

BEILINSON-FLACH ELEMENTS AND EULER SYSTEMS I: SYNTOMIC REGULATORS AND p -ADIC RANKIN L -SERIES

MASSIMO BERTOLINI, HENRI DARMON, AND VICTOR ROTGER

Abstract

This article is the first in a series devoted to the Euler system arising from p -adic families of *Beilinson-Flach elements* in the first K -group of the product of two modular curves. It relates the image of these elements under the p -adic syntomic regulator (as described by Besser (2012)) to the special values at the near-central point of Hida's p -adic Rankin L -function attached to two Hida families of cusp forms.

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1. Introduction

This article is the first in a series devoted to the Euler system of *Beilinson-Flach elements* in the motivic cohomology of a product of two modular curves. Its main result (see Theorem 4.2 and Corollary 4.4 of §4.2) is a p -adic analogue of the formula of Beilinson [Bei, Ch. 2, §6] expressing special values of Rankin L -series in terms of complex regulators. Beilinson's theorem (cf. §4.1 for an explicit version) relates:

- (1) the Rankin L -series $L(f \otimes g, s)$ attached to the convolution of weight 2 newforms f and g on $\Gamma_1(N)$, evaluated at the *near-central point* $s = 2$;
- (2) the image under the complex regulator of certain explicit elements in the motivic cohomology group $H_{\mathcal{M}}^3(X_1(N)^2, \mathbb{Q}(2))$ or, equivalently, in the higher Chow group $\mathrm{CH}^2(X_1(N)^2, 1) \otimes \mathbb{Q}$. These elements, whose definition is recalled in Section 3.1, are constructed from modular units and are referred to in the sequel as *Beilinson-Flach elements*.

In the p -adic setting, the complex L -series $L(f \otimes g, s)$ is replaced by Hida's p -adic Rankin L -series attached to two ordinary families of modular forms interpolating f and g , whose definition is briefly recalled in Section 2.2. The role of the complex regulator is played by the p -adic syntomic regulator on K_1 of a surface. Besser's description of it in terms of Coleman integration [Bes3], which is summarised in §3.3, is a key ingredient in the proof of Theorem 4.2.

Our approach also relies crucially on techniques developed in [DR] for relating p -adic Abel-Jacobi images of diagonal cycles to values of the Garrett-Rankin triple product p -adic L -function attached to a triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of Hida families of cusp forms. Corollary 4.4 deals with the setting where the cuspidal family \mathbf{h} in the triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is replaced by a Hida family of Eisenstein series. The reader will also note the close parallel between Theorem 4.2 and the main result of [BD], in which the p -adic regulators of certain elements in $K_2(X_1(N))$ are related to the value at $s = 2$ of the Mazur-Swinnerton-Dyer p -adic L -functions attached to weight two cusp forms. The results of the present article are in fact intermediate between those of [DR] and [BD], the latter treating the case where *both* \mathbf{g} and \mathbf{h} are replaced by Hida families of Eisenstein series—a setting in which the resulting p -adic Rankin L -function factors as a product of two Mazur-Kitagawa L -functions attached to \mathbf{f} .

We also remark that a function field analogue of Beilinson's Theorem involving Drinfeld modular curves is described in [Sre2], based on a description of non-Archimedean regulators given in [Sre1]. See also the related work of Ambrus Pál in the setting of the K_2 of Mumford curves [Pa].

Let us conclude this introduction by briefly discussing some eventual arithmetical applications of the main result of this paper.

I. The Euler system of Beilinson-Flach elements. The image of Beilinson-Flach elements under the p -adic étale regulator map gives rise to classes in the global cohomology group $H^1(\mathbb{Q}, V_f \otimes V_g(2))$, where V_f and V_g are the p -adic Galois representations attached to f and g , respectively. The work in preparation [BDR] explores the theme of the p -adic variation of the Beilinson-Flach classes attached to Hida families of cusp forms \mathbf{f} and \mathbf{g} . In particular, when \mathbf{g} specialises in weight one to a classical cusp form attached to an odd irreducible Artin representation ρ and \mathbf{f} specialises in weight two to the cusp form associated with an elliptic curve E over \mathbb{Q} , we expect the associated cohomology class to yield new cases of the Birch and Swinnerton-Dyer conjecture for the complex L -series $L(E, \rho, s)$, proving in particular that ρ does not occur in the representation $E(\overline{\mathbb{Q}}) \otimes \mathbb{C}$ when $L(E, \rho, 1) \neq 0$.

The idea of using Beilinson elements in Euler system arguments occurs much earlier in the work of Flach [Fl], who used them to construct classes in $H^1(\mathbb{Q}, \text{Sym}^2(E)(2))$ which are crystalline at p but ramified at a single prime $\ell \neq p$. Applying Kolyagin's method to these classes leads to the finiteness of the Shafarevich-Tate group of $\text{Sym}^2(E)(2)$ and an explicit annihilator of this group related to the special value $L(\text{Sym}^2(E), 2)$, which is critical in the sense of Deligne, unlike the special values $L(f \otimes g, 2)$ when f and g are distinct normalised newforms.

II. Hida's L -function for the symmetric square of a modular form. Theorem 4.2, specialised to the case $f = g$, is exploited by S. Dasgupta [Das] to study the Hida L -function $L(f \otimes f, s)$ and express it as the product of the Coates-Schmidt p -adic L -function attached to $\text{Sym}^2(f)$ and a Kubota-Leopoldt p -adic L -function. This factorisation, which can be viewed as another manifestation of the Artin formalism for p -adic L -series, is analogous to a formula of Gross [Gross] expressing the restriction to the cyclotomic line of the Katz two-variable p -adic L -function attached to an imaginary quadratic field as a product of two Kubota-Leopoldt L -functions. The Beilinson-Flach elements play the same role in Dasgupta's proof as elliptic units in the work of Gross.

III. p -adic L -functions and Euler systems over \mathbb{Z}_p^2 -extensions. The paper in preparation [LLZ] of A. Lei, D. Loeffler and S.L. Zerbes builds on the methods of this paper, in the setting where g varies over a collection of theta series attached to Hecke characters of an imaginary quadratic field K , to construct an Euler system for V_f over the various layers of the *two variable* \mathbb{Z}_p -extension K_∞ of K , thus supplying the global input for their extension

[LZ] of Perrin-Riou's machinery in which the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} is replaced by K_∞ .

2. Rankin L -series

Let f and g be normalised newforms of weights k, ℓ , levels N_f, N_g , and nebentypus characters χ_f, χ_g respectively. The p -adic representations V_f and V_g are part of a compatible system of representations which we continue to denote by the same symbol. Let

$$L(V_f \otimes V_g, s) = \prod_p \det((1 - \sigma_p p^{-s})|(V_f \otimes V_g)^{I_p})^{-1}$$

be the motivic L -function attached to the tensor product $V_f \otimes V_g$, where I_p denotes the inertia subgroup of a decomposition group at p , and σ_p a corresponding geometric Frobenius element.

The goal of this first chapter is to briefly recall the basic analytic properties of this L -series, describe Hida's construction of a p -adic avatar, and—in the special case where f and g are both of weight two—present parallel formulae for their special values at the near-central point $s = 2$, which is *not* critical in the sense of Deligne.

2.1. Complex L -series. Set $N := \text{lcm}(N_f, N_g)$ and replace χ_f and χ_g by their counterparts of modulus N sending any prime $r|N$ to 0. It is also convenient to replace f , as well as g , by a normalised eigenform of level N which

- (1) has the same eigenvalues for the Hecke operators T_r with $r \nmid N$;
- (2) is also an eigenvector for the Hecke operators U_r attached to the primes r dividing N .

This substitution having been made, let

$$(2.1) \quad f(z) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z}, \quad g(z) = \sum_{n=1}^{\infty} a_n(g) e^{2\pi i n z}$$

be the Fourier expansions of f and g , let K_f and $K_g \subset \bar{\mathbb{Q}}$ denote the subfields generated by the coefficients $a_n(f)$ and $a_n(g)$ respectively, and let K_{fg} denote the compositum of the two fields. The Hecke polynomials attached to f can be factored as

$$x^2 - a_p(f)x + \chi_f(p)p^{k-1} = (x - \alpha_p(f))(x - \beta_p(f)),$$

where $(\alpha_p(f), \beta_p(f)) = (a_p(f), 0)$ when $p|N$. Similar notation is adopted for g . The Rankin L -function attached to the pair (f, g) is defined by the formula

$$L(f \otimes g, s) := \prod_p L_{(p)}(f \otimes g, s), \quad \text{where}$$

$$L_{(p)}(f \otimes g, s) := (1 - \alpha_p(f)\alpha_p(g)p^{-s})^{-1}(1 - \alpha_p(f)\beta_p(g)p^{-s})^{-1} \\ \times (1 - \beta_p(f)\alpha_p(g)p^{-s})^{-1}(1 - \beta_p(f)\beta_p(g)p^{-s})^{-1}.$$

The Euler factors at p defining $L(V_f \otimes V_g, s)$ and $L(f \otimes g, s)$ agree for all $p \nmid N$, and hence the special values of $L(V_f \otimes V_g, s)$ and $L(f \otimes g, s)$ at integer points differ by elementary quantities in K_{fg}^\times . It will be more convenient, for the sequel, to focus attention on $L(f \otimes g, s)$. Assume without loss of generality that the forms f and g have been ordered in such a way that $k \geq \ell$.

