CHOW-HEEGNER POINTS ON CM ELLIPTIC CURVES AND VALUES OF p-ADIC L-FUNCTIONS

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Introduction

The theory of Heegner points supplies one of the most fruitful approaches to the Birch and Swinnerton-Dyer conjecture, leading to the best results for elliptic curves of analytic rank one. In spite of attempts to broaden the scope of the Heegner point construction ([BDG], [Da2], [Tr],...), all provable, systematic constructions of algebraic points on elliptic curves still rely on parametrisations of elliptic curves by modular or Shimura curves. The primary goal of this article is to explore new constructions of rational points on elliptic curves and abelian varieties in which, loosely speaking, Heegner divisors are replaced

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by higher-dimensional algebraic cycles on certain modular varieties. In general, the algebraicity of the resulting points depends on the validity of ostensibly difficult cases of the Hodge or Tate conjectures. One of the main results of this article (Theorem 8 of the introduction) illustrates how these algebraicity statements can sometimes be obtained unconditionally by exploiting the connection between the relevant "generalised Heegner cycles" and values of certain p-adic L-series.

In the course of our study, we are also led to a new proof of the main theorem of [Ru] and to a generalisation thereof (see Theorem 2 below), relating values of the Katz two-variable p-adic L-function to formal group logarithms of rational points on CM elliptic curves.

Rubin's Theorem. The main theorem in [Ru] (Cf. Corollary 10.3 of loc. cit.) concerns an elliptic curve A over $\mathbb{Q}$ with complex multiplication by the ring of integers of a quadratic imaginary field K. A classical result of Deuring identifies the Hasse-Weil L-series $L(A,s)$ of A with the L-series $L(\nu_A,s)$ attached to a Hecke character ν_A of K of infinity type $(1,0)$. When p is a prime which splits in K and does not divide the conductor of A, the Hecke L-function $L(\nu_A,s)$ has a p-adic analogue known as the Katz two-variable L-function attached to K. It is a p-adic analytic function, denoted $\nu \mapsto \mathscr{L}_p(\nu)$, on the space of Hecke characters equipped with its natural p-adic analytic structure. Section 2.1 recalls the definition of this L-function: the values $\mathscr{L}_p(\nu)$ at Hecke characters of infinity type $(1+j_1,-j_2)$ with $j_1,j_2 \geq 0$ are defined by interpolation of the classical L-values $L(\nu^{-1},0)$. Letting $\nu^* := \nu \circ c$, where c denotes complex conjugation on the ideals of K, it is readily seen by comparing Euler factors that $L(\nu,s) = L(\nu^*,s)$. A similar equality need not hold in the p-adic setting, because the involution $\nu \mapsto \nu^*$ corresponds to the map $(j_1,j_2) \mapsto (j_2,j_1)$ on weight space and therefore does not preserve the lower right quadrant of weights of Hecke characters that lie in the range of classical interpolation. Since ν_A lies in the domain of classical interpolation, the p-adic L-value $\mathscr{L}_p(\nu_A)$ is a simple multiple of $L(\nu_A^{-1},0) = L(A,1)$. Suppose that it vanishes. (This implies, by the Birch and Swinnerton-Dyer conjecture, that $A(\mathbb{Q})$ is infinite.) The value $\mathscr{L}_p(\nu_A^*)$ is a second, a priori more mysterious p-adic avatar of the leading term of $L(A,s)$ at $s=1$. Rubin's theorem gives a formula for this quantity:

Theorem 1 (Rubin)**.** *Let ν_A be a Hecke character of type $(1,0)$ attached to an elliptic curve $A/\mathbb{Q}$ with complex multiplication. Then there exists a global point $P \in A(\mathbb{Q})$ such that*

$$\mathscr{L}_p(\nu_A^*) = \Omega_p(A)^{-1} \log_{\omega_A}(P)^2 \pmod{\mathbb{Q}^\times}, \tag{1}$$

where

- *$\Omega_p(A)$ is the p-adic period attached to A as in Section 1.3;*
- *$\omega_A \in \Omega^1(A/\mathbb{Q})$ is a regular differential on A over $\mathbb{Q}$, and $\log_{\omega_A} : A(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p$ denotes the p-adic formal group logarithm with respect to ω_A.*

The point P is of infinite order if and only if $L(A,s)$ has a simple zero at $s=1$.

(For a more precise statement without the $\mathbb{Q}^\times$ ambiguity, see [Ru].) Formula (1) is peculiar to the p-adic world and suggests that p-adic L-functions encode arithmetic information that is not readily apparent in their complex counterparts. Inspired by Rubin's work, Perrin-Riou has formulated a p-adic Beilinson conjecture in [PR2] of which Theorem 1 should be a special case.

The proof of Theorem 1 given in [Ru] breaks up naturally into two parts:

(1) Rubin exploits the Euler system of elliptic units to construct a global cohomology class κ_A belonging to a pro-p Selmer group $\mathrm{Sel}_p(A/\mathbb{Q})$ attached to A. The close connection between elliptic units and the Katz L-function is then parlayed into the explicit evaluation of two natural p-adic invariants attached to κ_A: the p-adic formal group logarithm $\log_{A,p}(\kappa_A)$ and the cyclotomic p-adic height $\langle \kappa_A, \kappa_A \rangle$:

$$\log_{A,p}(\kappa_A) = (1-\beta_p^{-1})^{-1} \mathscr{L}_p(\nu_A^*) \Omega_p(A), \tag{2}$$

$$\langle \kappa_A, \kappa_A \rangle = (1-\alpha_p^{-1})^{-2} \mathscr{L}_p'(\nu_A) \mathscr{L}_p(\nu_A^*), \tag{3}$$

where

- α_p and β_p denote the roots of the Hasse polynomial $x^2 - a_p(A)x + p$, ordered in such a way that $\mathrm{ord}_p(\alpha_p) = 0$ and $\mathrm{ord}_p(\beta_p) = 1$;
- the quantity $\mathscr{L}_p'(\nu_A)$ denotes the derivative of $\mathscr{L}_p$ at ν_A in the direction of the cyclotomic character.

If $\mathscr{L}_p'(\nu_A)$ is non-zero, then an argument based on Perrin-Riou's p-adic analogue of the Gross-Zagier formula and the work of Kolyvagin implies that $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$ is a one-dimensional $\mathbb{Q}_p$-vector space with κ_A as a generator. (Cf. Thm. 8.1 and Cor. 8.3 of [Ru].) Equations (2) and (3) then make it possible to evaluate the ratio

$$\frac{\log^2_{A,p}(\kappa)}{\langle \kappa, \kappa \rangle} = \frac{(1-\beta_p^{-1})^{-2}\mathscr{L}_p(\nu_A^*)\Omega_p(A)^2}{(1-\alpha_p^{-1})^{-2}\mathscr{L}_p'(\nu_A)}, \tag{4}$$

a quantity which does not depend on the choice of generator κ of the $\mathbb{Q}_p$-vector space $\mathrm{Sel}_p(A/\mathbb{Q})\otimes\mathbb{Q}$.

(2) Independently of the construction of κ_A, the theory of Heegner points can be used to construct a canonical point $P \in A(\mathbb{Q})$, which is of infinite order when $\mathscr{L}_p'(\nu_A) \neq 0$. Its image $\kappa_P \in \mathrm{Sel}_p(A/\mathbb{Q})$ under the connecting homomorphism of Kummer theory supplies us with a second generator for $\mathrm{Sel}_p(A/\mathbb{Q}) \otimes \mathbb{Q}$. Furthermore, the p-adic analogue of the Gross-Zagier formula proved by Perrin-Riou in [PR1] shows that

$$\langle \kappa_P, \kappa_P \rangle = \mathscr{L}_p'(\nu_A)\Omega_p(A)^{-1} \pmod{\mathbb{Q}^\times}. \tag{5}$$

Rubin obtains Theorem 1 by setting $\kappa = \kappa_P$ in (4) and using (5) to eliminate the quantities involving $\langle \kappa_P, \kappa_P \rangle$ and $\mathscr{L}_p'(\nu_A)$.

The reader will note the key role that is played in Rubin's proof by both the Euler systems of elliptic units and of Heegner points. The new approach to Theorem 1 described in Chapter 2 relies solely on Heegner points, and requires neither elliptic units nor Perrin-Riou's p-adic height calculations. Instead, the key ingredient in this approach is the p-adic variant of the Gross-Zagier formula arising from the results of [BDP] which is stated in Theorem 2.9. This formula expresses p-adic logarithms of Heegner points in terms of the special values of a p-adic Rankin L-function attached to a cusp form f and an imaginary quadratic field K, and may be of some independent interest insofar as it exhibits a strong analogy with Rubin's formula but applies to arbitrary—not necessarily CM—elliptic curves over $\mathbb{Q}$. When f is the theta series attached to a Hecke character of K, Theorem 1 follows from the factorisation of the associated p-adic Rankin L-function into a product of two Katz L-functions, a factorisation which is a simple manifestation of the Artin formalism for these p-adic L-series.

