-dimensional Artin representations attached to g and h.

6.1. **Bounding Mordell-Weil groups.** For this section, let W be a general d-dimensional self-dual Artin representation, with coefficients in a finite extension L of $\mathbb{Q}$, and factoring through the Galois group of a finite extension H of $\mathbb{Q}$. Our goal is to present general results, based on local Tate duality and the global Poitou-Tate exact sequence, for bounding the W-isotypic part of $E(H)$, defined to be the L-vector space

$$E(H)_L^W := \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbb{Q})}(W, E(H) \otimes L).$$

As in the introduction, fix a rational prime p and an embedding $L \subset L_p$ of L into a finite extension of $\mathbb{Q}_p$. Associated to E and W are the continuous p-adic representations

$$V_p(E) := H^1_{\mathrm{et}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p)(1) \otimes_{\mathbb{Q}_p} L_p, \qquad W_p := W \otimes_L L_p, \qquad V_p(E) \otimes_{L_p} W_p$$

of $G_{\mathbb{Q}}$, which are L_p-vector spaces of dimensions 2, d and $2d$ respectively. Restriction to the absolute Galois group G_H induces an isomorphism

$$\begin{aligned} H^1(\mathbb{Q}, V_p(E)\otimes W_p) &\simeq (H^1(H, V_p(E))\otimes W_p)^{\mathrm{Gal}\,(H/\mathbb{Q})} \\ &= \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbb{Q})}(W_p, H^1(H, V_p(E))), \end{aligned} \tag{151}$$

where the self-duality of W_p is used to obtain the second equality. Thanks to this identification, the connecting homomorphism

$$\delta : E(H)\otimes L \longrightarrow H^1(H, V_p(E))$$

of Kummer theory gives rise to a homomorphism

$$\delta : E(H)_L^{W_p} \longrightarrow H^1(\mathbb{Q}, V_p(E)\otimes W_p). \tag{152}$$

For each rational prime ℓ, the maps (151) and (152) admit local counterparts

$$\begin{aligned} H^1(\mathbb{Q}_\ell, V_p(E)\otimes W_p) &\simeq \mathrm{Hom}_{\mathrm{Gal}\,(H/\mathbb{Q})}(W_p, \oplus_{\lambda|\ell} H^1(H_\lambda, V_p(E))), \\ \delta_\ell : \left(\oplus_{\lambda|\ell} E(H_\lambda)\right)_{L_p}^{W_p} &\longrightarrow H^1(\mathbb{Q}_\ell, V_p(E)\otimes W_p), \end{aligned}$$

for which the following diagram commutes:

$$\begin{array}{ccc} E(H)_L^W & \xrightarrow{\ \delta\ } & H^1(\mathbb{Q}, V_p(E)\otimes W_p) \\ \big\downarrow{\scriptstyle \mathrm{res}_\ell} & & \big\downarrow{\scriptstyle \mathrm{res}_\ell} \\ \left(\oplus_{\lambda|\ell} E(H_\lambda)\right)_{L_p}^{W_p} & \xrightarrow{\ \delta_\ell\ } & H^1(\mathbb{Q}_\ell, V_p(E)\otimes W_p). \end{array} \tag{153}$$

The Bloch-Kato submodule $H^1_{\mathrm{fin}}(\mathbb{Q}_\ell, V_p(E)\otimes W_p)$ of the local cohomology group $H^1(\mathbb{Q}_\ell, V_p(E)\otimes W_p)$ coincides with the image of the local connecting homomorphism δ_ℓ, and the *singular quotient* is defined to be

$$H^1_{\mathrm{sing}}(\mathbb{Q}_\ell, V_p(E)\otimes W_p) := \frac{H^1(\mathbb{Q}_\ell, V_p(E)\otimes W_p)}{H^1_{\mathrm{fin}}(\mathbb{Q}_\ell, V_p(E)\otimes W_p)}.$$

When $\ell = p$ is a prime of good reduction for E at which W is unramified, the subspace $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_p(E)\otimes W_p)$ consists of classes of cristalline extensions of Galois representations. For each rational prime ℓ, let

$$\mathrm{res}_\ell : H^1(\mathbb{Q}, V_p(E)\otimes W_p) \longrightarrow H^1(\mathbb{Q}_\ell, V_p(E)\otimes W_p)$$

denote the restriction map from the global to the local cohomology at ℓ. The composition

$$\partial_\ell : H^1(\mathbb{Q}, V_p(E)\otimes W_p) \longrightarrow H^1_{\mathrm{sing}}(\mathbb{Q}_\ell, V_p(E)\otimes W_p)$$

of res_ℓ with the natural projection to the singular quotient is called the *residue map at ℓ*.