2.1.1. Rankin’s method. We begin by recalling the general formula for $L(f \otimes g, s)$ coming out of Rankin’s method, involving the *non-holomorphic Eisenstein series*

$$(2.2) \quad \tilde{E}_{k-\ell, \chi}(z, s) = \sum'_{(m,n) \in N\mathbb{Z} \times \mathbb{Z}} \frac{\chi^{-1}(n)}{(mz+n)^{k-\ell}} \cdot \frac{y^s}{|mz+n|^{2s}}$$

of weight $k - \ell$, level N and character

$$\chi := \chi_f^{-1} \chi_g^{-1},$$

where the superscript $'$ in (2.2) indicates that the sum is taken over the non-zero lattice vectors $(m, n) \in N\mathbb{Z} \times \mathbb{Z}$. For fixed complex s with $\text{Re}(s) \gg 0$, the product $\tilde{E}_{k-\ell, \chi}(z, s) \times g(z)$ is a real-analytic \mathbb{C} -valued function on the Poincaré upper half-plane \mathcal{H} which transforms like a modular form of weight k , level N and character χ_f^{-1} and is of rapid decay at infinity. The space of such functions, denoted $S_k^{\text{ra}}(N, \chi_f^{-1})$, is equipped with the Petersson scalar product

$$\langle \cdot, \cdot \rangle_{k,N} : S_k^{\text{ra}}(N, \chi_f^{-1}) \times S_k^{\text{ra}}(N, \chi_f^{-1}) \longrightarrow \mathbb{C}$$

given by the formula

$$(2.3) \quad \langle f_1, f_2 \rangle_{k,N} := \int_{\Gamma_0(N) \backslash \mathcal{H}} y^k \overline{f_1(z)} f_2(z) \frac{dx dy}{y^2},$$

which is hermitian linear in the first argument and \mathbb{C} -linear in the second. Let $f^* \in S_k(N, \chi_f^{-1})$ denote the modular form obtained from f by applying complex conjugation to its Fourier coefficients.

Proposition 2.1 (Shimura). *For all $s \in \mathbb{C}$ with $\Re(s) \gg 0$,*

$$(2.4) \quad L(f \otimes g, s) = \frac{1}{2} \frac{(4\pi)^s}{\Gamma(s)} \left\langle f^*(z), \tilde{E}_{k-\ell, \chi}(z, s - k + 1) \cdot g(z) \right\rangle_{k,N}.$$

This well-known formula for the Rankin L -series is taken from equation (14) of [BD].

2.1.2. Critical values. Assume here and in §2.1.3 that $\ell < k$. The functional equation for $L(f \otimes g, s)$ arising from Proposition 2.1 reveals that the integer j is critical for $L(f \otimes g, s)$ if and only if it lies in the closed interval $[\ell, k - 1]$. We now describe a further closed formula for the value at an integer j belonging to the “right half critical segment” $[\frac{\ell+k-1}{2}, k - 1]$, which will be useful in deriving the algebraicity (up to periods) of $L(f \otimes g, j)$ predicted by the Deligne conjectures and ultimately in constructing Hida’s p -adic Rankin L -function by interpolating these quantities p -adically.

Having fixed an integer $j \in [\frac{\ell+k-1}{2}, k - 1]$, let $t \geq 0$ and $m \geq 1$ be given by

$$t := k - 1 - j, \quad m := k - \ell - 2t.$$

If $m \leq 2$, let us assume also that χ is non-trivial. Then the series

$$(2.5) \quad E_{m,\chi}(z) = 2^{-1}L(\chi, 1 - m) + \sum_{n=1}^{\infty} \sigma_{m-1,\chi}(n)q^n,$$

where $\sigma_{m-1,\chi}(n) = \sum_{d|n} \chi(d)d^{m-1}$ is the q -expansion of a *holomorphic* Eisenstein series of weight m and character χ .

The Shimura-Maass derivative operator

$$\delta_m := \frac{1}{2\pi i} \left(\frac{d}{dz} + \frac{im}{2y} \right)$$

transforms modular forms of weight m into (real analytic) modular forms of weight $m + 2$ which are *nearly holomorphic* in the sense of [Sh2], and its t -fold iterate $\delta_m^t := \delta_{m+2t-2} \cdots \delta_{m+2} \delta_m$ maps the space $M_m(N, \chi)$ to the space $M_{m+2t}^{\text{nh}}(N, \chi)$ of nearly holomorphic modular forms of weight $m + 2t$. Let

$$C(k, \ell, j) := \frac{(-1)^t 2^{k-1} (2\pi)^{k+m-1} \iota_\chi(iN)^{-m} \tau(\chi^{-1})}{(m+t-1)!(j-1)!}$$

be the elementary constant (in which $\iota_\chi = 1$ when χ is primitive) appearing in equation (18) of [BD]. The following formula for $L(f \otimes g, j)$ is obtained by setting $c = j$ in equation (18) of [BD]. (See also Theorem 2 of [Sh1].)

Proposition 2.2. *The special value $L(f \otimes g, j)$ is given by the formula*

$$(2.6) \quad L(f \otimes g, j) = C(k, \ell, j) \langle f^*(z), \delta_m^t E_{m,\chi}(z) \times g(z) \rangle_{k,N}.$$

2.1.3. Algebraicity and Deligne’s conjecture. Let $S_k^{\text{nh}}(N, \chi_f^{-1}; K_{fg}) \subset S_k^{\text{nh}}(N, \chi_f^{-1})$ denote the space of nearly holomorphic cusp forms which are *defined over* K_{fg} in the sense of Shimura (cf. Section 2.4 of [DR]). The cusp form

$$(2.7) \quad \Xi(f, g, j) := \delta_m^t E_{m,\chi} \times g \in S_k^{\text{nh}}(N, \chi_f^{-1})$$

appearing in Proposition 2.2 belongs to the K_{fg} -rational structure $S_k^{\text{nh}}(N, \chi_f^{-1}; K_{fg})$. Hence, its image

$$(2.8) \quad \Xi(f, g, j)^{\text{hol}} := \Pi_N^{\text{hol}}(\Xi(f, g, j))$$

under the holomorphic projection Π_N^{hol} of [DR] belongs to $S_k(N, \chi_f^{-1}; K_{fg})$, the space of holomorphic cusp forms with Fourier coefficients in K_{fg} . In particular, the ratio

$$(2.9) \quad \begin{aligned} L^{\text{alg}}(f \otimes g, j) &:= C(f, g, j)^{-1} \frac{L(f \otimes g, j)}{\langle f^*, f^* \rangle_{k, N}} \\ &= \frac{\langle f^*, \Xi(f, g, j) \rangle_{k, N}}{\langle f^*, f^* \rangle_{k, N}} = \frac{\langle f^*, \Xi(f, g, j)^{\text{hol}} \rangle_{k, N}}{\langle f^*, f^* \rangle_{k, N}} \end{aligned}$$

belongs to K_{fg} . This algebraicity result is consistent with Deligne’s conjecture which predicts that the period $C(f, g, j)\langle f^*, f^* \rangle_{k, N}$ is the “transcendental part” of the special value $L(f \otimes g, j)$. The associated “algebraic part” appearing in (2.9) will later be interpolated p -adically to obtain Hida’s p -adic Rankin L -function attached to f and g .

In order to do this, it will be convenient to give a more geometric description of the quantity $L^{\text{alg}}(f \otimes g, j)$ appearing in (2.9), in terms of the Poincaré duality on the de Rham cohomology of the modular curve $X_1(N)$ with values in appropriate sheaves with connection, as described in [DR, §2.3]. To lighten the notation, denote by Y and by X the open modular curve $Y_1(N)$ and the complete modular curve $X_1(N)$ respectively, classifying (generalised) elliptic curves equipped with an embedding of the finite flat group scheme μ_N of N -th roots of unity.

Let K be any field containing K_{fg} . Denote by $\mathcal{E} \rightarrow Y$ the universal elliptic curve over Y , and by ω the sheaf of relative differentials on \mathcal{E} over Y , extended to X as in [BDP, §1.1]. Recall the Kodaira-Spencer isomorphism $\omega^2 \simeq \Omega_X^1(\log \text{cusps})$, where $\Omega_X^1(\log \text{cusps})$ is the sheaf of regular differentials on Y with log poles at the cusps.