It is expected that the statement of Theorem 1 should generalise to the setting where ν_A is replaced by an algebraic Hecke character ν of infinity type $(1,0)$ of a quadratic imaginary field K (of arbitrary class number) satisfying

$$\nu|_{\mathbb{A}_\mathbb{Q}} = \varepsilon_K \cdot \mathbf{N}, \tag{6}$$

where ε_K denotes the quadratic Dirichlet character associated to $K/\mathbb{Q}$ and $\mathbf{N} : \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{R}^\times$ is the adèlic norm character. Chapter 2 treats this more general setting, which is not yet covered in the literature, although the original methods of [Ru] would probably extend as well with only technical complications. Assumption (6) implies that the classical functional equation relates $L(\nu, s)$ to $L(\nu, 2-s)$. Assume further that the sign w_ν in this functional equation satisfies

$$w_\nu = -1, \tag{7}$$

so that $L(\nu, s)$ vanishes to odd order at $s = 1$. For less serious reasons, it will also be convenient to make two further technical assumptions. Firstly, we assume that

$$\text{The discriminant } -D \text{ of } K \text{ is odd.} \tag{8}$$

Secondly, we note that assumption (6) implies that $\sqrt{-D}$ necessarily divides the conductor of ν, and we further restrict the setting by imposing the assumption that

$$\text{The conductor of } \nu \text{ is exactly divisible by } \sqrt{-D}. \tag{9}$$

The statement of Theorem 2 below requires some further notions which we now introduce. Let E_ν be the subfield of $\mathbb{C}$ generated by the values of the Hecke character ν, and let T_ν be its ring of integers. A general construction which is recalled in Sections 1.2 and 2.6 attaches to ν an abelian variety B_ν over K of dimension $[E_\nu : K]$, equipped with inclusions

$$T_\nu \subset \mathrm{End}_K(B_\nu), \qquad E_\nu \subset \mathrm{End}_K(B_\nu) \otimes \mathbb{Q}.$$

Given $\lambda \in T_\nu$, denote by $[\lambda]$ the corresponding endomorphism of B_ν, and set

$$\Omega^1(B_\nu/E_\nu)^{T_\nu} := \left\{\omega \in \Omega^1(B_\nu/E_\nu) \quad \text{such that} \quad [\lambda]^*\omega = \lambda\omega, \quad \forall \lambda \in T_\nu\right\}, \tag{10}$$

$$(B_\nu(K) \otimes E_\nu)^{T_\nu} := \{P \in B_\nu(K) \otimes_{\mathbb{Z}} E_\nu \quad \text{such that} \quad [\lambda]P = \lambda P, \quad \forall \lambda \in T_\nu\}. \tag{11}$$

The vector space $\Omega^1(B_\nu/E\nu)^{T_\nu}$ is one-dimensional over E_ν. The results of Gross-Zagier and Kolyvagin, which continue to hold in the setting of abelian variety quotients of modular curves, also imply that $(B_\nu(K) \otimes E_\nu)^{T_\nu}$ is one-dimensional over E_ν when $L(\nu, s)$ has a simple zero at $s = 1$.

After fixing a p-adic embedding $K \subset \mathbb{Q}_p$, the formal group logarithm on B_ν gives rise to a bilinear pairing

$$\begin{aligned} \langle\ ,\ \rangle : \Omega^1(B_\nu/K) \times B_\nu(K) &\longrightarrow \mathbb{Q}_p \\ (\omega, P) &\mapsto \log_\omega P, \end{aligned}$$

satisfying $\langle [\lambda]^*\omega, P\rangle = \langle \omega, [\lambda]P\rangle$ for all $\lambda \in T_\nu$. This pairing can be extended by E_ν-linearity to an $E_\nu \otimes \mathbb{Q}_p$-valued pairing between $\Omega^1(B_\nu/E_\nu)$ and $B_\nu(K) \otimes E_\nu$. When ω and P belong to these E_ν-vector spaces, we will continue to write $\log_\omega(P)$ for $\langle \omega, P\rangle$.

Theorem 2. *Let ν be an algebraic Hecke character of infinity type $(1, 0)$ satisfying (6), (7), (8) and (9) above. Then there exists $P_\nu \in B_\nu(K)$ such that*

$$\mathscr{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1} \log_{\omega_\nu}(P_\nu)^2 \pmod{E_\nu^\times},$$

where $\Omega_p(\nu^*) \in \mathbb{C}_p$ is the p-adic period attached to ν in Definition 1.14, and ω_ν is a non-zero element of $\Omega^1(B_\nu/E_\nu)^{T_\nu}$. The point P_ν is non-zero if and only if $L'(\nu, 1) \neq 0$.

Remark 3. Assumptions (8) and (9) do not reflect a serious limitation of our method of proof, but rather the fact that the main theorem of [BDP] on which it relies is only proved for imaginary quadratic fields of odd discriminant and with some restrictions on the conductor of ν. These assumptions could certainly be relaxed with some more work.

Remark 4. Equivalently, Theorem 2 could be stated by requiring that P_ν and ω_ν belong to $(B_\nu(K)\otimes E_\nu)^{T_\nu}$ and $\Omega^1(B_\nu/K)$ respectively.

Remark 5. The methods used in the proof of Theorem 2 also give information about the special values $\mathscr{L}_p(\nu^*)$ for Hecke characters ν of type $(1 + j, -j)$ satisfying (6) with $j \geq 0$. A discussion of this point will be taken up in future work.

Remark 6. The fact that our proof of Theorem 2 avoids the use of elliptic units raises the prospect of extending it to Hecke characters of more general CM fields.

Chow-Heegner points.

We now turn to the main goal of of this article: the study of generalisations of the Heegner point construction in which the role of Heegner divisors is played by null-homologous algebraic cycles of higher dimension.

We begin with a brief sketch of the classical picture which we aim to generalize. It is known thanks to [Wi], [TW], and [BCDT] that all elliptic curves over the rationals are modular. For an elliptic curve A of conductor N, this means that

$$L(A, s) = L(f, s), \tag{12}$$

where $f(z) = \sum a_n e^{2\pi i n z}$ is a cusp form of weight 2 on the Hecke congruence group $\Gamma_0(N)$. The modularity of A is established by showing that the p-adic Galois representation

$$V_p(A) := \left(\lim_{\leftarrow} A[p^n]\right) \otimes \mathbb{Q}_p = H^1_{\text{et}}(\bar{A}, \mathbb{Q}_p)(1)$$

is a constituent of the first p-adic étale cohomology of the modular curve $X_0(N)$. On the other hand, the Eichler-Shimura construction attaches to f an elliptic curve quotient A_f of the Jacobian $J_0(N)$ of $X_0(N)$ satisfying $L(A_f, s) = L(f, s)$. In particular, the semisimple Galois representations $V_p(A_f)$ and $V_p(A)$ are isomorphic. It follows from Faltings' proof of the Tate conjecture for abelian varieties over number fields that A is isogenous to A_f, and therefore there is a non-constant morphism

$$\Phi : J_0(N) \longrightarrow A \tag{13}$$

of algebraic varieties over $\mathbb{Q}$, inducing, for each $F \supset \mathbb{Q}$ a map $\Phi_F : J_0(N)(F) \longrightarrow A(F)$ on F-rational points.

A key application of Φ arises from the fact that $X_0(N)$ is equipped with a distinguished supply of algebraic points corresponding to the moduli of elliptic curves with complex multiplication by an order in a quadratic imaginary field K. The images under $\Phi_{\bar{\mathbb{Q}}}$ of the degree 0 divisors supported on these points produce elements of $A(\bar{\mathbb{Q}})$ defined over abelian extensions of K, which include the so-called *Heegner points*. The Gross-Zagier formula [GZ] relates the canonical heights of these points to the central critical derivatives of $L(A/K,s)$ and of its twists by abelian characters of K. This connection between algebraic points and Hasse-Weil L-series has led to the strongest known results on the Birch and Swinnerton-Dyer conjecture, most notably the theorem that

$$\mathrm{rank}(A(\mathbb{Q})) = \mathrm{ord}_{s=1} L(A,s) \quad \text{and} \quad \#\text{Ш}(A/\mathbb{Q}) < \infty, \qquad \text{when } \mathrm{ord}_{s=1}(L(A,s)) \leq 1,$$

which follows by combining the Gross-Zagier formula with a method of Kolyvagin (cf. [Gr2]). The theory of Heegner points is also the key ingredient in the proof of Theorems 1 and 2.

Given a variety X (defined over $\mathbb{Q}$, say), let $\mathrm{CH}^j(X)(F)$ denote the Chow group of codimension j algebraic cycles on X defined over a field F modulo rational equivalence, and let $\mathrm{CH}^j(X)_0(F)$ denote the subgroup of null-homologous cycles. Write $\mathrm{CH}^j(X)$ and $\mathrm{CH}^j(X)_0$ for the corresponding functors on $\mathbb{Q}$-algebras. Via the natural equivalence $\mathrm{CH}^1(X_0(N))_0 = J_0(N)$, the map Φ of (13) can be recast as a natural transformation

$$\Phi : \mathrm{CH}^1(X_0(N))_0 \longrightarrow A. \tag{14}$$

It is tempting to generalise (14) by replacing $X_0(N)$ by a variety X over $\mathbb{Q}$ of dimension $d > 1$, and $\mathrm{CH}^1(X_0(N))_0$ by $\mathrm{CH}^j(X)_0$ for some $0 \leq j \leq d$. Any element Π of the Chow group $\mathrm{CH}^{d+1-j}(X \times A)(\mathbb{Q})$ induces a natural transformation

$$\Phi : \mathrm{CH}^j(X)_0 \longrightarrow A \tag{15}$$

sending $\Delta \in \mathrm{CH}^j(X)_0(F)$ to

$$\Phi_F(\Delta) := \pi_{A,*}(\pi_X^*(\tilde{\Delta}) \cdot \tilde{\Pi}), \tag{16}$$

where π_X and π_A denote the natural projections from $X \times A$ to X and A respectively. We are mainly interested in the case where X is a Shimura variety or is closely related to a Shimura variety. (For instance, when X is the universal object or a self-fold fiber product of the universal object over a Shimura variety of PEL type.) The variety X is then referred to as a *modular variety* and the natural transformation Φ is called the *modular parametrisation of* A attached to the pair (X, Π).

Modular parametrisations acquire special interest when $\mathrm{CH}^j(X)_0(\bar{\mathbb{Q}})$ is equipped with a systematic supply of special elements, such as those arising from Shimura subvarieties of X. The images in $A(\bar{\mathbb{Q}})$ of such special elements under $\Phi_{\bar{\mathbb{Q}}}$ can be viewed as "higher-dimensional" analogues of Heegner points: they will be referred to as *Chow-Heegner points*. Given an elliptic curve A, it would be of interest to construct modular parametrisations to A in the greatest possible generality, study their basic properties, and explore the relations (if any) between the resulting systems of Chow-Heegner points and leading terms of L-series attached to A.