The Selmer group $\mathrm{Sel}_p(E, W)$ is defined in terms of these maps as

$$\mathrm{Sel}_p(E, W) = H^1_{\mathrm{fin}}(\mathbb{Q}, V_p(E)\otimes W_p) := \bigcap_\ell \ker\left(H^1(\mathbb{Q}, V_p(E)\otimes W_p) \xrightarrow{\partial_\ell} H^1_{\mathrm{sing}}(\mathbb{Q}_\ell, V_p(E)\otimes W_p)\right). \tag{154}$$

Lemma 6.1. *The local cohomology group $H^1(\mathbb{Q}_p, V_p(E)\otimes W_p)$ is a $2d$-dimensional L_p-vector space. The finite subspace $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_p(E)\otimes W_p)$ and the singular quotient $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_p(E)\otimes W_p)$ are each d-dimensional and in perfect duality under the local Tate pairing.*

Proof. This follows by choosing a $G_{\mathbb{Q}_p}$-stable lattice in $V_p(E)\otimes W_p$, applying (for instance) Theorem 2.17 of [DDT] to its finite $G_{\mathbb{Q}_p}$-stable quotients, and passing to the limit. $\square$

Proposition 6.2. *If the map ∂_p is surjective, then the map*

$$\mathrm{res}_p : \mathrm{Sel}_p(E, W) \longrightarrow H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_p(E)\otimes W_p) \tag{155}$$

is the zero map.

Proof. The standard calculations arising from the global Poitou-Tate exact sequence imply that the image of the restriction map

$$\mathrm{res}_p : \mathrm{Sel}_p(E, W) \longrightarrow H^1(\mathbb{Q}_p, V_p(E) \otimes W_p)$$

is d-dimensional. (This can be shown, for example, by choosing a $G_{\mathbb{Q}}$-stable lattice in $V_p(E) \otimes W_p$, applying Theorem 2.18 of [DDT] to its finite quotients, and taking the inverse limit.) It follows that the restriction of res_p to $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_p(E) \otimes W_p)$ is the zero map. □

Proposition 6.3. *Assume that the residue map ∂_p attached to the representation $V_p(E) \otimes W_p^\sigma$ is a surjective map of L_p-vector spaces, for all $\sigma \in \mathrm{Gal}\,(L/\mathbb{Q})$. Then $E(H)_L^W = 0$.*

Proof. By Proposition 6.2, the map res_p of (155) is the zero map, and therefore the commutativity of the diagram (153) implies that the natural map

$$\mathrm{res}_p : E(H)_L^{W^\sigma} \longrightarrow ((\oplus_{\mathfrak{p}|p} E(H_{\mathfrak{p}})) \otimes L_p)^{W^\sigma}$$

is the zero map, for each $\sigma \in \mathrm{Gal}\,(L/\mathbb{Q})$. Since the local connecting homomorphism δ_ℓ is injective, this implies that the vector space

$$\oplus_{\sigma \in \mathrm{Gal}\,(L/\mathbb{Q})} E(H)_L^{W^\sigma} = E(H)_L^{\tilde{W}}, \qquad \text{where } \tilde{W} = \oplus_{\sigma \in \mathrm{Gal}\,(L/\mathbb{Q})} W^\sigma,$$

has trivial image in the group $\oplus_{\mathfrak{p}|p} E(H_{\mathfrak{p}}) \otimes L$ of local points. But $\tilde{W}$ is a rational representation and hence admits an L-basis consisting of elements of $E(H)$. Since the natural map $E(H) \to \oplus_{\mathfrak{p}|p} E(H_{\mathfrak{p}})$ is injective modulo torsion, it follows that $\dim_{\mathbb{Q}} E(H)_{\mathbb{Q}}^{\tilde{W}} = 0$, and therefore the same is true of $\dim_L E(H)_L^W$. The proposition follows. □

6.2. **Proof of Theorem C.** The most interesting application of Theorem 5.3 arises without a doubt when $\ell = 1$. A simplified version of the following statement was stated in Theorem C of the introduction. As in loc. cit. let $g \in S_1(N_g, \chi)$ and $h \in S_1(N_h, \chi^{-1})$ be classical newforms of weight 1 and let $g_\alpha \in S_1(N_g p, \chi)$ and $h_\alpha \in S_1(N_h p, \chi^{-1})$ be ordinary p-stabilisations.