A modular form ϕ on $\Gamma_1(N)$ of weight $k = r + 2$ with Fourier coefficients in K corresponds to a global section of the sheaf $\omega^{r+2} = \omega^r \otimes \Omega_X^1(\log \text{cusps})$ over the base-change X_K of X to K . The sheaf ω^r can be viewed as a subsheaf of $\mathcal{L}_r := \text{Sym}^r \mathcal{L}$, where

$$\mathcal{L} := R^1 \pi_* (\mathcal{E} \rightarrow Y)$$

is the relative de Rham cohomology sheaf on Y , extended to X as in [BDP], equipped with the filtration

$$(2.10) \quad 0 \rightarrow \omega \rightarrow \mathcal{L} \rightarrow \omega^{-1} \rightarrow 0$$

arising from the Hodge filtration on the fibers. The sheaf \mathcal{L}_r is a coherent sheaf over X of rank $r+1$, endowed with the Gauss-Manin connection

$$\nabla : \mathcal{L}_r \longrightarrow \mathcal{L}_r \otimes \Omega_X^1(\log \text{cusps}).$$

Let $H_{\text{dR}}^1(X_K, \mathcal{L}_r, \nabla)$ be the de Rham cohomology of \mathcal{L}_r . It is equipped with the perfect Poincaré pairing

$$(2.11) \quad \langle \cdot, \cdot \rangle_{k,X} : H_{\text{dR}}^1(X_K, \mathcal{L}_r, \nabla) \times H_{\text{dR}}^1(X_K, \mathcal{L}_r, \nabla) \longrightarrow K,$$

which is compatible with the exact sequence

$$(2.12) \quad 0 \longrightarrow H^0(X_K, \omega^r \otimes \Omega_X^1) \longrightarrow H_{\text{dR}}^1(X_K, \mathcal{L}_r, \nabla) \longrightarrow H^1(X_K, \omega^{-r}) \longrightarrow 0,$$

in the sense that $H^0(X_K, \omega^r \otimes \Omega_X^1)$ is an isotropic subspace (cf. Sections 2 and 3 of [Col] for a more detailed account). In particular, Poincaré duality induces a perfect pairing

$$(2.13) \quad \langle \cdot, \cdot \rangle_{k,X} : H^1(X_K, \omega^{-r}) \times H^0(X_K, \omega^r \otimes \Omega_X^1) \longrightarrow K,$$

which is denoted by the same symbol by a slight abuse of notation.

The antiholomorphic differential η_f^{ah} defined by

$$(2.14) \quad \eta_f^{\text{ah}} := \frac{\bar{f}^*(z)d\bar{z}}{\langle f^*, f^* \rangle_{k,N}}$$

gives rise to a class in $H_{\text{dR}}^1(X_{\mathbb{C}}, \mathcal{L}_r, \nabla)$, whose image η_f in $H^1(X_{\mathbb{C}}, \omega^{-r})$ belongs to $H^1(X_K, \omega^{-r})$ (cf. Corollary 2.13 of [DR]). The following expression for the algebraic part $L^{\text{alg}}(f \otimes g, j)$ in terms of the class η_f follows directly from (2.9) in light of the discussion above:

Proposition 2.3. *The algebraic part $L^{\text{alg}}(f \otimes g, j)$ is equal to*

$$(2.15) \quad L^{\text{alg}}(f \otimes g, j) = \langle \eta_f, \Xi(f, g, j)^{\text{hol}} \rangle_{k,X}.$$

2.1.4. The value at the near central point. Consider now the case where $k = \ell = 2$ and $\chi_f \neq \chi_g^{-1}$, so that the character $\chi = \chi_f^{-1}\chi_g^{-1}$ is not the trivial one. The functional equation for $L(f \otimes g, s)$ relates $L(f \otimes g, s)$ to $L(f^* \otimes g^*, 3 - s)$, and this L -series has no critical points in the sense of Deligne. Proposition 2.5 below describes its value at the near-central point $s = 2$ in terms of logarithms of *modular units*.

Enlarge K so that it contains the field which is cut out by all the Dirichlet characters of modulus N , and let F be the field generated over K by the values of these characters. Let $\text{Eis}_{\ell}(\Gamma_1(N); F)$ denote the subspace of $M_{\ell}(\Gamma_1(N); F)$ spanned by the weight ℓ Eisenstein series with coefficients in F . The logarithmic derivative gives a surjective homomorphism

$$(2.16) \quad \mathcal{O}(Y_K)^{\times} \otimes F \xrightarrow{\text{dlog}} \text{Eis}_2(\Gamma_1(N); F),$$

whose kernel is the subspace $K^\times \otimes F$ spanned by the non-zero constant functions.

Definition 2.4. Let u_χ be the modular unit satisfying

$$(2.17) \quad \text{dlog}(u_\chi) = E_{2,\chi}$$

whose value at ∞ is 1 in the sense of [Br, §5].

Proposition 2.5. *Given weight two eigenforms f and g as above,*

$$(2.18) \quad L(f \otimes g, 2) = 16\pi^3 N^{-2} \tau(\chi^{-1}) \langle f^*(z), \log |u_\chi(z)| \cdot g(z) \rangle_{2,N}.$$

Proof. By Proposition 2.1,

$$(2.19) \quad L(f \otimes g, 2) = \frac{1}{2} (4\pi)^2 \left\langle f^*(z), \tilde{E}_{0,\chi}(z, 1) \cdot g(z) \right\rangle_{2,N}.$$

A direct calculation (cf. equation (26) of [BD]) shows that

$$(2.20) \quad \frac{1}{2\pi i} \frac{d}{dz} \tilde{E}_{0,\chi}(z, 1) = -\frac{1}{4\pi} \tilde{E}_{2,\chi}(z) = 2\pi N^{-2} \tau(\chi^{-1}) E_{2,\chi}(z).$$

Having normalized u_χ as in Definition 2.4, one obtains the equality

$$(2.21) \quad \tilde{E}_{0,\chi}(z, 1) = 2\pi N^{-2} \tau(\chi^{-1}) \log |u_\chi(z)|,$$

which is compatible with (2.17). Combining (2.19) with (2.21) completes the proof of the proposition. \square

2.2. p -adic L -series. Let $p \geq 3$ be a prime, and fix an embedding of K into \mathbb{C}_p . This section recalls the definition of the Rankin p -adic L -function associated by Hida [Hi] to the convolution of two Hida families of cusp forms. For the sake of brevity, we proceed here, just as in [BD], by specialising the approach and notations of [DR], which constructs the p -adic L -function associated to a triple product of three Hida families $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of cusp forms. The setting considered here consists, essentially, of letting \mathbf{h} be a Hida family of Eisenstein series.

2.2.1. Ordinary projections. Let f, g be eigenforms of level N , weights $k > \ell$ and nebentypus χ_f, χ_g as in (2.1). Let also $j \in [\frac{\ell+k-1}{2}, k-1]$ be an integer and set $t = k-1-j \geq 0$ and $m = k-\ell-2t \geq 1$ as in §2.1.2. The following ordinariness assumption is important for the constructions described in this section.

Assumption 2.6. *The cuspidal eigenforms f and g are ordinary at p , and $p \nmid N$.*

Under this assumption, the f -isotypic part of the exact sequence (2.12) with $K = \mathbb{C}_p$ admits a canonical *unit root* splitting, arising from the action of Frobenius on de Rham cohomology. Let η_f^{ur} be the lift of η_f to the unit root subspace $H_{\text{dR}}^1(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)^{f,\text{ur}}$. The right-hand side of (2.15) is then equal to

$$(2.22) \quad \langle \eta_f, \Xi(f, g, j)^{\text{hol}} \rangle_{k,X} = \langle \eta_f^{\text{ur}}, \Xi(f, g, j)^{\text{hol}} \rangle_{k,X}.$$

Now let e_{ord} be Hida’s ordinary projector to $H_{\text{dR}}^1(Y_K, \mathcal{L}_r, \nabla)^{\text{ord}}$. By Proposition 2.11 of [DR], the right-hand side of (2.22) can be rewritten after viewing $\Xi(f, g, j)^{\text{hol}}$ as an overconvergent p -adic modular form and setting $\Xi(f, g, j)^{\text{ord}} := e_{\text{ord}}\Xi(f, g, j)^{\text{hol}}$ as

$$(2.23) \quad \langle \eta_f^{\text{ur}}, \Xi(f, g, j)^{\text{hol}} \rangle_{k, X} = \langle \eta_f^{\text{ur}}, \Xi(f, g, j)^{\text{ord}} \rangle_{k, X}.$$

By Proposition 2.8 of [DR],

$$(2.24) \quad \Xi(f, g, j)^{\text{ord}} = e_{\text{ord}}(d^t E_{m, \chi} \cdot g),$$

where $d = q \frac{d}{dq}$ is Serre’s derivative operator on p -adic modular forms.

Given a p -adic modular form $\phi = \sum c_n q^n$, let $\phi^{[p]} := \sum_{p \nmid n} c_n q^n$ denote its “ p -depletion” and set

$$(2.25) \quad \Xi(f, g, j)^{\text{ord}, p} := e_{\text{ord}}(d^t E_{m, \chi}^{[p]} \cdot g).$$

Proposition 2.7. *Let e_{f^*} be the projector to the f^* -isotypic subspace of $H_{\text{dR}}^1(Y_K, \mathcal{L}_{k-2}, \nabla)$. Then*

$$e_{f^*}\Xi(f, g, j)^{\text{ord}, p} = \frac{\mathcal{E}(f, g, j)}{\mathcal{E}(f)} \cdot e_{f^*}\Xi(f, g, j)^{\text{ord}},$$

where

$$\begin{aligned} \mathcal{E}(f, g, j) &= (1 - \beta_p(f)\alpha_p(g)p^{t-k+1})(1 - \beta_p(f)\beta_p(g)p^{t-k+1}) \\ &\quad \times (1 - \beta_p(f)\alpha_p(g)\chi(p)p^{t-k+m})(1 - \beta_p(f)\beta_p(g)\chi(p)p^{t-k+m}), \\ \mathcal{E}(f) &= 1 - \beta_p(f)^2\chi_f^{-1}(p)p^{-k}. \end{aligned}$$

Proof. This follows from Corollary 4.17 of [DR], in light of Proposition 2.8 of [DR]. □

2.2.2. Hida’s p -adic L-series. Let \mathbf{f} and \mathbf{g} be Hida families of ordinary p -adic modular forms of tame level N , indexed by weight variables k and ℓ in suitable neighborhoods $U_{\mathbf{f}}$ and $U_{\mathbf{g}}$ of $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$, contained in a single residue class modulo $p-1$. (These families may be obtained, as shall be the case considered in §4.2, by deforming two given ordinary classical eigenforms f and g of possibly equal weights.) Assume likewise that the parameter $j = k - 1 - t$ belongs to a single residue class modulo $p-1$ so that the same holds true for the weight $m = k - \ell - 2t$ of the Eisenstein series $E_{m, \chi}$.