We develop this loosely formulated program in the simple but non-trivial setting where A is an elliptic curve with complex multiplication by an imaginary quadratic field K of odd discriminant $-D$, and X is a suitable family of $2r$-dimensional abelian varieties fibered over a modular curve.

For the Introduction, suppose for simplicity that K has class number one and that A is the canonical elliptic curve over $\mathbb{Q}$ of conductor D^2 attached to the Hecke character defined by

$$\nu_A((a)) = \varepsilon_K(a \bmod \sqrt{-D})a.$$

(These assumptions will be significantly relaxed in the body of the paper.) Given a regular differential $\omega_A \in \Omega^1(A/\mathbb{Q})$, let $[\omega_A]$ denote the corresponding class in the de Rham cohomology of A.

Fix an integer $r \geq 0$, and consider the Hecke character $\psi = \nu_A^{r+1}$. The binary theta series

$$\theta_\psi := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \nu_A^{r+1}(\mathfrak{a}) q^{\mathfrak{a}\bar{\mathfrak{a}}}$$

attached to ψ is a modular form of weight $(r+2)$ on a certain modular curve C (which is a quotient of $X_1(D)$ or $X_0(D^2)$ depending on whether r is odd or even), and has rational Fourier coefficients. Such a modular form gives rise to a regular differential $(r+1)$-form ω_{θ_ψ} on the rth *Kuga-Sato variety* over C, denoted W_r. Let $[\omega_{\theta_\psi}]$ denote the class of ω_{θ_ψ} in the de Rham cohomology $H^{r+1}_{\mathrm{dR}}(W_r/\mathbb{Q})$. The classes of ω_{θ_ψ} and of the antiholomorphic $(r+1)$-form $\bar{\omega}_{\theta_\psi}$ generate the θ_ψ-isotypic component of $H^{r+1}_{\mathrm{dR}}(W_r/\mathbb{C})$ under the action of the Hecke correspondences.

For all $1 \le j \le r+1$, let $p_j : A^{r+1} \longrightarrow A$ denote the projection onto the j-th factor, and let

$$[\omega_A^{r+1}] := p_1^*[\omega_A] \wedge \cdots \wedge p_{r+1}^*[\omega_A] \quad \in \quad H^{r+1}_{\mathrm{dR}}(A^{r+1}).$$

Our construction of Chow-Heegner points is based on the following conjecture which is formulated (for more general K, without the class number one hypothesis) in Section 3.

Conjecture 7. *There is an algebraic cycle class* $\Pi^? \in \mathrm{CH}^{r+1}(W_r \times A^{r+1})(K) \otimes \mathbb{Q}$ *satisfying*

$$\Pi^{?*}_{\mathrm{dR}}([\omega_A^{r+1}]) = [\omega_{\theta_\psi}],$$

where

$$\Pi^{?*}_{\mathrm{dR}} : H^{r+1}_{\mathrm{dR}}(A^{r+1}/K) \longrightarrow H^{r+1}_{\mathrm{dR}}(W_r/K)$$

is the map on de Rham cohomology induced by $\Pi^?$.

The rationale for Conjecture 7 is explained in Section 3.4, where it is shown to follow from the Tate or Hodge conjectures on algebraic cycles.

We next make the simple (but key) remark that the putative cycle $\Pi^?$ is also an element of $\mathrm{CH}^{r+1}(X_r \times A)$, where X_r is the $(2r+1)$-dimensional variety

$$X_r := W_r \times A^r.$$

Viewed in this way, the cycle $\Pi^?$ gives rise to a modular parametrisation

$$\Phi^? : \mathrm{CH}^{r+1}(X_r)_0 \longrightarrow A$$

as in (15). It is defined over K, and

$$\Phi^{?*}_{\mathrm{dR}}(\omega_A) = \omega_{\theta_\psi} \wedge \eta_A^r, \tag{17}$$

where η_A is the unique element of $H^1_{\mathrm{dR}}(A/K)$ satisfying

$$[\lambda]^*\eta_A = \bar{\lambda}\eta, \text{ for all } \lambda \in \mathcal{O}_K, \qquad \langle \omega_A, \eta_A \rangle = 1. \tag{18}$$

The article [BDP] introduced and studied a collection of null-homologous, r-dimensional algebraic cycles on X_r, referred to as *generalised Heegner cycles*. These cycles, whose precise definition is recalled in Section 3.5, extend the notion of Heegner cycles on Kuga-Sato varieties considered in [Scho], [Ne] and [Zh]. They are indexed by isogenies $\varphi : A \longrightarrow A'$, and are defined over abelian extensions of K. It can be shown that they generate a subgroup of $\mathrm{CH}^{r+1}(X_r)_0(K^{\mathrm{ab}})$ of infinite rank. The map $\Phi^?_{K^{\mathrm{ab}}}$ should transform these generalised Heegner cycles into points of $A(K^{\mathrm{ab}})$. It is natural to expect that the resulting collection $\{\Phi^?_{K^{\mathrm{ab}}}(\Delta_\varphi)\}_{\varphi: A \longrightarrow A'}$ of Chow-Heegner points generates an infinite rank subgroup of $A(K^{\mathrm{ab}})$, and that it gives rise to an "Euler system" in the sense of Kolyvagin. In the classical situation where $r = 0$, the variety X_r is just a modular curve and the existence of $\Pi^?$ follows from Faltings' proof of the Tate conjecture for curves. When $r \ge 1$, the very existence of the collection of Chow-Heegner points relies, ultimately, on producing the algebraic cycle $\Pi^?$ unconditionally.

Section 4 offers some theoretical evidence for the existence of $\Phi^?$ arising from p-adic methods. It rests on the fact that the following p-adic analogues of $\Phi^?_F$ can be constructed without invoking the Hodge or Tate conjectures.

(a) The map Φ^{et}_F: Let F be any field containing K. The Chow group $\mathrm{CH}^{r+1}(X_r)_0(F)$ of null-homologous cycles is equipped with the p-adic étale Abel-Jacobi map over F:

$$\mathrm{AJ}^{\mathrm{et}}_F : \mathrm{CH}^{r+1}(X_r)(F)_0 \longrightarrow H^1(F, H^{2r+1}_{\mathrm{et}}(\overline{X}_r, \mathbb{Q}_p)(r+1)), \tag{19}$$

where $H^1(F,M)$ denotes the continuous Galois cohomology of a $G_F := \mathrm{Gal}(\bar{F}/F)$-module M. As is explained in Section 3.4, the Tate cycle over $\mathbb{Q}$ whose existence is used to justify Conjecture 7 gives rise to a G_F-equivariant projection

$$\pi : H^{2r+1}(\overline{X}_r, \mathbb{Q}_p)(r+1) \longrightarrow H^1_{\mathrm{et}}(\overline{A}, \mathbb{Q}_p)(1) = V_p(A), \tag{20}$$

where $V_p(A)$ is the p-adic Galois representation arising from the p-adic Tate module of A. As it stands, the maps π and Φ_F^{et} are only well-defined up to multiplication by $\mathbb{Q}_p^\times$. We normalise π by embedding F into a p-adic completion F_p, and requiring that the map $\pi_{\mathrm{dR}} : H^{2r+1}_{\mathrm{dR}}(X_r/F_p) \longrightarrow H^1_{\mathrm{dR}}(A/F_p)$ obtained by applying to π the comparison functor between p-adic étale cohomology and deRham cohomology over p-adic fields satisfies

$$\pi_{\mathrm{dR}}(\omega_{\theta_\psi} \wedge \eta_A^r) = \omega_A, \tag{21}$$

where ω_{θ_ψ} and ω_A are as in Conjecture 7, and η_A is defined in (18).

The maps (19) and (20) can be combined to give a map

$$\Phi_F^{\mathrm{et}} : \mathrm{CH}^{r+1}(X_r)(F)_0 \longrightarrow H^1(F, V_p(A)), \tag{22}$$

which is the counterpart of the conjectural map $\Phi_F^?$ in p-adic étale cohomology. More precisely, with π chosen to satisfy (21), the map Φ_F^{et} is related to $\Phi_F^?$ (when the latter can be shown to exist) by the commutative diagram

$$\begin{tikzcd} & A(F) \otimes \mathbb{Q} \arrow[d, "\delta"] \\ \mathrm{CH}^{r+1}(X_r)_0(F) \arrow[ur, dashed, "\Phi_F^{(?)}"] \arrow[r, "\Phi_F^{\mathrm{et}}"] & H^1(F, V_p(A)), \end{tikzcd} \tag{23}$$

where

$$\delta : A(F) \otimes \mathbb{Q} \longrightarrow H^1(F, V_p(A)) \tag{24}$$

is the projective limit of the connecting homomorphisms arising in the p^n-descent exact sequences of Kummer theory.

(b) The map $\Phi_F^{(p)}$: When F is a number field, (23) suggests that the image of Φ_F^{et} is contained in the Selmer group of A over F, and this can indeed be shown to be the case. This means that the image of $\Phi_{F_v}^{\mathrm{et}}$ is contained in the images of the local connecting homomorphisms $\delta_v : A(F_v) \longrightarrow H^1(F_v, V_p(A))$ for all the completions of F. In particular, replacing F by its p-adic completion F_p, we can *define* the map $\Phi_F^{(p)}$ by the commutativity of the following local counterpart of the diagram (23):

$$\begin{tikzcd} & A(F_p) \otimes \mathbb{Q} \arrow[d, "\delta"] \\ \mathrm{CH}^{r+1}(X_r)_0(F_p) \arrow[ur, bend left, "\Phi_F^{(p)}"] \arrow[r, "\Phi_{F_p}^{\mathrm{et}}"] & H^1(F_p, V_p(A)). \end{tikzcd} \tag{25}$$

As will be explained in more detail in Section 4, the map $\Phi_F^{(p)}$ can also be defined by p-adic integration, via the comparison theorems between the p-adic étale cohomology and the de Rham cohomology of varieties over p-adic fields.