Let $\underline{g}$ and $\underline{h}$ be Hida families specialising to $g_y = g_\alpha$ and $h_z = h_\alpha$ at suitable weight one points y and z of the parameter spaces $\Omega_{\underline{g}}$ and $\Omega_{\underline{h}}$ respectively, and fix test vectors $\breve{f}^*$, $\underline{\breve{g}}$ and $\underline{\breve{h}}$ as in (92) and (93). Let

$$\kappa(f, g_\alpha, h_\alpha) \in H^1(\mathbb{Q}, V_{fgh}(N))$$

be the global cohomology class introduced in (76).

Theorem 6.4. *The central critical value $L(f, g, h, 1)$ is non-zero if and only if the global cohomology class $\kappa(f, g_\alpha, h_\alpha)$ is* not *crystalline at p.*

Proof. Since the triplet $(2, 1, 1)$ is unbalanced, with 2 as its dominant weight, the point (y, z) lies in the region of interpolation defining the p-adic L-function $\mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \underline{\breve{g}\breve{h}})$. By [DR13, Theorems 4.2 and 4.7], $L(f, g, h, 1) \neq 0$ if and only if there exists a choice $(\breve{f}^*, \breve{g}_\alpha, \breve{h}_\alpha)$ of test vectors such that $\mathscr{L}_p{}^{f_\alpha}(\breve{f}^*, \underline{\breve{g}\breve{h}})(y, z) \neq 0$. By Theorem 5.3, this in turn is equivalent to the non-vanishing of the value of $\mathcal{L}(\kappa_p^f(\breve{f}^*, \underline{\breve{g}\breve{h}}))$ at the point (y, z), which Proposition 5.2 recasts as a non-zero multiple of $\exp_p^*(\kappa_p^f(\breve{f}^*, \breve{g}_\alpha, \breve{h}_\alpha))$. Because the dual exponential map appearing in (143) is an isomorphism, one concludes that $L(f, g, h, 1) \neq 0$ if and only if there is a triple $(\breve{f}^*, \breve{g}_\alpha, \breve{h}_\alpha)$ of test vectors for which the local class $\kappa_p^f(\breve{f}^*, \breve{g}_\alpha, \breve{h}_\alpha)$ is non-trivial.

As $G_{\mathbb{Q}_p}$-modules, $V_f(Np)^- \otimes V_{gh}^{\beta\beta}(N)$ is isomorphic to the direct sum of a finite number of copies of $K_x\{\psi_f \Psi_{g_\alpha}^{-1} \Psi_{h_\alpha}^{-1}\}$, and the local class $\kappa_p^f(f, g_\alpha, h_\alpha)$ is determined by the collection of its projections to the cohomology groups $H^1(\mathbb{Q}_p, K_x\{\psi_f \Psi_{g_\alpha}^{-1} \Psi_{h_\alpha}^{-1}\})$ indexed by the set of all possible triples $(\breve{f}^*, \breve{g}_\alpha, \breve{h}_\alpha)$ of test vectors. The non-vanishing of $\kappa_p^f(f, g_\alpha, h_\alpha)$ is therefore equivalent to the non-vanishing of the local class $\kappa_p^f(\breve{f}^*, \breve{g}_y, \breve{h}_z) \in H^1(\mathbb{Q}_p, K_x\{\psi_f \Psi_{g_\alpha}^{-1} \Psi_{h_\alpha}^{-1}\})$ introduced in (95) for some such triple. Theorem 6.4 now follows from Proposition 2.8. □

6.3. **Proof of Theorem A and its corollaries.** We now specialise the results of the previous section to the setting where $W = V_{gh}$ in order to prove Theorem A of the introduction and its corollaries. Henceforth, the Artin representation V_{gh} is viewed as having coefficients in a finite extension L of $\mathbb{Q}$, just as the representation W of the previous section. In the notation of §1.6 we thus have $L_p = K_{fgh}$.

Proposition 6.5. *Assume $L(f,g,h,1) \neq 0$. For all pairs of eigenvalues*

$$\lambda \in \{(\alpha_g, \alpha_h), (\alpha_g, \beta_h), (\beta_g, \alpha_h), (\beta_g, \beta_h)\},$$

there exists a global class $\kappa_\lambda \in H^1(\mathbb{Q}, V_{fgh})$ whose natural image in $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{fgh})$ is non-zero and belongs to $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_f \otimes V_{gh}^\lambda)$.