For $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$ and $\ell \in U_{\mathbf{g}} \cap \mathbb{Z}^{\geq 2}$, let

$$f_k \in S_k(N, \chi_f), \quad g_\ell \in S_\ell(N, \chi_g)$$

be the classical cusp forms whose p -stabilisations are the weight k and ℓ specialisations of \mathbf{f} and \mathbf{g} respectively. (We denote by χ_f , resp. χ_g , the common character of the modular forms f_k , resp. g_k .)

The collection of p -adic modular forms $\Xi(f_k, g_\ell, j)^{\text{ord}, p}$ (defined as in equation (2.25)) indexed by

$$(2.26) \quad \{(k, \ell, j), \quad k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}, \quad \ell \in U_{\mathbf{g}} \cap \mathbb{Z}^{\geq 2}, \quad \frac{\ell + k - 1}{2} \leq j \leq k - 1\}$$

has Fourier coefficients which extend analytically to $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$ as functions in k, ℓ and j . Hence, they can be viewed as a (three-variable) Λ -adic family of modular forms of level N in the sense of [DR, §2.7].

Set

$$\mathcal{E}^*(f_k) := 1 - \beta_p(f_k)^2 \chi_f^{-1}(p) p^{1-k}.$$

Proposition 4.10 of [DR] shows that the expression

$$(2.27) \quad L_p(\mathbf{f}, \mathbf{g})(k, \ell, j) := \frac{1}{\mathcal{E}^*(f_k)} \langle \eta_{f_k}^{\text{ur}}, \Xi(f_k, g_\ell, j)^{\text{ord}, p} \rangle_{k, X},$$

defined on the triples (k, ℓ, j) in the set in (2.26), extends to an analytic function $L_p(\mathbf{f}, \mathbf{g})$ on $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$, which we refer to as the Hida p -adic Rankin L -function attached to \mathbf{f} and \mathbf{g} . This appellation is justified by noting that for all triples (k, ℓ, j) in the range of “classical interpolation”, i.e. belonging to (2.26), the function $L_p(\mathbf{f}, \mathbf{g})(k, \ell, j)$ satisfies the interpolation property

$$L_p(\mathbf{f}, \mathbf{g})(k, \ell, j) = \frac{\mathcal{E}(f_k, g_\ell, j)}{\mathcal{E}^*(f_k)\mathcal{E}(f_k)} L^{\text{alg}}(f_k \otimes g_\ell, j).$$

This follows from a direct calculation combining (2.27), Proposition 2.7, (2.23), (2.22) and (2.15).

Note that the point $(2, 2, 2)$ lies outside the region of classical interpolation for this function. (In fact, there are no critical values for the pair of weights $(2, 2)$.) Corollary 4.4 of Section 4.2 relates the value of $L_p(\mathbf{f}, \mathbf{g})$ at $(2, 2, 2)$ to the p -adic regulator attached in Section 3.3 to the triple of modular forms $(f = f_2, g = g_2, E_{2, X})$.

Generalising our setting somewhat, we do not assume now that the modular form $g \in S_2(N, \chi_g)$ is ordinary, so that g may not necessarily be viewed as the weight 2 specialisation of a Hida family. In this case, the above construction still allows us to define a two-variable p -adic L -function $L_p(\mathbf{f}, g)(k, j)$ on $U_{\mathbf{f}} \times \mathbb{Z}_p$ by the equation

$$(2.28) \quad L_p(\mathbf{f}, g)(k, j) := \frac{1}{\mathcal{E}^*(f_k)} \langle \eta_{f_k}^{\text{ur}}, \Xi(f_k, g, j)^{\text{ord}, p} \rangle_{k, X},$$

for $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$ and $(k + 1)/2 \leq j \leq k - 1$. Theorem 4.2 relates $L_p(\mathbf{f}, g)(2, 2)$ to the p -adic regulator attached to $(f, g, E_{2, X})$.

3. Beilinson-Flach elements

3.1. Definition and basic properties. Let S be a quasi-projective variety over a field K , and $K_j(S)$ denote Quillen’s algebraic K -groups of S . The motivic cohomology groups $H_{\mathcal{M}}^i(S, \mathbb{Q}(n)) = K_{2n-i}^{(n)}(S)$ of S were defined by Beilinson [Bei, §2] as the n -th graded piece of the Adams filtration on $K_{2n-i}(S) \otimes \mathbb{Q}$. In parallel with Beilinson’s motivic cohomology groups, Bloch [Bl] introduced the higher Chow groups $\text{CH}^i(S, n)$ of S .

In this note we shall focus on the smooth projective surface $S := X \times X$, where X is the modular curve over the field K of §2.1.4. For $i = 3$ and $n = 2$, $H_{\mathcal{M}}^3(S, \mathbb{Q}(2)) = K_1^{(2)}(S)$ is identified with $\text{CH}^2(S, 1) \otimes \mathbb{Q}$. The higher Chow group $\text{CH}^2(S, 1)$ may be explicitly described (cf. also [Sc]) as the first homology of the *Gersten complex*

$$(3.1) \quad K_2(K(S)) \xrightarrow{\partial} \bigoplus_{Z \subset S} K(Z)^\times \xrightarrow{\text{div}} \bigoplus_{P \in S} \mathbb{Z},$$

where

- (1) $K_2(K(S))$ denotes the second Milnor K -group of the rational function field $K(S)$, and ∂ is the map whose “component at Z ” is the tame symbol attached to the valuation ord_Z ;
- (2) the group

$$\Theta := \bigoplus_{Z \subset S} K(Z)^\times$$

is the set of finite formal linear combinations $\sum_i (Z_i, u_i)$, where the Z_i are irreducible curves in S and u_i is a rational function on Z_i ;

- (3) the map div is the divisor map, and the direct sum defining its target is taken over all closed points $P \in S_K$.

Given a closed point $P \in X$ and a rational function u on X , an element of Θ of the form $(\{P\} \times X, u)$ (resp. of the form $(X \times \{P\}, u)$) is said to be *vertical* (resp. *horizontal*). A linear combination of vertical and horizontal terms is said to be *negligible*. Similar definitions apply to the tensor product $\Theta \otimes F$ over any field F .

Let $\Delta \subset S$ be a copy of the curve X diagonally embedded in S . Let F denote the field introduced in §2.1.4, let $u \in \mathcal{O}(Y_K)^\times \otimes F$ be a modular unit with coefficients in F , and consider the element $(\Delta, u) \in \Theta \otimes F$.

Lemma 3.1. *There exists a negligible element $\theta_u \in \Theta \otimes F$ satisfying*

$$\text{div}(\theta_u) = \text{div}(\Delta, u).$$

Proof. Let $D_u = \text{div}(\Delta, u) \in \coprod_{P \in S} F$ be the image of the element $(\Delta, u) \in \Theta$ under the divisor map. Since D_u is an F -linear combination of elements

of the form $(c_1, c_1) - (c_2, c_2)$ where c_1 and c_2 are cusps of the modular curve X_K , it is enough to construct a negligible element $\theta \in \Theta \otimes \mathbb{Q}$ satisfying

$$(3.2) \quad \operatorname{div}(\theta) = (c_1, c_1) - (c_2, c_2).$$

By the Manin-Drinfeld theorem, there is an element $\alpha \in \mathcal{O}(Y_K)^\times \otimes \mathbb{Q}$ whose divisor is $c_1 - c_2$, and the negligible element given by

$$\theta = (\{c_1\} \times X, \alpha) + (X \times \{c_2\}, \alpha)$$

satisfies (3.2). The lemma follows. □

Thanks to Lemma 3.1, we can associate to any element of the form $(\Delta, u) \in \Theta \otimes F$ the element

$$(3.3) \quad \Delta_u := \text{class of } (\Delta, u) - \theta_u \quad \text{in } H^3_{\mathcal{M}}(S, F(2)).$$

These elements were introduced by Beilinson in [Bei, Ch. 2, §6]. A variant ([Fl, Prop. 2.1]) of the above construction was later exploited by Flach in [Fl] to prove the finiteness of the Tate-Shafarevic group of the symmetric square of an elliptic curve, using the method of Kolyvagin. We call Δ_u the *Beilinson-Flach element* attached to the modular unit $u \in \mathcal{O}(Y_K)^\times \otimes F$. Strictly speaking, Δ_u is not a well-defined element in $H^3_{\mathcal{M}}(S, F(2))$, as it is only well-defined modulo the F -vector space generated by the classes of negligible elements. However, this inherent ambiguity will not lead to problems because the image of Δ_u under the relevant piece of the regulator maps will turn out to depend only on u and not on the choice of θ_u made in defining Δ_u . See Proposition 3.3 below for more details.