The main result of Chapter 4 relates the Selmer classes of the form $\Phi_F^{\mathrm{et}}(\Delta)$ when F is a global field and Δ is a generalised Heegner cycle, to global points in $A(F)$. We will only state a special case of the main result, postponing the more general statement to Section 4.2. Assume for Theorem 8 below that the field K has odd discriminant, that the sign in the functional equation for $L(\nu_A, s)$ is -1, so that the Hasse-Weil L-series $L(A/\mathbb{Q}, s) = L(\nu_A, s)$ vanishes to odd order at $s = 1$, and that the integer r is odd. In that case, the theta series θ_ψ belongs to the space $S_{r+2}(\Gamma_0(D), \varepsilon_K)$ of cusp forms on $\Gamma_0(D)$ of weight $r+2$ and character $\varepsilon_K := \left(\frac{\cdot}{D}\right)$. In particular, the variety W_r is essentially the rth Kuga-Sato variety over the modular curve $X_0(D)$. Furthermore, the L-series $L(\nu_A^{2r+1}, s)$ has sign 1 in its functional equation, and $L(\nu_A^{2r+1}, s)$ therefore vanishes to *even order* at the central point $s = r + 1$.

Theorem 8. *Let Δ be the generalised Heegner cycle in $\mathrm{CH}^{r+1}(X_r)_0(K)$ attached to the identity isogeny $1 : A \longrightarrow A$. The cohomology class $\Phi_K^{\mathrm{et}}(\Delta)$ belongs to $\delta(A(K) \otimes \mathbb{Q})$. More precisely, there is a point $P_D \in A(K) \otimes \mathbb{Q}$ (depending on D but not on r) such that*

$$\Phi_K^{\mathrm{et}}(\Delta) = \sqrt{-D} \cdot m_{D,r} \cdot \delta(P_D),$$

where $m_{D,r} \in \mathbb{Z}$ is given by

$$m_{D,r}^2 = \frac{2r!(2\pi\sqrt{D})^r}{\Omega(A)^{2r+1}} L(\nu_A^{2r+1}, r+1),$$

and $\Omega(A)$ is a complex period attached to A. The point P_D is of infinite order if and only if

$$L'(\nu_A, 1) \neq 0.$$

Remark 9. When $L(A, s)$ has a simple zero at $s = 1$, it is known a priori that the Selmer group $\mathrm{Sel}_p(A/K)$ is of rank one over $K \otimes \mathbb{Q}_p$, and agrees with $\delta(A(K) \otimes \mathbb{Q}_p)$. It follows directly that

$$\Phi_K^{\mathrm{et}}(\Delta) \text{ belongs to } \delta(A(K) \otimes \mathbb{Q}_p).$$

The first part of Theorem 8 is stronger in that it involves the rational vector space $A(K)\otimes\mathbb{Q}$ rather than its p-adification. This stronger statement is not a formal consequence of the one-dimensionality of the Selmer group. Indeed, its proof relies on invoking Theorem 2 after relating the local point $\Phi_{K_p}^{(p)}(\Delta) \in A(K_p)$ to the special value $\mathscr{L}_p(\nu_A^*)$ that arises in that theorem.

Section 5.1 describes a complex homomorphism

$$\Phi_{\mathbb{C}} : \mathrm{CH}^{r+1}(X_r)_0(\mathbb{C}) \longrightarrow A(\mathbb{C})$$

which is defined analytically by integration of differential forms on $X_r(\mathbb{C})$, without invoking Conjecture 7, but agrees with $\Phi_{\mathbb{C}}^?$ when the latter exists. This map is defined from the complex Abel-Jacobi map on cycles introduced and studied by Griffiths and Weil, and is the complex analogue of homomorphism $\Phi_F^{(p)}$ defined in (25). The existence of the global map $\Phi_K^?$ predicted by the Hodge or Tate conjecture would imply the following algebraicity statement:

Conjecture 10. *Let H be a subfield of K^{ab} and let $\Delta_\varphi \in \mathrm{CH}^{r+1}(X_r)_0(H)$ be a generalised Heegner cycle defined over H. Then (after fixing an an embedding of H into $\mathbb{C}$),*

$$\Phi_{\mathbb{C}}(\Delta_\varphi) \text{ belongs to } A(H),$$

and

$$\Phi_{\mathbb{C}}(\Delta_\varphi^\sigma) = \Phi_{\mathbb{C}}(\Delta_\varphi)^\sigma \quad \text{for all } \sigma \in \mathrm{Gal}(H/K).$$

While ostensibly weaker than Conjecture 7, Conjecture 10 has the virtue of being more readily amenable to experimental verification. Section 5 explains how the images of generalised Heegner cycles under $\Phi_{\mathbb{C}}$ can be computed numerically to high accuracy, and illustrates, for a few such Δ_φ, how the points $\Phi_{\mathbb{C}}(\Delta_\varphi)$ can be recognized as algebraic points defined over the predicted class fields. In particular, extensive numerical verifications of Conjecture 10 are carried out, for fairly large values of r. This conjecture appears to lie deeper than its p-adic counterpart, and we were unable to provide any theoretical evidence for it beyond the fact that it follows from the Hodge or Tate conjectures. It might be argued that calculations of the sort that are performed in Section 5 provide independent numerical confirmation of the these conjectures for certain specific Hodge and Tate cycles on the $(2r+2)$-dimensional varieties $W_r \times A^{r+1}$, for which the corresponding algebraic cycles seem hard to produce unconditionally.

Conventions regarding number fields and embeddings: Throughout this article, all number fields that arise are viewed as embedded in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. A complex embedding $\bar{\mathbb{Q}} \longrightarrow \mathbb{C}$ and p-adic embeddings $\bar{\mathbb{Q}} \longrightarrow \mathbb{C}_p$ for each rational prime p are also fixed from the outset, so that any finite extension of $\mathbb{Q}$ is simultaneously realised as a subfield of $\mathbb{C}$ and of $\mathbb{C}_p$.

1. Hecke characters and periods

1.1. **Algebraic Hecke characters.** Let K and E be number fields. Given a $\mathbb{Z}$-linear combination

$$\phi = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}[\mathrm{Hom}(K, \bar{\mathbb{Q}}]$$

of embeddings of K into $\bar{\mathbb{Q}}$, we define

$$\alpha^\phi := \prod_\sigma (\sigma\alpha)^{n_\sigma},$$

for all $\alpha \in K^\times$. Let $I_\mathfrak{f}$ denote the group of fractional ideals of K which are prime to a given integral ideal $\mathfrak{f}$ of K, and let

$$J_\mathfrak{f} := \{(\alpha) \text{ such that } \alpha \gg 0 \text{ and } \alpha - 1 \in \mathfrak{f}\} \quad \subset \quad I_\mathfrak{f}.$$

Definition 1.1. An E-valued *algebraic Hecke character* (or simply Hecke character) of K of infinity type ϕ on $I_\mathfrak{f}$ is a homomorphism

$$\chi : I_\mathfrak{f} \to E^\times$$

such that

$$\chi((\alpha)) = \alpha^\phi, \quad \text{for all } (\alpha) \in J_\mathfrak{f}. \tag{26}$$

The largest integral ideal $\mathfrak{f}$ satisfying (26) is called the *conductor* of χ, and is denoted $\mathfrak{f}_\chi$.

The most basic examples of algebraic Hecke characters are the norm characters on $\mathbb{Q}$ and on K respectively, which are given by

$$\mathbf{N}(a) = |a|, \qquad \mathbf{N}_K := \mathbf{N} \circ \mathrm{N}^K_{\mathbb{Q}}.$$

Note that the infinity type ϕ of a Hecke character χ must be trivial on all totally positive units congruent to $1 \mod \mathfrak{f}$. Hence the existence of such a χ implies there is an integer $w(\chi)$ (called the *weight* of χ or of ϕ) such that for any choice of embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$,

$$n_\sigma + n_{\bar{\sigma}} = w(\chi), \quad \text{for all } \sigma \in \mathrm{hom}(K, \bar{\mathbb{Q}}).$$

Let $U'_\mathfrak{f} \subset \mathbb{A}_K^\times$ be the subgroup defined by

$$U'_\mathfrak{f} := \left\{ (x_v) \in \mathbb{A}_K^\times \text{ such that } \begin{array}{ll} x_v \equiv 1 \pmod{\mathfrak{f}}, & \text{for all } v | \mathfrak{f}, \\ x_v > 0, & \text{for all real } v \end{array} \right\},$$

and let $U_\mathfrak{f}$ denote its maximal compact subgroup:

$$U_\mathfrak{f} := \{(x_v) \in U'_\mathfrak{f} \text{ such that } x_v \in \mathcal{O}_{K_v}^\times, \text{ for all non-archimedean } v\}.$$

A Hecke character χ may also be viewed as a character on $\mathbb{A}_K^\times / U_\mathfrak{f}$ (denoted by the same symbol by a common abuse of notation),

$$\chi : \mathbb{A}_K^\times / U_\mathfrak{f} \to E^\times, \qquad \text{satisfying} \qquad \chi|_{K^\times} = \phi. \tag{27}$$

To wit, given $x \in \mathbb{A}_K^\times$, we define $\chi(x)$ by choosing $\alpha \in K^\times$ such that αx belongs to $U'_\mathfrak{f}$, and setting

$$\chi(x) = \chi(i(\alpha x)) \phi(\alpha)^{-1}, \tag{28}$$

where the symbol $i(x)$ denotes the fractional ideal of K associated to x. This definition is independent of the choice of α by (26). In the opposite direction, given an idèle character χ of conductor $\mathfrak{f}$ as in (27), we can set

$$\chi(\mathfrak{a}) = \chi(x), \quad \text{for any } x \in U'_\mathfrak{f} \text{ such that } i(x) = \mathfrak{a}.$$

The subfield of E generated by the values of χ on $I_\mathfrak{f}$ is easily seen to be independent of the choice of $\mathfrak{f}$ and will be denoted E_χ.