Proof. Up to re-ordering the eigenvalues of Frob_p acting on V_g and V_h, it may be assumed without loss of generality that $\lambda = (\alpha_g, \alpha_h)$. By Theorem 6.4, the associated cohomology class $\kappa(f, g_\alpha, h_\alpha)$ is non-cristalline, and therefore there exists a Galois-equivariant surjection $j : V_{fgh}(N) \longrightarrow V_{fgh}$ for which $\kappa_\lambda := j(\kappa(f, g_\alpha, h_\alpha))$ is also non-cristalline. Since the natural image of κ_λ in $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{fgh})$ belongs to $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_f \otimes V_{gh}^\lambda)$ by Proposition 2.8, the proposition follows. $\square$

Theorem 6.6. *If $L(E, V_{gh}, 1) \neq 0$, there exists a rational prime p for which the natural map*

$$\partial_p : H^1(\mathbb{Q}, V_f \otimes V_{gh}) \longrightarrow H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_f \otimes V_{gh})$$

is surjective.

Proof. The Chebotarev density theorem shows the existence of a prime $p \geq 5$ whose associated Frobenius element acts on both V_g and V_h with distinct eigenvalues, and such that $a_p(f) \neq 0$ (so that f is ordinary at p). By Proposition 6.5, the non-vanishing of $L(E, V_{gh}, 1)$ implies the existence of global classes

$$\kappa_{\alpha_g,\alpha_h}, \quad \kappa_{\alpha_g,\beta_h}, \quad \kappa_{\beta_g,\alpha_h}, \quad \kappa_{\beta_g,\beta_h} \in H^1(\mathbb{Q}, V_{fgh}) \tag{156}$$

whose image under ∂_p is non-zero in $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_{fgh})$. Since $\partial_p(\kappa_\lambda)$ belongs to $H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_f \otimes V_{gh}^\lambda)$ for each pair $\lambda \in \{(\alpha_g, \alpha_h), (\alpha_g, \beta_h), (\beta_g, \alpha_h), (\beta_g, \beta_h)\}$, the four classes are linearly independent and generate this singular quotient. $\square$

We can now prove Theorem A of the introduction:

Theorem 6.7. *Assume that ϱ_{gh} is regular. If $L(E, \varrho_{gh}, 1) \neq 0$, then $E(H)_L^{\varrho_{gh}} = 0$.*

Proof. For all $\sigma \in \mathrm{Gal}\,(L/\mathbb{Q})$, the non-vanishing of $L(E, \varrho_{gh}, 1)$ at the central point implies that the same is true for $L(E, V_{gh}^\sigma, 1)$, for all $\sigma \in \mathrm{Gal}\,(L/\mathbb{Q})$. Furthermore, there exists a prime p for which V_{gh}^σ is regular, for all such σ. Theorem 6.6 implies that the map

$$\partial_p : H^1(\mathbb{Q}, V_p(E) \otimes V_{gh}^\sigma) \longrightarrow H^1_{\mathrm{sing}}(\mathbb{Q}_p, V_p(E) \otimes V_{gh}^\sigma)$$

is surjective for each $\sigma \in \mathrm{Gal}\,(L/\mathbb{Q})$. The theorem now follows from Proposition 6.3. $\square$

Turning to Corollary A1 of the introduction, let K be a quadratic field of discriminant prime to the conductor N_E of E. If ψ is a finite order ray class character of K, let ψ' denote the character obtained from it by composing with the automorphism of $K/\mathbb{Q}$. A ring class character is a character of K with trivial central character, which therefore satisfies $\psi' = \psi^{-1}$. Let Π_K denote the set of all ring class characters of conductor prime to N_E. When K is a real quadratic field, the set Π_K can be partitioned into two subsets Π_K^+ and Π_K^-, consisting of totally even and totally odd ring class characters, respectively. Let $\Pi_{E/K}$, $\Pi_{E/K}^+$ and $\Pi_{E/K}^-$ denote the subsets of Π_K, Π_K^+ and Π_K^- respectively consisting of the characters for which $L(E/K, \psi, 1) \neq 0$.