3.2. Complex regulators. Fix an embedding of K into the field of complex numbers. Following the definitions in [Bei, §2], [DS, §2], the *complex regulator* on $H^3_{\mathcal{M}}(S_{\mathbb{C}}, \mathbb{Q}(2))$ may be regarded as a map

$$(3.4) \quad \mathbf{reg}_{\mathbb{C}} : H^3_{\mathcal{M}}(S_{\mathbb{C}}, \mathbb{Q}(2)) \longrightarrow (\operatorname{Fil}^1 H^2_{\text{dR}}(S/\mathbb{C}))^\vee,$$

where here the superscript \vee denotes the complex linear dual. It sends the class of $\theta = \sum_i (Z_i, u_i)$ to the element $\mathbf{reg}_{\mathbb{C}}(\theta)$ defined by

$$\mathbf{reg}_{\mathbb{C}}(\theta)(\omega) = \frac{1}{2\pi i} \sum_i \int_{Z_i - Z_i^{\text{sing}}} \omega \log |u_i|.$$

Recall the modular unit u_χ associated to the Dirichlet character χ , and the class $\eta_f^{\text{ah}} \in H^1_{\text{dR}}(X/\mathbb{C})$ attached to the cusp form f . Moreover, write as customary $\omega_g \in \operatorname{Fil}^1 H^1_{\text{dR}}(X/\mathbb{C})$ for the class associated to the regular differential $2\pi i g(z) dz$.

The tensor product $\omega_g \otimes \eta_f^{\text{ah}}$ of these classes gives rise, via the Künneth decomposition of $H^2_{\text{dR}}(S/\mathbb{C})$, to an element of $\operatorname{Fil}^1 H^2_{\text{dR}}(S/\mathbb{C})$.

Proposition 3.2. *With notation as above, we have*

$$\begin{aligned} \mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ah}}) &= (-2i)[\Gamma_0(N) : \Gamma_1(N)(\pm 1)] \langle f^*, f^* \rangle_{2,N}^{-1} \\ &\quad \times \langle f^*(z), \log |u_\chi(z)| \cdot g(z) \rangle_{2,N}. \end{aligned}$$

Proof. Since the differential $\omega_g \otimes \eta_f^{\text{ah}}$ vanishes identically on the horizontal and vertical curves on S , the negligible element θ_{u_χ} arising in the definition of Δ_{u_χ} does not contribute to the value of the regulator at that class. Hence

$$\begin{aligned} \mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ah}}) &= \int_{X(\mathbb{C})} \frac{\bar{f}^*(z)}{\langle f^*, f^* \rangle_{2,N}} g(z) \log |u_\chi(z)| dz d\bar{z} \\ &= (-2i)[\Gamma_0(N) : \Gamma_1(N)(\pm 1)] \langle f^*, f^* \rangle_{2,N}^{-1} \langle f^*(z), \log |u_\chi(z)| \cdot g(z) \rangle_{2,N}, \end{aligned}$$

where the last equality follows from the explicit formula for the Petersson scalar product on $S_k^{\text{ra}}(N, \chi_f^{-1})$. \square

3.3. p -adic regulators. Let K_p be a finite extension of \mathbb{Q}_p containing K and fix an embedding of K_p in \mathbb{C}_p . Write \mathcal{O}_p , resp. k_p , for the ring of integers, resp. the residue field of K_p . Let \mathcal{X} denote the (Deligne-Rapoport) smooth model of X over \mathcal{O}_p , and $\tilde{\mathcal{X}}/k_p$ its special fiber. Define $\mathcal{S} = \mathcal{X} \times \mathcal{X}$, which is a smooth projective model of S_{K_p} over \mathcal{O}_p .

In analogy with the complex regulator (3.4), there is a p -adic syntomic regulator map

$$(3.5) \quad \mathbf{reg}_p : H_{\mathcal{M}}^3(S_{K_p}, \mathbb{Q}(2)) \longrightarrow (\text{Fil}^1 H_{\text{dR}}^2(S/K_p))^\vee := \text{Hom}(\text{Fil}^1 H_{\text{dR}}^2(S/K_p), K_p)$$

arising from the syntomic Chern character in K -theory (cf. [Gros], [Ni], [Bes3]).

After possibly enlarging the field K_p , let $\{P_1, \dots, P_t\} \subset \mathcal{X}(\mathcal{O}_p)$ be a set of points consisting of the cusps and of a choice of a lift of every supersingular point in $\tilde{\mathcal{X}}(\bar{\mathbb{F}}_p)$. Set

$$\mathcal{X}' = \mathcal{X} \setminus \{P_1, \dots, P_t\}, \quad X' = \mathcal{X}' \times_{\text{spec } \mathcal{O}_p} \text{spec } K_p.$$

Let $\text{red} : \mathcal{X}(\mathcal{O}_p) \longrightarrow \tilde{\mathcal{X}}(k_p)$ denote the reduction map and let $\mathcal{A} \subset X(K_p)$ be the affinoid subspace of the rigid analytic variety underlying X defined by

$$\mathcal{A} := X(K_p) - \text{red}^{-1}(\{\tilde{P}_1, \dots, \tilde{P}_t\}), \quad \tilde{P}_j := \text{red}(P_j).$$

Fix a system $\{\mathcal{W}_\epsilon\}_{\epsilon > 0}$ of wide open neighborhoods of \mathcal{A} as in [DR, §2.1] and denote by Φ the canonical lift of Frobenius on X as in [DR, §2.2]. As explained there, restriction from X' to \mathcal{W}_ϵ gives rise to an isomorphism

$$(3.6) \quad H_{\text{dR}}^1(X') \xrightarrow{\text{comp}_\epsilon} H_{\text{rig}}^1(\mathcal{W}_\epsilon)$$

between the de Rham cohomology of the open curve X' and the rigid cohomology $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$ of \mathcal{W}_ϵ . The inclusion $X' \subset X$ yields by restriction a

monomorphism

$$H_{\text{dR}}^1(X) \hookrightarrow H_{\text{dR}}^1(X'),$$

and the image of $H_{\text{dR}}^1(X)$ under comp_ϵ consists of those classes in $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$ whose annular residues about all the points $\{P_i\}$ vanish. The lift Φ of Frobenius induces a linear endomorphism of $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$ which preserves the subspace $H_{\text{dR}}^1(X)$.

Label now two copies of X as X_1 and X_2 , denote by Φ_1 and Φ_2 the corresponding canonical lifts of Frobenius on the system of wide open neighborhoods \mathcal{W}_ϵ , and write $\Phi_{12} := (\Phi_1, \Phi_2)$ for the associated lift of Frobenius on the product $X_1 \times X_2$.

Choose a polynomial $P(x) \in \mathbb{C}_p[x]$ such that

- (i) $P(\Phi_{12})$ annihilates the class of $\omega_g \otimes \frac{du_\chi}{u_\chi}$ in $H_{\text{rig}}^2(\mathcal{W}_\epsilon^2)$;
- (ii) $P(\Phi)$ is an invertible endomorphism on $H_{\text{dR}}^1(X')$.

Such a polynomial exists, since the eigenvalues of Φ_{12} acting on the space spanned by the Frobenius translates of $\omega_g \otimes \frac{du_\chi}{u_\chi}$ have complex absolute value $p^{3/2}$, while Φ acts on $H_{\text{dR}}^1(X')$ with eigenvalues of complex absolute value $p^{1/2}$ and p .

Thanks to (i), there exists a rigid analytic one-form

$$(3.7) \quad \varrho_P \in \Omega^1(\mathcal{W}_\epsilon^2) \text{ such that } d(\varrho_P) = P(\Phi_{12}) \left(\omega_g \otimes \frac{du_\chi}{u_\chi} \right).$$

This form, which depends on the choice of P , is determined only up to *closed* forms in $\Omega^1(\mathcal{W}_\epsilon^2)$ by (3.7).

In order to adapt our calculations to Besser’s in [Bes2] and [Bes3], it will be convenient to fix a particular choice of polynomial P and form ϱ_P . (In the next section we shall exploit the fact that the computations performed there hold independently of the choice of P and will work with a different polynomial so that we can take advantage of the results obtained in [DR].)