The *central character* of a Hecke character of K is defined as follows:

Definition 1.2. The *central character* of a Hecke character η of K is the finite order character of $\mathbb{Q}$ given by

$$\eta|_{\mathbb{A}_\mathbb{Q}^\times} = \varepsilon_\eta \cdot \mathbf{N}^{w(\eta)}.$$

The infinity type ϕ defines a homomorphism $\mathrm{Res}_{K,\mathbb{Q}}(\mathbf{G}_m) \to \mathrm{Res}_{E/\mathbb{Q}} \mathbf{G}_m$ of algebraic groups and therefore induces a homomorphism

$$\phi_{\mathbb{A}} : \mathbb{A}_K^\times \to \mathbb{A}_E^\times$$

on adelic points. Given a Hecke character χ with values in E and a place λ of E (either finite or infinite), we may use $\phi_{\mathbb{A}}$ to define an idèle class character

$$\chi_\lambda : \mathbb{A}_K^\times / K^\times \to E_\lambda^\times,$$

by setting

$$\chi_\lambda(x) = \chi(x)/\phi_{\mathbb{A}}(x)_\lambda.$$

If λ is an infinite place, the character χ_λ is a Grossencharacter of K of type A_0. If λ is a finite place, then χ_λ factors through G_K^{ab} and gives a Galois character valued in $E_\lambda^\times$, satisfying

$$\chi_\lambda(\mathrm{Frob}_{\mathfrak{p}}) = \chi(\mathfrak{p})$$

for any prime ideal $\mathfrak{p}$ of K not dividing $\mathfrak{f}\lambda$.

Definition 1.3. Let $E = \prod_i E_i$ be a product of number fields. An *E-valued algebraic Hecke character* is a character

$$\chi : I_{\mathfrak{f}} \to E^\times$$

whose projection to each component E_i is an algebraic Hecke character in the sense defined above.

The L-functions attached to any algebraic Hecke character χ are defined by

$$L(\chi, s) = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\right)^{-1}, \qquad L_{\mathfrak{f}}(\chi, s) = \prod_{\mathfrak{p}\nmid\mathfrak{f}} \left(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\right)^{-1}.$$

Note that $L(\chi, s) = L_{\mathfrak{f}_\chi}(\chi, s)$ when χ is a primitive Hecke character.

Remark 1.4. The definition of $L(\chi, s)$ as a $\mathbb{C}$-valued function relies on the fact that E_χ is given as a subfield of $\mathbb{C}$.

1.2. **Abelian varieties associated to characters of type $(1, 0)$.** In this section, we limit the discussion to the case where K is an imaginary quadratic field. Let $\tau : K \mapsto \mathbb{C}$ be the given complex embedding of K. A Hecke character of infinity type $\phi = n_\tau \tau + n_{\bar\tau} \bar\tau$ will also be said to be of infinity type $(n_\tau, n_{\bar\tau})$.

Let ν be a Hecke character of K of infinity type $(1, 0)$ and conductor $\mathfrak{f}_\nu$, let $E_\nu \supset K$ denote the subfield of $\bar{\mathbb{Q}}$ generated by its values, and let T_ν be the ring of integers of E_ν. The Hecke character ν gives rise to a compatible system of one-dimensional ℓ-adic representations of G_K with values in $(E \otimes \mathbb{Q}_\ell)^\times$, denoted $\rho_{\nu,\ell}$, satisfying

$$\rho_{\nu,\ell}(\sigma_{\mathfrak{a}}) = \nu(\mathfrak{a}), \qquad \text{for all } \mathfrak{a} \in I_{\mathfrak{f}_\nu \ell},$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(\bar{K}/K)$ denotes frobenius conjugacy class attached to $\mathfrak{a}$. The theory of complex multiplication realises the representations $\rho_{\nu,\ell}$ on the division points of so-called CM abelian varieties, which are defined as follows.

Definition 1.5. A *CM abelian variety* over K is a pair (B, E) where

(1) B is an abelian variety over K;
(2) E is a product of CM fields equipped the structure of a K-algebra and an inclusion

$$i : E \longrightarrow \mathrm{End}_K(B) \otimes \mathbb{Q},$$

satisfying $\dim_K(E) = \dim B$;
(3) for all $\lambda \in K \subset E$, the endomorphism $i(\lambda)$ acts on the cotangent space $\Omega^1(B/K)$ as multiplication by λ.

The abelian varieties (B, E) over K with complex multiplication by a fixed E form a category denoted $\mathcal{CM}_{K,E}$ in which a morphism from B_1 to B_2 is a morphism $j : B_1 \longrightarrow B_2$ of abelian varieties over K for which the diagrams

$$\begin{tikzcd} B_1 \arrow[r, "j"] \arrow[d, "e"] & B_2 \arrow[d, "e"] \\ B_1 \arrow[r, "j"] & B_2 \end{tikzcd}$$

commute, for all $e \in E$ which belong to both $\mathrm{End}_K(B_1)$ and $\mathrm{End}_K(B_2)$. An isogeny in $\mathcal{CM}_{K,E}$ is simply a morphism in this category arising from an isogeny on the underlying abelian varieties.

If (B, E) is a CM abelian variety, its endomorphism ring over K contains a finite index subring T^0 of the integral closure T of $\mathbb{Z}$ in E. After replacing B by the K-isogenous abelian variety $\hom_{T_0}(T, B)$, we can assume that $\mathrm{End}_K(B)$ contains T. This assumption, which is occasionally convenient, will consistently be made from now on.

Let (B, E) be a CM-abelian variety with E a field, and let $E' \supset E$ be a finite extension of E with ring of integers T'. The abelian variety $B \otimes_T T'$ is defined to be the variety whose L-rational points, for any $L \supset K$, are given by

$$(B \otimes_T T')(L) = (B(\bar{\mathbb{Q}}) \otimes_T T')^{\mathrm{Gal}(\bar{\mathbb{Q}}/L)}.$$

This abelian variety is equipped with an action of T' by K-rational endomorphisms, described by multiplication on the right, and therefore $(B \otimes_T T', E')$ is an object of $\mathcal{CM}_{K,E'}$. Note that $B \otimes_T T'$ is isogenous to $t := \dim_E(E')$ copies of B, and that the action of T on $B \otimes_T T'$ agrees with the "diagonal" action of T on B^t.

Let ℓ be a rational prime. For each CM abelian variety (B, E), let

$$T_\ell(B) := \lim_{\leftarrow, n} B[\ell^n](\bar{K}), \qquad V_\ell(B) := T_\ell(B) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

be the ℓ-adic Tate module and ℓ-adic representation of G_K attached to B. The $\mathbb{Q}_\ell$-vector space $V_\ell(B)$ is a free $E \otimes \mathbb{Q}_\ell$-module of rank one via the action of E by endomorphisms. The natural action of $G_K := \mathrm{Gal}(\bar{K}/K)$ on $V_\ell(B)$ commutes with this $E \otimes \mathbb{Q}_\ell$-action, and the collection $\{V_\ell(B)\}$ thus gives rise to a compatible system of one-dimensional ℓ-adic representations of G_K with values in $(E \otimes \mathbb{Q}_\ell)^\times$, denoted $\rho_{B,\ell}$. We note in passing that for any extension $E' \supset E$ where T' the integral closure of T in E',

$$T_\ell(B \otimes_T T') = T_\ell(B) \otimes_T T', \qquad V_\ell(B \otimes_T T') = V_\ell(B) \otimes_E E'.$$

The following result is due to Casselman (cf. Theorem 6 of [Shi]).

Theorem 1.6. *Let ν be a Hecke character of K of type $(1, 0)$ as above, and let $\rho_{\nu,\ell}$ be the associated one-dimensional ℓ-adic representation with values in $(E_\nu \otimes \mathbb{Q}_\ell)$. Then*

(1) *There exists a CM abelian variety (B_ν, E_ν) satisfying*

$$\rho_{B_\nu,\ell} \simeq \rho_{\nu,\ell}.$$

(2) *The CM abelian variety B_ν is unique up to isogeny over K. More generally, if (B, E) is any CM abelian variety with $E \supset E_\nu$ satisfying $\rho_{B,\ell} \simeq \rho_{\nu,\ell} \otimes_{E_\nu} E$ as $(E \otimes \mathbb{Q}_\ell)[G_K]$-modules, then there is an isogeny in $\mathcal{CM}_{K,E}$ from B to $B_\nu \otimes_{T_\nu} T$.*

Let ψ be a Hecke character of infinity type $(1, 0)$, and let χ be a finite order Hecke character. In comparing the abelian varieties B_ψ and $B_{\psi\chi^{-1}}$ attached to these two characters, it is useful to introduce a CM abelian variety $B_{\psi,\chi}$ over K, which we now proceed to describe.

Let $E_{\psi,\chi}$ be the subfield of $\bar{\mathbb{Q}}$ generated by E_ψ and E_χ, and let $T_{\psi,\chi} \subset E_{\psi,\chi}$ be its ring of integers. We also write H_χ for the abelian extension of K which is cut out by χ viewed as a Galois character of G_K. Consider first the abelian variety over K with endomorphism by $T_{\psi,\chi}$:

$$B^0_{\psi,\chi} := B_\psi \otimes_{T_\psi} T_{\psi,\chi}.$$

The natural inclusion $i_\psi : T_\psi \longrightarrow T_{\psi,\chi}$ induces a morphism $i : B_\psi \longrightarrow B^0_{\psi,\chi}$ with finite kernel, which is compatible with the T_ψ-actions on both sides and is given by

$$i(P) = P \otimes 1.$$

Lemma 1.7. *Let F be any number field containing $E_{\psi,\chi}$. The restriction map i^* induces an isomorphism*

$$i^*: \Omega^1(B^0_{\psi,\chi}/F)^{T_{\psi,\chi}} \longrightarrow \Omega^1(B_\psi/F)^{T_\psi} \tag{29}$$

of one-dimensional F-vector spaces.