To the pair (E, K) one can associate the sign $\mathrm{sgn}(E, K) \in \{\pm 1\}$ of the functional equation for the L-function $L(E/K, s)$. It turns out that this is the very same sign that occurs in the functional equation of $L(E/K, \psi, s)$ for any $\psi \in \Pi_K$. In particular, $\Pi_{E/K}$ is empty when $\mathrm{sgn}(E, K) = -1$.

When $\mathrm{sgn}(E, K) = 1$ and K is imaginary quadratic, a non-vanishing theorem of Cornut and Vatsal shows that $\Pi_{E/K}$ is infinite. If K is real quadratic, a simple argument involving congruences for L-values shows that $\Pi_{E/K}^+$ is either empty or infinite, and likewise for $\Pi_{E/K}^-$. The scenario where $\Pi_{E/K}^\pm = \emptyset$ is highly unlikely, since it would provide us with a systematic supply of ring class characters for which the (primitive) L-function $L(E/K, \psi, s)$ admits at least a double zero at the center.

Definition 6.8. The pair (E,K) is said to satisfy the *non-vanishing hypothesis* if $\Pi^+_{E/K}$ and $\Pi^-_{E/K}$ are both non-empty (and hence, infinite).

When K is real, a proof of this non-vanishing hypothesis does not seem out of reach of current techniques in analytic number theory, although the unavailability of an "anti-cyclotomic $\mathbb{Z}_p$-extension" of K prevents a straightforward application of the methods of Cornut and Vatsal.

Lemma 6.9. *For all $\psi \in \Pi_K$, there exists a ray class character ψ_0 of K of conductor prime to N_E such that $\psi = \psi_0/\psi_0'$.*

Proof. Let M/K be the cyclic extension of K (of degree n, say) which is cut out by the character ψ. This field is Galois over $\mathbb{Q}$, and $\mathrm{Gal}\,(M/\mathbb{Q})$ is isomorphic to the dihedral group D_n of order $2n$, furnished with a natural embedding into the semi-direct product $(\mathbb{C}^\times \rtimes \mathbb{Z}/2\mathbb{Z})$ via the character ψ. Embed D_n into $\mathrm{PGL}_2(\mathbb{C})$ by sending $\lambda \in \mathbb{C}^\times$ to the class of any diagonal matrix of the form $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$, where $\lambda_1/\lambda_2 = \lambda$ modulo the center, and sending some reflection in D_n to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This yields a projective representation $\bar r : G_{\mathbb{Q}} \to \mathrm{PGL}_2(\mathbb{C})$. By a classical result of Tate on the vanishing of $H^2(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1))$, the projective representation $\bar r$ may be lifted to a linear representation $r : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$. By construction, the representation r has dihedral projective image, and hence is induced from some character ψ_0 of K. The restriction of r to G_K is then equal to the direct sum $\psi_0 \oplus \psi_0'$. It follows that the restriction of $\bar r$ to G_K is identified with ψ_0/ψ_0', and therefore that $\psi_0/\psi_0' = \psi$. To show that ψ_0 can be chosen to be unramified at the primes of K dividing N_E, observe that for each prime $\ell | N_E$, the linear representation r maps the inertia group at ℓ to the subgroup $\mathbb{C}^\times$ of scalar matrices, since $\bar r$ is unramified at ℓ. Now choose a Dirichlet character $\chi : G_{\mathbb{Q}} \longrightarrow \mathbb{C}^\times$ whose restriction to I_ℓ agrees with the restriction of r, for all $\ell | N_E$, and replace r by $r \otimes \chi^{-1}$. This substitution does not affect the projective representation attached to r, but leads to a representation r which is unramified at the primes dividing N_E; hence the same is true for the resulting character ψ_0. □

Note that, if K is a real quadratic field, the character ψ_0 is of mixed signature (resp. totally even or odd) when ψ is totally odd (resp. totally even).

By considering the case where α is further restricted to be a quadratic character of K, it is clear that the non-vanishing hypothesis is satisfied if there are two quadratic Dirichlet characters χ (of the two possible signatures) for which

$$L(E, \chi, 1) \neq 0, \quad \text{and} \quad L(E, \chi\chi_K, 1) \neq 0, \tag{157}$$

where χ_K is the (fixed) Dirichlet character attached to K. Such simultaneous non-vanishing results for pairs of quadratic Dirichlet characters with fixed product seem to lie just beyond the reach of currently available techniques. However, a character χ satisfying (157) is widely expected to exist, and a quick inspection is usually enough to produce such a χ, for any given E and K. For example, if E is the elliptic curve of smallest conductor 11 and $K = \mathbb{Q}(\sqrt{5})$ is the real quadratic field of smallest discriminant 5, a cursory inspection of the tables of Cremona reveals that $L(E, \chi, 1) \neq 0$ and $L(E, \chi\chi_K, 1) \neq 0$ when

(1) χ is the trivial character;
(2) χ is the odd quadratic character of conductor 4.