Let $P_g(t) \in \mathbb{C}_p[t]$ be a polynomial such that $P_g(\Phi)$ annihilates the class of ω_g in $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$. Specifically, we may set $P_g(t) := t^2 - a_p(g)t + \chi_g(p)p$, and let $F_g \in \mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon)$ be a Coleman integral of ω_g , that is to say, a rigid analytic function such that

$$(3.8) \quad pdF_g = p\omega_{g^{[p]}} = P_g(\Phi)\omega_g$$

(cf. for example equation (127) of [DR]). Likewise, let $P_{E_\chi}(t) \in \mathbb{C}_p[t]$ be a polynomial such that $P_{E_\chi}(\Phi)$ annihilates the class of the Eisenstein series $E_\chi = \frac{du_\chi}{u_\chi}$ in $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$. Here we make the specific choice $P_{E_\chi}(t) := t^h - p^h$, where h is the order of the root of unity $\chi(p)$ (in other words, Φ^h/p^h fixes the class of E_χ). Although a more optimal choice for $P_{E_\chi}(t)$ would have been the linear polynomial $t - \chi(p)p$, we made here a choice corresponding to the

one made in the definition of the modified syntomic regulator $\text{reg}(u_\chi)$ of the function u_χ (cf. [Bes2, Prop. 10.3]). The rigid analytic function

$$(3.9) \quad F_{E_\chi} := p^{-h} P_{E_\chi}(\Phi) \log(u_\chi) \in \mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon)$$

is a Coleman integral of E_χ , satisfying

$$p^h dF_{E_\chi} = P_{E_\chi}(\Phi) E_\chi.$$

Given two choices as above of polynomials $P_g(t) = \prod_i (t - \alpha_i)$ and $P_{E_\chi}(t) = \prod_j (t - \beta_j)$, it is clear that the polynomial

$$(3.10) \quad P(t) := P_g(t) \star P_{E_\chi}(t) := \prod_{i,j} (t - \alpha_i \beta_j)$$

satisfies (i) above. Moreover, as explained in [Bes1, Lemma 4.2, (4)], there exist polynomials $a(t_1, t_2), b(t_1, t_2)$ such that $P(t_1 \cdot t_2) = p^{-1} a(t_1, t_2) P_g(t_1) + p^{-h} b(t_1, t_2) P_{E_\chi}(t_2) \in \mathbb{C}_p[t_1, t_2]$, and one checks that

$$(3.11) \quad \varrho_P = a(\Phi_1, \Phi_2) \left(F_g \otimes \frac{du_\chi}{u_\chi} \right) + b(\Phi_1, \Phi_2) (\omega_g \otimes F_{E_\chi}) \in \Omega^1(\mathcal{W}_\epsilon^2)$$

then satisfies (3.7).

There is a certain degree of ambiguity in (3.11): neither the Coleman primitives F_g, F_{E_χ} nor the polynomials $a(t_1, t_2), b(t_1, t_2)$ are unique. But all solutions of the differential equation (3.7) are of the form (3.11); moreover, given one such ϱ_P , all of them can be written as $\varrho_P + \varrho_0$ with ϱ_0 a closed 1-form on \mathcal{W}_ϵ^2 .

We can single out a canonical choice of ϱ_P (up to exact 1-forms on \mathcal{W}_ϵ^2) by setting $F_g(\infty) = F_{E_\chi}(\infty) = 0$ in (3.11); more precisely, in doing this, two different choices of pairs $(a(t_1, t_2), b(t_1, t_2)), (a'(t_1, t_2), b'(t_1, t_2))$ allowed by [Bes1, Lemma 4.2, (4)] give rise to forms $\varrho_{P,a,b}, \varrho_{P,a',b'}$ such that $\varrho_0 = \varrho_{P,a,b} - \varrho_{P,a',b'}$ is exact on \mathcal{W}_ϵ^2 and therefore the class of ϱ_0 in $H_{\text{rig}}^1(\mathcal{W}_\epsilon^2)$ vanishes.

Imposing $F_g(\infty) = 0$ amounts to normalizing the q -expansion of F_g to be

$$(3.12) \quad F_g(q) = \sum_{p \nmid n} \frac{a_n(g)}{n} q^n,$$

and the condition $F_{E_\chi}(\infty) = 0$ is equivalent to normalizing the modular unit u_χ as was done in Definition 2.4. This way F_{E_χ} also equals the modified syntomic regulator $\text{reg}(u_\chi)$ of u_χ defined in [Bes2, Prop. 10.3].

Let $\Delta \subset \mathcal{W}_\epsilon^2$ denote the diagonal and define

$$(3.13) \quad \xi'_P := [\varrho_P|_\Delta] \in H_{\text{rig}}^1(\mathcal{W}_\epsilon) \simeq H_{\text{dR}}^1(X').$$

The above discussion shows that the class ξ'_P in $H^1_{\text{rig}}(\mathcal{W}_\epsilon) = \frac{\Omega^1(\mathcal{W}_\epsilon)}{d\mathcal{O}(\mathcal{W}_\epsilon)}$ is well-defined. Moreover, in view of condition (ii), we can now set

$$(3.14) \quad \xi' := P(\Phi)^{-1} \cdot \xi'_P \in H^1_{\text{dR}}(X'),$$

which is directly seen to be independent of the choice of P .

Finally, let $\text{spl}_X : H^1_{\text{dR}}(X') \rightarrow H^1_{\text{dR}}(X)$ denote the Frobenius equivariant splitting of the short exact sequence

$$(3.15) \quad 0 \rightarrow H^1_{\text{dR}}(X) \rightarrow H^1_{\text{dR}}(X') \rightarrow K_p(-1)^{t-1} \rightarrow 0$$

and set $\xi := \text{spl}_X(\xi') \in H^1_{\text{dR}}(X)$.

Proposition 3.3. *With notation as above, we have*

$$\text{reg}_p(\Delta_{u_x})(\omega_g \otimes \eta_f^{\text{ur}}) = \langle \eta_f^{\text{ur}}, \xi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the pairing on $H^1_{\text{dR}}(X)$ induced by Poincaré duality.

Proof. Thanks to the work of Besser [Bes3], the p -adic syntomic regulator (3.5) admits the following description in terms of Coleman integration. Let $\theta = \sum_i (Z_i, u_i)$ be an element in $K_1^{(2)}(S)$ and write $\iota_i : Z_i \hookrightarrow S$ for the embedding of Z_i in S given by inclusion. Assume for simplicity that the curves Z_i are all non-singular and that θ is *integral*, by which we mean that for each i :

- the curve Z_i admits a smooth integral model \mathcal{Z}_i over \mathcal{O}_p and
- the divisor of u_i , when regarded as a function on \mathcal{Z}_i , does not contain the special fiber.

Note that these conditions are satisfied in our setting.

Under this assumption, θ lies in the image of the natural restriction map $K_1(\mathcal{S}) \rightarrow K_1(S)$.

Let $\Omega^{\text{II}}(X_{K_p})$ denote the space of differential forms of the second kind on X_{K_p} , that is to say, the space of meromorphic 1-forms whose residue at any point of the curve is zero. There is an exact sequence

$$0 \rightarrow K_p(X)^\times \xrightarrow{d} \Omega^{\text{II}}(X_{K_p}) \rightarrow H^1_{\text{dR}}(X/K_p) \rightarrow 0$$

and for any $\eta \in \Omega^{\text{II}}(X_{K_p})$ we write $[\eta]$ for its class in $H^1_{\text{dR}}(X/K_p)$.

Instead of invoking the description of the p -adic syntomic regulator in terms of Besser-de Jeu’s global triple index as stated in the main theorem of [Bes3], it will be more convenient for us to exploit [Bes3, Prop. 6.3], which provides a formula for (3.5) in the language of Besser’s finite polynomial cohomology [Bes1]. In order to state this formula, let H^*_{ms} and H^*_{fp} denote, respectively, Besser’s modified version of syntomic cohomology and finite polynomial cohomology; cf. e.g. [Bes3, §2] for a quick review of both and their interactions.

Let $\omega \in \Omega^1(X_{K_p})$ be a regular form on X and $\eta \in \Omega^{\text{II}}(X_{K_p})$ be a differential of the second kind, regular on some affine curve $X^0 \subset X$. Write

$$\omega_1 = \pi_1^*(\omega) \in \Omega^1(S), \quad \eta_2 = \pi_2^*(\eta) \in \Omega^{\text{II}}(S)$$

for the pull-back of ω and η under the projection of S into the first and second component, respectively.

Then the class $\omega_1 \wedge [\eta_2]$ is an element of $\text{Fil}^1 H_{\text{dR}}^2(S)$ and, according to [Bes3, Theorem 1.1, Prop. 6.3]:

$$(3.16) \quad \mathbf{reg}_p(\theta)(\omega_1 \otimes [\eta_2]) = \sum_i \langle \iota_i^* \tilde{\eta}_2, \iota_i^* \tilde{\omega}_1 \cup \mathbf{reg}(u_i) \rangle_{\mathcal{Z}_i^0, \text{fp}},$$

where

- $\mathcal{Z}_i^0 = Z_i \cap (X \times X^0)$, \mathcal{Z}_i^0 is the model for Z_i^0 deduced from \mathcal{Z}_i ,
- $\mathbf{reg}(u_i) \in H_{\text{ms}}^1(\mathcal{Z}_i^0, 1) \subseteq H_{\text{fp}}^1(\mathcal{Z}_i^0, 1, 2)$ is the regulator of the function u_i as defined in [Bes2, Prop. 10.3],
- $\iota_i^* \tilde{\omega}_1 \in H_{\text{fp}}^1(\mathcal{Z}_i, 1, 1)$ is a Coleman primitive of $\iota_i^* \omega \in \Omega^1(Z_i)$,
- $\iota_i^* \tilde{\eta}_2 \in H_{\text{fp},c}^1(\mathcal{Z}_i^0, 0, 1)$ is the single lift of $\iota_i^*([\eta_2])$ under the isomorphism

$$(3.17) \quad \mathfrak{p} : H_{\text{fp},c}^1(\mathcal{Z}_i^0, 0, 1) \xrightarrow{\sim} H_{\text{dR}}^1(Z_i)$$

of [Bes3, Lemma 6.2], and

$$(3.18) \quad \langle \cdot, \cdot \rangle_{\mathcal{Z}_i^0, \text{fp}} : H_{\text{fp},c}^1(\mathcal{Z}_i^0, 0, 1) \times H_{\text{fp}}^2(\mathcal{Z}_i^0, 2, 3) \longrightarrow H_{\text{fp},c}^3(\mathcal{Z}_i^0, 2, 4) \simeq H_{\text{dR},c}^2(\mathcal{Z}_i^0)^{\text{tr}} \simeq K_p$$

is the pairing induced by Poincaré duality in finite polynomial cohomology. Here $H_{\text{fp},c}^*$ stands for finite polynomial cohomology with compact support, as introduced in [Bes3, §4]. The cup-product (3.18) is constructed in [Bes3], where it is also shown that it satisfies the projection formula.