Proof. The fact that $B^0_{\psi,\chi}$ and B_ψ are CM abelian varieties over F implies that source and target in (29) are both one-dimensional over F. To see that i^* is injective, let ω be an element of $\ker(i^*) \cap \Omega^1(B^0_{\psi,\chi}/F)^{T_{\psi,\chi}}$. Then since ω is stable under the action of $T_{\psi,\chi}$ by endomorphisms, it follows that

$$\omega \text{ belongs to } \cap_{\lambda \in T_{\psi,\chi}} [\lambda] \ker(i^*) = 0.$$

Hence $\omega = 0$ and the lemma follows. □

We can now denote by $\omega^0_{\psi,\chi} \in \Omega^1(B^0_{\psi,\chi}/\bar{\mathbb{Q}})^{T_{\psi,\chi}}$ the unique regular differential satisfying

$$i^*(\omega_{\psi,\chi}) = \omega_\psi, \text{ where } \omega_\psi \in \Omega^1(B_\psi/E_\psi)^{T_\psi}. \tag{30}$$

It follows from Lemma 1.7 that $\omega^0_{\psi,\chi}$ exists and is unique (once ω_ψ has been chosen), and that $\omega^0_{\psi,\chi}$ belongs to $\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})$.

The character $\chi^{-1}: \mathrm{Gal}(H_\chi/K) \longrightarrow T_\chi^\times$ can be viewed as a one-cocycle in

$$H^1(\mathrm{Gal}(H_\chi/K), T^\times_{\psi,\chi}) \subset H^1(\mathrm{Gal}(H_\chi/K), \mathrm{Aut}(B^0_{\psi,\chi})).$$

Let

$$B_{\psi,\chi} := (B^0_{\psi,\chi})^{\chi^{-1}} \tag{31}$$

denote the twist of $B^0_{\psi,\chi}$ by this cocycle. There is a natural identification $B^0_{\psi,\chi}(\bar{K}) = B_{\psi,\chi}(\bar{K})$ of sets, arising from an isomorphism of varieties over H_χ, where H_χ is the extension of K cut out by χ. The actions of G_K on $B^0_{\psi,\chi}(\bar{K})$ and $B_{\psi,\chi}(\bar{K})$, denoted $*_0$ and $*$ respectively, are related by

$$\sigma * P = (\sigma *_0 P) \otimes \chi^{-1}(\sigma), \qquad \text{for all } \sigma \in G_K. \tag{32}$$

In particular, for any $L \supset K$, we have:

$$B_{\psi,\chi}(L) \quad = \quad \left\{ P \in B_\psi(\bar{\mathbb{Q}}) \otimes_{T_\psi} T_{\psi,\chi} \text{ such that } \sigma P = P \otimes \chi(\sigma), \quad \forall \sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/L) \right\}. \tag{33}$$

Likewise, the natural actions of G_K on $\Omega^1(B^0_{\psi,\chi}/\bar{K})$ and on $\Omega^1(B_{\psi,\chi}/\bar{K})$ are related by

$$\sigma * \omega = [\chi^{-1}(\sigma)]^*(\sigma *_0 \omega) \qquad \text{for all } \sigma \in G_K. \tag{34}$$

The isomorphism of $B^0_{\psi,\chi}$ and $B_{\psi,\chi}$ as CM abelian varieties over H_χ gives natural identifications

$$\Omega^1(B^0_{\psi,\chi}/H_\chi) = \Omega^1(B_{\psi,\chi}/H_\chi), \qquad \Omega^1(B^0_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}} = \Omega^1(B_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}},$$

where $E'_{\psi,\chi}$ denotes the subfield of $\bar{\mathbb{Q}}$ generated by H_χ and $E_{\psi,\chi}$. Let $\omega^0_{\psi,\chi}$ and $\omega_{\psi,\chi}$ be $E_{\psi,\chi}$ vector space generators of $\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$ and $\Omega^1(B_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$ respectively. Since they both generate $\Omega^1(B_{\psi,\chi}/E'_{\psi,\chi})^{T_{\psi,\chi}}$ as an $E'_{\psi,\chi}$-vector space, they necessarily differ by a non-zero scalar in $E'_{\psi,\chi}$.

To spell out the relation between $\omega^0_{\psi,\chi}$ and $\omega_{\psi,\chi}$ more precisely, it will be useful to introduce the notion of a *generalised Gauss sum* attached to any finite order character χ of G_K. Given such a character, let $E \subset \bar{\mathbb{Q}}$ denote the field generated by the values of χ, and let

$$E\{\chi\} := \{\lambda \in EH_\chi \text{ such that } \lambda^\sigma = \chi(\sigma)\lambda, \quad \forall \sigma \in \mathrm{Gal}(EH_\chi/E)\}.$$

This set is a one-dimensional E-vector space in a natural way. It is not closed under multiplication, but $E\{\chi_1\} \cdot E\{\chi_2\} = E\{\chi_1\chi_2\}$.

Definition 1.8. An E-vector space generator of $E\{\chi\}$ is called a *Gauss sum* attached to the character χ, and is denoted $\mathfrak{g}(\chi)$.

By definition, the Gauss sum $\mathfrak{g}(\chi)$ belongs to $E\{\chi\} \cap (EH_\chi)^\times$, but is only well-defined up to multiplication by $E^\times$.

The following lemma pins down the relationship between the differentials $\omega^0_{\psi,\chi}$ and $\omega_{\psi,\chi}$.

Lemma 1.9. *For all Hecke characters ψ and χ as above,*

$$\omega_{\psi,\chi} = \mathfrak{g}(\chi)\omega^0_{\psi,\chi} \pmod{E^\times_{\psi,\chi}}.$$

Proof. Let $\lambda \in (H_\chi E_{\psi,\chi})^\times$ be the scalar satisfying

$$\omega_{\psi,\chi} = \lambda \omega^0_{\psi,\chi}. \tag{35}$$

Since $\omega_{\psi,\chi}$ is an $E_{\psi,\chi}$-rational differential on $B_{\psi,\chi}$, for all $\sigma \in \mathrm{Gal}(\bar{K}/E_{\psi,\chi})$ we have

$$\omega_{\psi,\chi} = \sigma * \omega_{\psi,\chi} = [\chi^{-1}(\sigma)]^* \sigma *_0 \omega_{\psi,\chi} = \chi^{-1}(\sigma)\lambda^\sigma \omega^0_{\psi,\chi}, \tag{36}$$

where the second equality follows from equation (34) and the last from the fact that the differential $\omega^0_{\psi,\chi}$ belongs to $\Omega^1(B^0_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$. Comparing (35) and (36) gives $\lambda^\sigma = \chi(\sigma)\lambda$, and hence $\lambda = \mathfrak{g}(\chi)$ (mod $E^\times_{\psi,\chi}$). The lemma follows. □

The following lemma relates the abelian varieties $B_{\psi,\chi}$ and B_ν, where $\nu = \psi\chi^{-1}$.

Lemma 1.10. *There is an isogeny defined over K:*

$$i_\nu : B_{\psi,\chi} \longrightarrow B_\nu \otimes_{T_\nu} T_{\psi,\chi}$$

which is compatible with the action of $T_{\psi,\chi}$ by endomorphisms on both sides.

Proof. The pair $(B^0_{\psi,\chi}, E_{\psi,\chi})$ is a CM abelian variety having ψ (viewed as taking values in $E_{\psi,\chi}$) as its associated Hecke character. The Hecke character attached to the Galois twist $B_{\psi,\chi}$ is therefore $\psi\chi^{-1} = \nu$. The second part of Theorem 1.6 implies that $B_{\psi,\chi}$ and $B_\nu \otimes_{T_\nu} T_{\psi,\chi}$ are isogenous over K as CM abelian varieties, and the lemma follows. □

1.3. **Complex periods and special values of L-functions.** This section attaches certain periods to the quadratic imaginary field K and to Hecke characters of this field.

We begin by defining a complex period attached to K. This period depends on the following choices:

1. An elliptic curve A with complex multiplication by $\mathcal{O}_K$, defined over a finite extension F of K. (Note that F necessarily contains the Hilbert class field of K.)
2. A regular differential $\omega_A \in \Omega^1(A/F)$.
3. A non-zero element γ of $H_1(A(\mathbb{C}), \mathbb{Q})$.

The complex period attached to this data is defined by

$$\Omega(A) := \frac{1}{2\pi i}\int_\gamma \omega_A \qquad (\mathrm{mod}\ \bar{\mathbb{Q}}^\times) \quad (\mathrm{mod}\ F^\times). \tag{37}$$

Note that $\Omega(A)$ depends on the pair (ω, γ). A different choice of ω or γ has the effect of multiplying $\Omega(A)$ by a scalar in $F^\times$, and therefore $\Omega(A)$ can be viewed as a well-defined element of $\mathbb{C}^\times/F^\times$.