Hence the pair (E,K) satisfies the non-vanishing hypothesis.

Theorem 6.10. *Let E be an elliptic curve over $\mathbb{Q}$ and let K be a quadratic field for which* $\mathrm{sign}(E,K) = 1$. *When K is real, assume that (E,K) satisfies the non-vanishing hypothesis. Then for all ring class characters $\psi \in \Pi_K$,*

$$L(E/K, \psi, 1) \neq 0 \quad \Rightarrow \quad E(H)^\psi = 0.$$

Proof. If ψ is a quadratic (genus) character, the statement already follows from the approach of Gross-Zagier and Kolyvagin (or Kato), since the induced representation V_ψ decomposes as a sum of two quadratic Dirichlet characters.

Assume henceforth that $\psi^2 \neq 1$, which amounts to saying that V_ψ is irreducible. Choose a character $\alpha \in \Pi_{E/K}$ which differs from ψ and ψ^{-1}, and for which $V_\psi \oplus V_\alpha$ is regular. If K is real, assume α to be of opposite signature to ψ. This choice is possible thanks to the non-vanishing hypothesis, and implies that complex conjugation acts on $V_\psi \oplus V_\alpha$ with eigenvalues $(1, 1, -1, -1)$.

Invoking Lemma 6.9, let ψ_0 and α_0 be ray class characters of K of conductors prime to N_E and satisfying $\psi_0/\psi_0' = \psi$ and $\alpha_0/\alpha_0' = \alpha$, and set

$$\psi_g := \psi_0\alpha_0, \qquad \psi_h := (\psi_0'\alpha_0)^{-1}.$$

The two-dimensional representations

$$\varrho_g := \mathrm{Ind}_K^{\mathbb{Q}}\psi_g, \qquad \varrho_h := \mathrm{Ind}_K^{\mathbb{Q}}\psi_h$$

satisfy the following properties:

- $\det(\varrho_g) = \det(\varrho_h)^{-1}$, because the central characters of ψ_g and ψ_h are inverses of each other,
- ϱ_g and ϱ_h are odd: this is automatic when K is imaginary; when K is real this follows because ψ and α have opposite signature, hence exactly one of ψ_0 and α_0 is of mixed signature, implying that both ψ_g and ψ_h are of mixed signature.
- ϱ_g and ϱ_h are irreducible: this amounts to saying that $\psi_g \neq \psi_g'$ and $\psi_h \neq \psi_h'$; tracing the definitions, this holds because α has been chosen to be different from ψ and ψ^{-1};
- the conductors of ϱ_g and ϱ_h are prime to N_E, and hence the same is true for the levels of the associated weight one modular forms.

The tensor product ϱ_{gh} decomposes as

$$V_{gh} = V_{\psi_g} \otimes V_{\psi_h} = V_{\psi_g\psi_h} \oplus V_{\psi_g\psi_h'} = V_\psi \oplus V_\alpha. \tag{158}$$

Thorem 6.10 now follows from the regularity of V_g and V_h and from Theorem 6.7, since

$$L(E, V_{gh}, 1) = L(E, V_\psi, 1)L(E, V_\alpha, 1) = L(E/K, \psi, 1)L(E/K, \alpha, 1) \neq 0.$$

□

We now turn to the proof of Corollary A2 of the introduction:

Theorem 6.11. *Let E be an elliptic curve over $\mathbb{Q}$ and let K be a non-real quintic extension of $\mathbb{Q}$ with Galois group A_5, of discriminant prime to the conductor of E. Then*

$$\mathrm{ord}_{s=1}L(E/K, s) = \mathrm{ord}_{s=1}L(E/\mathbb{Q}, s) \quad \Rightarrow \quad \mathrm{rank}(E(K)) = \mathrm{rank}(E(\mathbb{Q})).$$

Proof. Let $\tilde{K}$ denote the Galois closure of K and fix an embedding of the group A_5 into $\mathrm{PGL}_2(\mathbb{C})$. There are exactly two conjugacy classes of such embeddings, and the further choice of an isomorphism $\mathrm{Gal}\,(\tilde{K}/\mathbb{Q}) = A_5$ gives rise to a projective representation

$$\bar{\varrho}_g : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}\,(\tilde{K}/\mathbb{Q}) \simeq A_5 \subset \mathrm{PGL}_2(\mathbb{C}).$$

Tate's lifting theorem produces a linear lift $\varrho_g : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ of $\bar{\varrho}_g$, which is odd since K is not totally real. An argument similar to the one concluding the proof of Lemma 6.9 also shows that ϱ_g can be chosen to unramified at the primes dividing N_E.