At the time [Bes3] was written, the results were subject to the compatibility of pushforward maps in syntomic and motivic cohomology, as specified in [Bes3, Conjecture 4.2]. At present this compatibility has been checked by Déglise and Mazzari [DM], and thus (3.16) holds unconditionally.

Let us now apply (3.16) to the Beilinson-Flach element Δ_{u_x} that was introduced in (3.3). Recall that the curves in $X \times X$ on which Δ_{u_x} is supported are the images of X under the diagonal embedding $\iota_{12}(x) = (x, x)$ and the various horizontal and vertical embeddings $\iota_{1,c}(x) = (x, c)$ and $\iota_{2,c}(x) = (c, x)$, where c is a cusp on the modular curve X .

We firstly claim that the terms on the right-hand side of (3.16) corresponding to $\iota_{1,c}$ and $\iota_{2,c}$ vanish and the one corresponding to ι_{12} is independent of the choices of lifts to finite polynomial cohomology.

To see that, put $\omega = \omega_g$ and $\eta = \eta_f^{\text{ur}}$ and recall $X' = \mathcal{X}' \times K_p$ is the curve obtained from X by removing a finite set of points including all the cusps. Note first that $\iota_{1,c}^*([\eta_2]) = 0 \in H_{\text{dR}}^1(X)$, because the composition $\pi_2 \circ \iota_{1,c}$ is the constant function c on X . Hence, since the map p in (3.17) is an isomorphism, the class of the lift $\iota_{1,c}^*(\tilde{\eta}_2)$ is also trivial and

$$\langle \iota_{1,c}^* \tilde{\eta}_2, \iota_{1,c}^* \tilde{\omega}_1 \cup \text{reg}(u) \rangle_{\mathcal{X}', \text{fp}} = 0,$$

for any rational function u .

We similarly have $\iota_{2,c}^*(\omega_1) = 0 \in \Omega^1(X)$ because $\pi_1 \circ \iota_{2,c} = c$. Notice however that a lift of 0 to $H_{\text{fp}}^1(\mathcal{X}, 1, 1)$ is not necessarily trivial, but represented by a pair in $\mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^1(X)$ of the form $[(\lambda, 0)]$, where λ is a constant. Then, if u is a modular unit on X , the cup-product

$$\iota_{2,c}^* \tilde{\omega}_1 \cup \text{reg}(u) \in H_{\text{fp}}^2(\mathcal{X}', 2, 3) \simeq H_{\text{dR}}^1(X') \simeq H_{\text{rig}}^1(\mathcal{W}_\epsilon)$$

may be represented by the pair $(\lambda \frac{du}{u}|_{\mathcal{W}_\epsilon}, 0)$. But then

$$(3.19) \quad \langle \iota_{2,c}^* \tilde{\eta}_2, \iota_{2,c}^* \tilde{\omega}_1 \cup \text{reg}(u) \rangle_{\mathcal{X}', \text{fp}} = \lambda \langle \eta_f^{\text{ur}}, \frac{du}{u} \rangle_{\text{dR}} = 0$$

because the cusp form f is orthogonal to the Eisenstein series $\frac{du}{u}$. This accounts for the vanishing of the horizontal and vertical terms and explains why we call them negligible.

As for the diagonal term, let us show that $\langle \iota_{12}^* \tilde{\eta}_2, \iota_{12}^* \tilde{\omega}_1 \cup \text{reg}(u_\chi) \rangle_{\mathcal{X}', \text{fp}}$ is independent of the choices of lifts to finite polynomial cohomology. Since $\pi_1 \circ \iota_{12}$ and $\pi_2 \circ \iota_{12}$ are both the identity map on X , this is just $\langle \tilde{\eta}_f^{\text{ur}}, \tilde{\omega}_g \cup \text{reg}(u_\chi) \rangle_{\mathcal{X}', \text{fp}}$. Again there is a single choice for $\tilde{\eta}_f^{\text{ur}}$, but the Coleman integral F_g of ω_g is only well-defined up to a constant. The difference between any two choices is then equal to

$$\langle \tilde{\eta}_f^{\text{ur}}, [(\lambda, 0)] \cup \text{reg}(u_\chi) \rangle_{\mathcal{X}', \text{fp}} = \lambda \left\langle \eta_f^{\text{ur}}, \frac{du_\chi}{u_\chi} \right\rangle_{\text{dR}}$$

for some $\lambda \in K_p$, and the same orthogonality argument between cusp and Eisenstein forms again shows that this is 0. The claim follows.

Summing up, we obtain from (3.16) that

$$(3.20) \quad \text{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}) = \langle \tilde{\eta}_f^{\text{ur}}, \tilde{\omega}_g \cup \text{reg}(u_\chi) \rangle_{\mathcal{X}', \text{fp}}.$$

Recall that $\tilde{\omega}_g$ may be represented by the pair (F_g, ω_g) where $F_g \in \mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon)$ is a Coleman integral of ω_g , which in light of the above claim we are entitled to normalize as was done in (3.12). Besides, by [Bes2, Prop. 10.3] the class $\text{reg}(u_\chi)$ is represented by the pair $(F_{E_\chi}, \frac{du_\chi}{u_\chi}) \in \mathcal{O}_{\text{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^1(X')$ where F_{E_χ} is the Coleman integral of $\frac{du_\chi}{u_\chi}$ introduced in (3.9) and normalized as we explained right after (3.12).

By definition, $\tilde{\omega}_g \cup \text{reg}(u_\chi)$ is the restriction to the diagonal of $\pi_1^* \tilde{\omega}_g \wedge \pi_2^* \text{reg}(u_\chi)$. Note that the polynomial P defined in equation (3.10) satisfies the properties (i) and (ii) above. The class $\pi_1^* \tilde{\omega}_g \wedge \pi_2^* \text{reg}(u_\chi)$ in $H_{\text{fp}}^2(\mathcal{X}'^2, 2, 3)$ may then be represented by the pair

$$(3.21) \quad (\varrho_P, \pi_1^* \omega_g \wedge \pi_2^* \frac{du_\chi}{u_\chi}) \in \Omega_{\text{rig}}^1(\mathcal{W}_\epsilon^2) \oplus \Omega^2(X'^2),$$

where ϱ_P is the form introduced in (3.11), which satisfies

$$(3.22) \quad d\varrho_P = P(\Phi_{12})(\pi_1^* \omega_g \wedge \pi_2^* \frac{du_\chi}{u_\chi}).$$

Let us again remark that this differential equation does not determine ϱ_P uniquely, but that the above normalizations of F_g and F_{E_χ} completely determine it up to exact 1-forms on \mathcal{W}_ϵ^2 . Obviously, when we restrict (3.21) to the diagonal, this ambiguity does not affect the class we obtain in $H_{\text{fp}}^2(\mathcal{X}', 2, 3)$, because exact 1-forms on \mathcal{W}_ϵ vanish in $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$.

In conclusion, the class $\tilde{\omega}_g \cup \text{reg}(u_\chi)$ in $H_{\text{fp}}^2(\mathcal{X}', 2, 3)$ may be represented by the pair

$$(\iota_{12}^*(\varrho_P), 0) \in \Omega_{\text{rig}}^1(\mathcal{W}_\epsilon) \oplus \Omega^2(X'),$$

where ϱ_P is as above and $\iota_{12}^*(\varrho_P)$ is the form denoted by ξ'_P in (3.13).