For any Hecke character ψ of K, recall that ψ^* is the Hecke character defined as in the Introduction by $\psi^*(x) = \psi(\bar{x})$. Suppose that ψ is of infinity type $(1,0)$, and let $E_\psi \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ denote the field generated by the values of ψ (or, equivalently, ψ^*). Choose (arbitrary) non-zero elements

$$\omega_\psi \in \Omega^1(B_\psi/E_\psi)^{T_\psi}, \qquad \gamma \in H_1(B_\psi(\mathbb{C}), \mathbb{Q}),$$

where B_ψ is the CM abelian variety attached to ψ by Theorem 1.6, and $\Omega^1(B_\psi/E_\psi)^{T_\psi}$ is defined in equation (10) of the Introduction. The period $\Omega(\psi^*)$ attached to ψ^* is defined by setting

$$\Omega(\psi^*) = \frac{1}{2\pi i}\int_\gamma \omega_\psi \qquad (\mathrm{mod}\ E_\psi^\times).$$

Note that the complex number $\Omega(\psi^*)$ does not depend, up to multiplication by $E_\psi^\times$, on the choices of ω_ψ and γ that were made to define it.

Lemma 1.11. *If ψ is a Hecke character of infinity type $(1,0)$, and χ is a finite order character, then*

$$\Omega(\psi^*\chi) = \Omega(\psi^*)\mathfrak{g}(\chi^*)^{-1} \quad (\mathrm{mod}\ E^\times_{\psi,\chi}). \tag{38}$$

Proof. Choose a non-zero generator γ of $H_1(B^0_{\psi,\chi}(\mathbb{C}), \mathbb{Q}) = H_1(B_{\psi,\chi}(\mathbb{C}), \mathbb{Q})$ (viewed as a one-dimensional $E_{\psi,\chi}$ vector space via the endomorphism action). By definition,

$$\Omega((\psi\chi^{-1})^*) = \int_\gamma \omega_{\psi,\chi} = \mathfrak{g}(\chi)\int_\gamma \omega^0_{\psi,\chi} = \mathfrak{g}(\chi)\Omega(\psi^*) \quad (\mathrm{mod}\ E^\times_{\psi,\chi}),$$

where the second equality follows from Lemma 1.9. The result now follows after substituting χ^{*-1} for χ. □

One can also attach a period $\Omega(\psi^*)$ to an arbitrary Hecke character of K following the article [GS]. (See also Sections 3.1 and 3.2.) These more general periods are known to satisfy the following multiplicativity relations: see Section 1 of [Har] for instance.

Proposition 1.12. *Let ψ be a Hecke character of infinity type (k,j). Then*

(1) *The ratio*
$$\frac{\Omega(\psi^*)}{(2\pi i)^j \Omega(A)^{k-j}}$$
is algebraic.

(2) *For all ψ and ψ',*
$$\Omega(\psi\psi') = \Omega(\psi)\Omega(\psi') \pmod{E^\times_{\psi,\psi'}},$$
where $E_{\psi,\psi'}$ is the subfield of $\bar{\mathbb{Q}}$ generated by E_ψ and $E_{\psi'}$.

The following theorem is due to Goldstein and Schappacher [GS] (and Blasius for general CM fields).

Theorem 1.13. *Suppose that ψ has infinity type (k,j) with $k > j$, and that m is a critical integer for $L(\psi^{-1}, s)$. Then*
$$\frac{L(\psi^{-1},m)}{(2\pi i)^m \Omega(\psi^*)} \text{ belongs to } E_\psi,$$
and for all $\tau \in \mathrm{Gal}(E_\psi/K)$,
$$\left(\frac{L(\psi^{-1},m)}{(2\pi i)^m \Omega(\psi^*)}\right)^\tau = \frac{L((\psi^{-1})^\tau,m)}{(2\pi i)^m \Omega((\psi^*)^\tau)}.$$

1.4. **p-adic periods.** Fix a prime p that splits in K. We will need p-adic analogs of the periods $\Omega(A)$ and $\Omega(\nu^*)$. One way to define these p-adic periods is to use the comparison isomorphism between p-adic étale cohomology over p-adic fields and de Rham cohomology, as hinted in Blasius's article [Bl]. It is also possible to supply a more direct definition, as will be done in this section.

The p-adic analogue $\Omega_p(A)$ of $\Omega(A)$ is obtained by considering the base change $A_{\mathbb{C}_p}$ of A to $\mathbb{C}_p$ (via our fixed embedding of F into $\mathbb{C}_p$). Assume that A has good reduction at the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$, i.e., that $A_{\mathbb{C}_p}$ extends to a smooth proper model $A_{\mathcal{O}_{\mathbb{C}_p}}$ over $\mathcal{O}_{\mathbb{C}_p}$. The p-adic completion $\hat{A}_{\mathcal{O}_{\mathbb{C}_p}}$ of A along its special fiber is isomorphic to $\hat{\mathbf{G}}_m$. Following [deS] II, §4.4, choose an isomorphism $\iota_p : \hat{A} \longrightarrow \hat{\mathbf{G}}_m$ over $\mathcal{O}_{\mathbb{C}_p}$, and define $\Omega_p(A) \in \mathbb{C}_p^\times$ by the rule
$$\omega_A = \Omega_p(A) \cdot \iota_p^*(du/u), \tag{39}$$
where u is the standard coordinate on $\hat{\mathbf{G}}_m$. The invariant $\Omega_p(A) \in \mathbb{C}_p^\times$ thus defined depends on the choices of ω_A and ι_p, but only up to multiplication by a scalar in $F^\times$. Observe also that $\Omega(A)$ and $\Omega_p(A)$ each depend linearly in the same way on the choice of the global differential ω_A.

The p-adic period $\Omega_p(A)$ can be used to define p-adic analogues of the complex periods $\Omega(\nu)$ that appear in the statement of Theorem 1.13.

Definition 1.14. The p-adic period attached to a Hecke character ν of type $(1,0)$, denoted $\Omega_p(\nu^*)$, is defined to be
$$\Omega_p(\nu^*) := \Omega_p(A) \cdot \frac{\Omega(\nu^*)}{\Omega(A)}.$$
More generally, for any character ν of infinity type (k,j), we define
$$\Omega_p(\nu^*) := \Omega_p(A)^{k-j} \cdot \frac{\Omega(\nu^*)}{(2\pi i)^j \Omega(A)^{k-j}}.$$

It can be seen from this definition that the period $\Omega_p(\nu)$, like its complex counterpart $\Omega(\nu)$, is well-defined up to multiplication by a scalar in $E_\nu^\times$. The following p-adic analogue of Lemma 1.11 is a direct consequence of this lemma combined with the definition of $\Omega_p(\psi)$:

Lemma 1.15. *If ψ is a Hecke character of infinity type $(1,0)$, and χ is a finite order character, then*
$$\Omega_p(\psi^*\chi) = \Omega_p(\psi^*)\mathfrak{g}(\chi^*)^{-1} \pmod{E^\times_{\psi,\chi}}. \tag{40}$$

Likewise, Proposition 1.12 implies:

Proposition 1.16. *Let ψ be a Hecke character of infinity type (k, j). Then*

(1) *The ratio*

$$\frac{\Omega_p(\psi^*)}{(2\pi i)^j \Omega_p(A)^{k-j}}$$

is algebraic.

(2) *For all ψ and ψ',*

$$\Omega_p(\psi\psi') = \Omega_p(\psi)\Omega_p(\psi') \pmod{E_{\psi,\psi'}^\times}. \tag{41}$$

2. p-adic L-functions and Rubin's formula

2.1. **The Katz p-adic L-function.** Throughout this chapter, we will fix a prime p that is split in K. Let $\mathfrak{c}$ be an integral ideal of K which is prime to p, and let $\Sigma(\mathfrak{c})$ denote the set of all Hecke characters of K of conductor dividing $\mathfrak{c}$. Denote by $\mathfrak{p}$ the prime above p corresponding to the chosen embedding $K \hookrightarrow \bar{\mathbb{Q}}_p$.

A character $\nu \in \Sigma(\mathfrak{c})$ is called a *critical character* if $L(\nu^{-1}, 0)$ is a critical value in the sense of Deligne, i.e., if the Γ-factors that arise in the functional equation for $L(\nu^{-1}, s)$ are non-vanishing and have no poles at $s = 0$. The set $\Sigma_{\rm crit}(\mathfrak{c})$ of critical characters can be expressed as the disjoint union

$$\Sigma_{\rm crit}(\mathfrak{c}) = \Sigma_{\rm crit}^{(1)}(\mathfrak{c}) \cup \Sigma_{\rm crit}^{(2)}(\mathfrak{c}),$$

where

$$\Sigma_{\rm crit}^{(1)}(\mathfrak{c}) = \{\nu \in \Sigma(\mathfrak{c}) \quad \text{of type } (\ell_1, \ell_2) \text{ with } \ell_1 \le 0, \quad \ell_2 \ge 1\},$$
$$\Sigma_{\rm crit}^{(2)}(\mathfrak{c}) = \{\nu \in \Sigma(\mathfrak{c}) \quad \text{of type } (\ell_1, \ell_2) \text{ with } \ell_1 \ge 1, \quad \ell_2 \le 0\}.$$

The possible infinity types of Hecke characters in these two critical regions are sketched in Figure 1. Note in particular that when $\mathfrak{c} = \bar{\mathfrak{c}}$, the regions $\Sigma_{\rm crit}^{(1)}(\mathfrak{c})$ and $\Sigma_{\rm crit}^{(2)}(\mathfrak{c})$ are interchanged by the involution $\nu \mapsto \nu^*$, where ν^* is defined as in the Introduction by $\nu^*(a) = \nu(\bar{a})$, with $a \mapsto \bar{a}$ denoting the conjugation on $\mathbb{A}_K$.

The set $\Sigma_{\rm crit}(\mathfrak{c})$ is endowed with a natural p-adic topology arising from the compact open topology on the space of functions on a certain subset of $\mathbb{A}_K^\times$, as described in Section 5.4 of [BDP]. The subsets $\Sigma_{\rm crit}^{(1)}(\mathfrak{c})$ and $\Sigma_{\rm crit}^{(2)}(\mathfrak{c})$ are dense in the completion $\hat{\Sigma}_{\rm crit}(\mathfrak{c})$ relative to this topology.