Assume that the field H cut out by ϱ_g is a cyclic extension of $\tilde{K}$ of degree a power of 2. The image of $\mathrm{Gal}\,(H/\tilde{K})$ under ϱ_g consists of scalar matrices whose diagonal entries are 2-power roots of unity. Let σ be an automorphism of $\mathbb{C}$ which agrees with complex conjugation on the 2-power roots of unity and sends $\sqrt{5}$ to $-\sqrt{5}$, and let ϱ_h be the Artin representation obtained by applying σ to the matrix entries of ϱ_g. The tensor product ϱ_{gh} is a four-dimensional representation which factors through $\mathrm{Gal}\,(\tilde{K}/\mathbb{Q}) = A_5$ since it is trivial on $\mathrm{Gal}\,(H/\tilde{K})$. To identify this representation, observe that there are precisely two distinct irreducible two-dimensional representations, denoted ϱ_1 and ϱ_2, of the non-trivial central extension $\tilde{A}_5$ of A_5, whose traces lie in $\mathbb{Q}(\sqrt{5})$ and which are conjugate to each other over this field. Viewing $\varrho_1 \otimes \varrho_2$ as a representation of $\mathrm{Gal}\,(\tilde{K}/\mathbb{Q})$, one has

$$\varrho_{gh} = \varrho_1 \otimes \varrho_2 = (\mathrm{Ind}_K^{\mathbb{Q}}1) - 1,$$

where the last equality of virtual representations can be seen, for instance, by consulting the Atlas of finite simple groups. The Artin formalism implies that

$$r_{\mathrm{an}}(E, \varrho_{gh}) = r_{\mathrm{an}}(E/K) - r_{\mathrm{an}}(E/\mathbb{Q}), \qquad r(E, \varrho_{gh}) = r(E/K) - r(E/\mathbb{Q}).$$

By hypothesis, $r_{\mathrm{an}}(E, \varrho_{gh}) = 0$. Theorem 6.7 can now be invoked to conclude that $r(E, \varrho_{gh}) = 0$, i.e., $r(E, K) = r(E, \mathbb{Q})$. Theorem 6.11 follows. □

6.4. **Proof of Theorems B and D.** We conclude with the proof of Theorem D of the introduction, which in turn implies Theorem B. Assume for this section that

(i) $L(E, \varrho_{gh}, 1) = 0$;
(ii) the Artin representations ϱ_g and ϱ_h are both regular at p.

Let V_{fgh} and $V_{fgh}(N) = V_f(Np) \otimes V_g(Np) \otimes V_h(Np)$ denote the K_{fgh}-vector spaces obtained by specialising of $\mathbb{V}_{f\underline{gh}}$ and $V_{f\underline{gh}}(N)$ at the weight one point (y, z) attached to (f, g, h). As discussed right after (76), the global cohomology classes

$$\kappa(f, g_\alpha, h_\alpha), \quad \kappa(f, g_\alpha, h_\beta) \quad \in \quad H^1(\mathbb{Q}, V_{fgh}(N)) \tag{159}$$

are both locally trivial at primes $\ell \neq p$. Hence the vanishing of $L(E, \varrho_{gh}, 1)$ implies, by Theorem 6.4, that $\kappa(f, g_\alpha, h_\alpha)$ and $\kappa(f, g_\alpha, h_\beta)$ belong to the Selmer group $\mathrm{Sel}_p(E, V_{gh}(N))$, which can be identified with a finite sum of copies of the Selmer group $\mathrm{Sel}_p(E, \varrho_{gh})$.