As in [Bes1, (14)] there is a commutative diagram

$$(3.23) \quad \begin{array}{ccc} H_{\text{fp},c}^1(\mathcal{X}', 0, 1) \times H_{\text{dR}}^1(X') & \xrightarrow{\text{Id} \times \text{i}} & H_{\text{fp},c}^1(\mathcal{X}', 0, 1) \times H_{\text{fp}}^2(\mathcal{X}', 2, 3) \\ \downarrow \text{p} \times \text{Id} & & \downarrow \langle \cdot, \cdot \rangle_{\text{fp}} \\ H_{\text{dR},c}^1(X')^{w=1} \times H_{\text{dR}}^1(X') & \xrightarrow{\langle \cdot, \cdot \rangle_{\text{dR}}} & H_{\text{dR},c}^2(X') \simeq H_{\text{fp},c}^3(\mathcal{X}', 2, 4), \end{array}$$

where $H_{\text{dR},c}^1(X')^{w=1}$ stands for the pure submodule of weight 1 of $H_{\text{dR},c}^1(X')$. In fact both maps

$$H_{\text{dR}}^1(X') \xrightarrow{\text{i}} H_{\text{fp}}^1(\mathcal{X}', 2, 3) \quad \text{and} \quad H_{\text{fp},c}^1(\mathcal{X}', 0, 1) \xrightarrow{\text{p}} H_{\text{dR},c}^1(X')^{w=1}$$

are isomorphisms, as follows from [Bes3, (2.7) and the first assertion of Lemma 2.8].

By definition of i , the preimage of $\tilde{\omega}_g \cup \text{reg}(u_\chi) = [(\xi'_P, 0)]$ under i is the class in $H_{\text{rig}}^1(\mathcal{W}_\epsilon)$ of the 1-form $P(\Phi)^{-1}(\xi'_Q) = \xi'$. To conclude, we now deduce from the commutativity of (3.23) that

$$\langle \eta_f^{\text{ur}}, \xi' \rangle_{\text{dR}} = \langle \tilde{\eta}_f^{\text{ur}}, \text{i}(\xi') \rangle_{\text{fp}} = \langle \tilde{\eta}_f^{\text{ur}}, \tilde{\omega}_g \cup \text{reg}(u_\chi) \rangle_{\text{fp}}.$$

Since the class η_f^{ur} is orthogonal to the complement of $H_{\text{dR}}^1(X)$ in $H_{\text{dR}}^1(X')$ under the Frobenius equivariant splitting of (3.15), we have $\langle \eta_f^{\text{ur}}, \xi' \rangle_{\text{dR}} = \langle \eta_f^{\text{ur}}, \xi \rangle_{\text{dR}}$, and the proposition follows. \square

4. The Beilinson formula

Let $f \in S_2(N, \chi_f)$, $g \in S_2(N, \chi_g)$ be eigenforms of weight 2 as in Section 2.1.4. Recall that f and g are not assumed to be newforms. Moreover, we insist on the condition $\chi_f \neq \chi_g^{-1}$, which implies that $\chi = \chi_f^{-1}\chi_g^{-1}$ is non-trivial.

4.1. The complex setting. In [Bei, Ch. 2, §6], Beilinson relates the image of Δ_{u_χ} under the complex regulator map to the value at $s = 2$ of the Rankin L -series attached to $f \otimes g$. The following explicit version of Beilinson’s theorem is a slight generalisation of the results of [BaSr].

Proposition 4.1. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$\frac{L(f \otimes g, 2)}{\langle f^*, f^* \rangle_{2,N}} = (8i)\pi^3[\Gamma_0(N) : \Gamma_1(N)(\pm 1)]^{-1}N^{-2}\tau(\chi^{-1})\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ah}}).$$

Proof. This follows by combining the explicit formula for $L(f \otimes g, 2)$ obtained in Proposition 2.5 with the explicit expression for $\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})$ given in Proposition 3.2. □

4.2. The p -adic setting. Let $p \geq 3$ be a prime which does not divide N . Assume that f is ordinary at p (with respect to a fixed embedding of the field K_f in \mathbb{C}_p). Let \mathbf{f} be the Hida family whose specialisation in weight 2 is the p -stabilisations of f , and let $L_p(\mathbf{f}, g)(k, j)$ be the p -adic L -function defined in Section 2.2.2.

Let $\mathcal{E}(f)$, $\mathcal{E}^*(f)$ and $\mathcal{E}(f, g, 2)$ be the p -adic multipliers defined in Sections 2.2.1 and 2.2.2. Recall that

$$\begin{aligned} \mathcal{E}(f, g, 2) &= (1 - \beta_p(f)\alpha_p(g)p^{-2})(1 - \beta_p(f)\beta_p(g)p^{-2}) \\ &\quad \times (1 - \beta_p(f)\alpha_p(g)\chi(p)p^{-1})(1 - \beta_p(f)\beta_p(g)\chi(p)p^{-1}). \end{aligned}$$

The following p -adic Beilinson formula is the main result of this paper.

Theorem 4.2. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$L_p(\mathbf{f}, g)(2, 2) = \frac{\mathcal{E}(f, g, 2)}{\mathcal{E}(f) \cdot \mathcal{E}^*(f)} \times \mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}).$$

Proof. By the description of the p -adic L -function given in equation (2.28),

$$L_p(\mathbf{f}, g)(k, j) = \frac{1}{\mathcal{E}^*(f_k)} \langle \eta_{f_k}^{\text{ur}}, \Xi(f_k, g, j)^{\text{ord}, p} \rangle_{k, X}$$

for all triples (k, ℓ, j) belonging to the set (2.26). Since the terms in the above expression vary analytically, taking the limit to $k = \ell = j = 2$ yields, in light

of equation (2.25),

$$(4.1) \quad L_p(\mathbf{f}, g)(2, 2) = \frac{1}{\mathcal{E}^*(f)} \left\langle \eta_f^{\text{ur}}, e_{\text{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g) \right\rangle_{2,X}.$$

On the other hand, by Proposition 3.3,

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}) = \langle \eta_f^{\text{ur}}, \xi \rangle.$$

Since

$$\Phi(\eta_f^{\text{ur}}) = \alpha_p(f)\eta_f^{\text{ur}}, \quad \langle \Phi(\eta_f^{\text{ur}}), \Phi(\xi) \rangle = p\langle \eta_f^{\text{ur}}, \xi \rangle, \quad \alpha_p(f)\beta_p(f) = \chi_f(p)p,$$

we deduce by multi-linearity that

$$\langle \eta_f^{\text{ur}}, \xi \rangle = P(\chi_f^{-1}(p)\beta_p(f))^{-1} \langle \eta_f^{\text{ur}}, \xi'_P \rangle.$$

Since f is an ordinary eigenform, the quantity $\langle \eta_f^{\text{ur}}, \xi'_P \rangle$ depends only on the f^* -isotypical ordinary projection of ξ'_P , that is, $\langle \eta_f^{\text{ur}}, \xi'_P \rangle = \langle \eta_f^{\text{ur}}, e_{f^*}e_{\text{ord}}\xi'_P \rangle$.

Choose the polynomial $P(x)$ satisfying conditions (i) and (ii) to be

$$P(x) := (x - \alpha_p(g)) \cdot (x - \alpha_p(g)\chi(p)p) \cdot (x - \beta_p(g)) \cdot (x - \beta_p(g)\chi(p)p).$$

This choice of P has the advantage of allowing us to directly invoke the calculations already performed in [DR, Prop. 5.4]. They give

$$e_{f^*}e_{\text{ord}}\xi'_P = \chi_f(p)^{-2}p^4\mathcal{E}(f) \cdot e_{f^*}e_{\text{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g).$$

A direct calculation shows that

$$\mathcal{E}(f, g, 2) = p^{-4}\chi_f(p)^{-2}P(\chi_f^{-1}(p)\beta_p(f)).$$

By combining the above remarks, we find the following expression for the p -adic regulator:

$$(4.2) \quad \mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}) = \frac{\mathcal{E}(f)}{\mathcal{E}(f, g, 2)} \times \left\langle \eta_f^{\text{ur}}, e_{f^*}e_{\text{ord}}(d^{-1}E_{2,\chi}^{[p]} \cdot g) \right\rangle_{2,X}.$$

The theorem follows by comparing equations (4.1) and (4.2). □

Remark 4.3. Note that the modular form g that arises in Theorem 4.2 is fixed throughout the argument and is thus not required to be ordinary at p . Assume now that both f and g are ordinary at p (with respect to a fixed embedding of the field K_{fg} in \mathbb{C}_p). Let \mathbf{f} and \mathbf{g} be the Hida families whose specialisations in weight 2 are the p -stabilisations of f and g , respectively, and let $L_p(\mathbf{f}, \mathbf{g})(k, \ell, j)$ be the p -adic L -function defined in Section 2.2.2. The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.4. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$L_p(\mathbf{f}, \mathbf{g})(2, 2, 2) = \frac{\mathcal{E}(f, g, 2)}{\mathcal{E}(f) \cdot \mathcal{E}^*(f)} \times \mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\text{ur}}).$$

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UNIVERSITÄT DUISBURG-ESSEN, FAKULTÄT FÜR MATHEMATIK, MATHEMATIKARRÉE,
THEA-LEYMANN-STRASSE 9, 45326 ESSEN, GERMANY
E-mail address: massimo.bertolini@uni-due.de

MCGILL UNIVERSITY, BURNSIDE HALL, ROOM 1111, MONTRÉAL, QUEBEC H3A 0G4,
CANADA
E-mail address: darmon@math.mcgill.ca

UNIVERSITAT POLITÈCNICA DE CATALUNYA, MA II, DESPATX 413, C. JORDI GIRONA 1-3,
08034 BARCELONA, SPAIN
E-mail address: victor.rotger@upc.edu