Lemma 6.12. *The images of $\kappa(f, g_\alpha, h_\alpha)$ and $\kappa(f, g_\alpha, h_\beta)$ in $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_f(Np) \otimes V_g^{\alpha_g}(Np) \otimes V_h(Np))$ belong to $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_f(Np) \otimes V_g^{\alpha_g}(Np) \otimes V_h^{\beta_h}(Np))$ and to $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_f(Np) \otimes V_g^{\alpha_g}(Np) \otimes V_h^{\alpha_h}(Np))$ respectively.*

Proof. This follows directly from Proposition 2.7. $\square$

Theorem 6.13. *Assume that $\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}) \neq 0$ for some choice*

$$\breve{f} \in S_2(Np)[f], \qquad \breve{g}^* \in S_1(N, \chi^{-1})^\vee[g], \qquad \breve{h} \in S_1(N, \chi^{-1})[h]$$

of test vectors. Then there exist $G_\mathbb{Q}$-equivariant projections $j_1, j_2 : V_{fgh}(N) \to V_{fgh}$ for which the global classes

$$\kappa_{\alpha\alpha} := j_1(\kappa(f, g_\alpha, h_\alpha)), \qquad \kappa_{\alpha\beta} := j_2(\kappa(f, g_\alpha, h_\beta)) \tag{160}$$

are linearly independent in $\mathrm{Sel}_p(E, \varrho_{gh})$.

Proof. Let $\underline{g}$ and $\underline{h}$ be Hida families specialising to g_α and to h_α respectively in weight one, i.e., for which $g_y = g_\alpha$ and $h_z = h_\alpha$ for suitable weight one points $(y, z) \in \Omega_{\underline{g}} \times \Omega_{\underline{h}}$. By Proposition 5.1 combined with Theorem 5.3, the non-vanishing of

$$\mathscr{L}_p{}^{g_\alpha}(\breve{f}, \breve{g}^*, \breve{h}) := \mathscr{L}_p{}^{g_\alpha}(\breve{f}, \underline{\breve{g}}^*, \underline{\breve{h}})(y, z)$$

implies that the local class $\kappa_p^g(\breve{f}, \breve{g}_\alpha^*, \breve{h}_\alpha)$ has non-zero p-adic logarithm. It follows a fortiori from (94) and (95) that the local class $\kappa_p^g(f, \underline{gh})(y, z)$ is non-zero, and therefore, that the natural image of the global class $\kappa(f, g_\alpha, h_\alpha)$ in $H^1(\mathbb{Q}_p, V_f(Np) \otimes V_g^{\alpha_g}(Np) \otimes V_h(Np))$ is non-zero as well. The same argument in which h_α is replaced by h_β implies (in light of equation (9) of the introduction) the same conclusion for the global class $\kappa(f, g_\alpha, h_\beta)$.

After choosing a basis of $\mathrm{Hom}_{G_\mathbb{Q}}(V_{fgh}(N), V_{fgh})$ (of cardinality t, say), the resulting isomorphism $V_{fgh}(N) \longrightarrow \oplus_{i=1}^t V_{fgh}$ induces an isomorphism

$$H^1(\mathbb{Q}_p, V_f(Np) \otimes V_g^{\alpha_g}(Np) \otimes V_h(Np)) \longrightarrow \oplus_{i=1}^t H^1(\mathbb{Q}_p, V_f \otimes V_g^{\alpha_g} \otimes V_h)$$

on the quotients of the local cohomology groups at p. Hence there are $G_\mathbb{Q}$-equivariant homomorphisms $j_1, j_2 : V_{fgh}(N) \to V_{fgh}$ for which the global classes $\kappa_{\alpha\alpha}$ and $\kappa_{\alpha\beta}$ defined in (160) have non-trivial image in the quotient $H^1(\mathbb{Q}_p, V_f \otimes V_g^{\alpha_g} \otimes V_h)$.

As argued above right after (159), the global classes $\kappa(f, g_\alpha, h_\alpha)$ and $\kappa(f, g_\alpha, h_\beta)$ lie in the Selmer group $\mathrm{Sel}_p(E, V_{gh}(N))$ and hence $\kappa_{\alpha\alpha}$ and $\kappa_{\alpha\beta}$ likewise belong to $\mathrm{Sel}_p(E, \varrho_{gh})$. Lemma 6.12 further implies that their natural images in $H^1(\mathbb{Q}_p, V_f \otimes V_g^{\alpha_g} \otimes V_h)$ belong to the complementary subspaces $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_f \otimes V_g^{\alpha_g} \otimes V_h^{\beta_h})$ and $H^1_{\mathrm{fin}}(\mathbb{Q}_p, V_f \otimes V_g^{\alpha_g} \otimes V_h^{\alpha_h})$ respectively. These images are therefore linearly independent, and the theorem follows. $\square$

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