Recall that $\mathfrak{p}$ is the prime above p induced by our chosen embedding of K into $\mathbb{C}_p$. The following proposition on the existence of the p-adic L-function is due to Katz. The statement below is a restatement of [deS] (II, Thm. 4.14) with a minor correction.

Proposition 2.1. *There exists a p-adic analytic function $\nu \mapsto \mathscr{L}_p(\nu)$ (valued in $\mathbb{C}_p$) on $\hat{\Sigma}_{\rm crit}(\mathfrak{c})$ which is determined by the interpolation property:*

$$\frac{\mathscr{L}_p(\nu)}{\Omega_p(A)^{\ell_1-\ell_2}} = \left(\frac{\sqrt{D}}{2\pi}\right)^{\ell_2} (\ell_1 - 1)!(1 - \nu(\mathfrak{p})/p)(1 - \nu^{-1}(\bar{\mathfrak{p}}))\frac{L_{\mathfrak{c}}(\nu^{-1}, 0)}{\Omega(A)^{\ell_1-\ell_2}}, \tag{42}$$

for all critical characters $\nu \in \Sigma_{\rm crit}^{(2)}(\mathfrak{c})$ of infinity type (ℓ_1, ℓ_2).

The right hand side of (42) belongs to $\bar{\mathbb{Q}}$, by Part 1 of Proposition 1.12 and Theorem 1.13 with $m = 0$. Equation (42) should be interpreted to mean that the left hand side also belongs to $\bar{\mathbb{Q}}$, viewed as a subfield of $\mathbb{C}_p$ under the chosen embeddings, and agrees with the right hand side. Note that although both sides of (42) depend on the choice of the differential ω_A that was made in the definition of the periods $\Omega(A)$ and $\Omega_p(A)$, the quantity $\mathscr{L}_p(\nu)$, just like its complex counterpart $L_{\mathfrak{c}}(\nu^{-1}, 0)$, does not depend on this choice.

We are mainly interested in the behavior of $\mathscr{L}_p(\nu)$ at the so-called *self-dual* Hecke characters, which are defined as follows:

Definition 2.2. A Hecke character $\nu \in \Sigma_{\rm crit}(\mathfrak{c})$ is said to be *self-dual* or *anticyclotomic* if

$$\nu\nu^* = \mathrm{N}_K.$$

Note that a self-dual character is necessarily of infinity type $(1 + j, -j)$ for some $j \in \mathbb{Z}$. There is also a restriction on the *central character* of such a ν. More precisely, it is clear that $\varepsilon_{\bar{\nu}} = \overline{\varepsilon_\nu}$, while $\varepsilon_{\nu^*} = \varepsilon_\nu$. If ν is a self-dual character, it follows that for any $x \in \mathbb{A}_K^\times$,

$$\nu(\mathrm{N}_{\mathbb{Q}}^K(x)) = \nu(x\bar{x}) = (\nu\nu^*)(x) = \mathbf{N}_K(x) = \mathbf{N}(\mathrm{N}_{\mathbb{Q}}^K(x)).$$

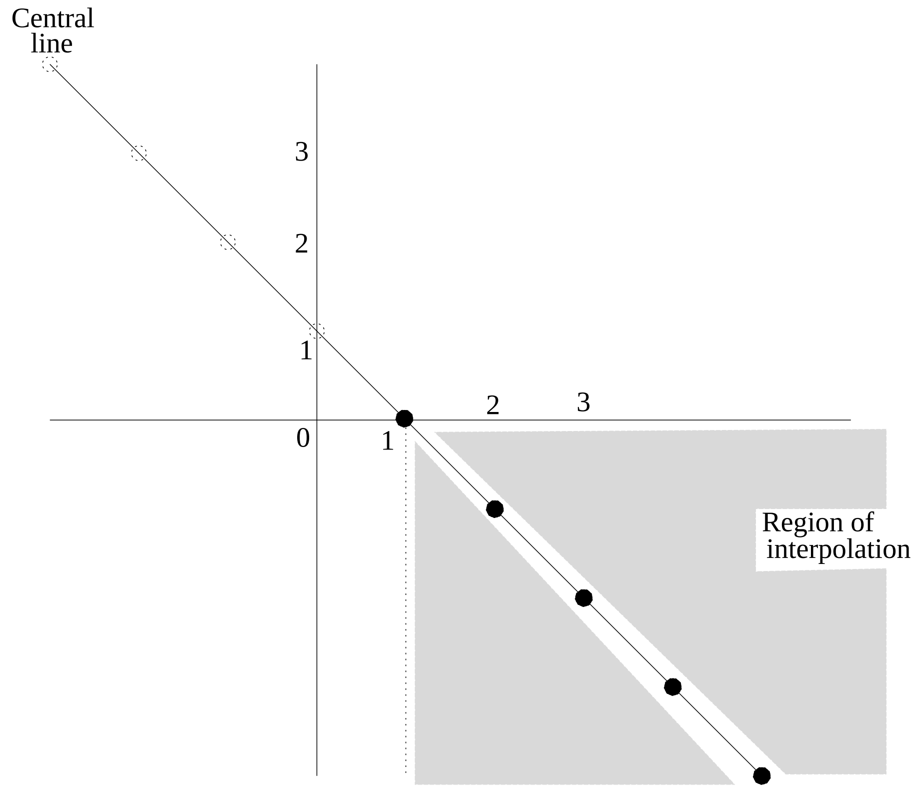

FIGURE 1. Critical infinity types for the Katz p-adic L-function

Hence

$$\nu|_{\mathrm{N}_{\mathbb{Q}}^{K}\mathbb{A}_K^\times} = \mathbf{N} \qquad \text{and} \qquad \varepsilon_\nu|_{\mathrm{N}_{\mathbb{Q}}^{K}\mathbb{A}_K^\times} = 1.$$

This implies that the central character of a self-dual character is either 1 or ε_K, where ε_K denotes the quadratic Dirichlet character corresponding to the extension $K/\mathbb{Q}$.

The reason for the terminology in Definition 2.2 is that the functional equation for the L-series $L(\nu^{-1}, s)$ relates $L(\nu^{-1}, s)$ to $L(\nu^{-1}, -s)$, and therefore $s = 0$ is the *central critical point* for this complex L-series. Also since the conductor of a self-dual character is clearly invariant under complex conjugation, we will assume henceforth that $\mathfrak{c} = \bar{\mathfrak{c}}$. Denote by $\Sigma_{\mathrm{sd}}(\mathfrak{c})$ the set of self-dual Hecke characters, and write

$$\Sigma_{\mathrm{sd}}^{(1)}(\mathfrak{c}) = \Sigma_{\mathrm{crit}}^{(1)}(\mathfrak{c}) \cap \Sigma_{\mathrm{sd}}(\mathfrak{c}), \qquad \Sigma_{\mathrm{sd}}^{(2)}(\mathfrak{c}) = \Sigma_{\mathrm{crit}}^{(2)}(\mathfrak{c}) \cap \Sigma_{\mathrm{sd}}(\mathfrak{c}).$$

In particular, the possible infinity types of characters in $\Sigma_{\mathrm{sd}}^{(2)}(\mathfrak{c})$ correspond to the black dots in Figure 1.

The following is merely a restatement of Proposition 2.1 for self-dual characters.

Proposition 2.3. *For all characters $\nu \in \Sigma_{\mathrm{sd}}^{(2)}(\mathfrak{c})$ of infinity type $(1 + j, -j)$ with $j \geq 0$,*

$$\frac{\mathscr{L}_p(\nu)}{\Omega_p(A)^{1+2j}} = (1 - \nu^{-1}(\bar{\mathfrak{p}}))^2 \times \frac{j!(2\pi)^j L_{\mathfrak{c}}(\nu^{-1}, 0)}{\sqrt{D}^j\, \Omega(A)^{1+2j}}. \tag{43}$$

The following theorem is the p-adic counterpart of Theorem 1.13.

Theorem 2.4. *Suppose that $\nu \in \Sigma_{\mathrm{sd}}^{(2)}(\mathfrak{c})$ is of infinity type $(1 + j, -j)$, with $j \geq 0$. Then*

$$\frac{\mathscr{L}_p(\nu)}{\Omega_p(\nu^*)} \text{ belongs to } E_\nu.$$

Proof. By definition of $\Omega_p(\nu^*)$,

$$\begin{aligned}\frac{\mathscr{L}_p(\nu)}{\Omega_p(\nu^*)} &= \frac{\mathscr{L}_p(\nu)}{\Omega_p(A)^{1+2j}} \times \frac{(2\pi i)^{-j}\Omega(A)^{1+2j}}{\Omega(\nu^*)} \\ &= \frac{L(\nu^{-1},0)}{(2\pi i)^{-j}\Omega(A)^{1+2j}} \times \frac{(2\pi i)^{-j}\Omega(A)^{1+2j}}{\Omega(\nu^*)} = \frac{L(\nu^{-1},0)}{\Omega(\nu^*)} \pmod{E_\nu^\times},\end{aligned}$$

where the penultimate equality follows from the interpolation property of the Katz p-adic L-function in Proposition 2.3. The result is now a direct consequence of Theorem 1.13 with $m = 0$. □

Theorem 2.4 expresses $\mathscr{L}_p(\nu)$ as an E_ν-multiple of a p-adic period $\Omega_p(\nu^*)$, when ν lies in the range $\Sigma_{\mathrm{sd}}^{(2)}(\mathfrak{c})$ of classical interpolation for the Katz p-adic L-function. Our main interest is in obtaining analogous results for certain critical characters in $\Sigma_{\mathrm{sd}}^{(1)}(\mathfrak{c})$. These characters are outside the range of interpolation, and so (43) does not directly say anything about these values. Our approach to studying them relies on a different kind of p-adic L-function, namely one attached to Rankin-Selberg L-series, which we define and study in the following section.

2.2. **p-adic Rankin L-series.** In this section, we consider p-adic L-functions obtained by interpolating special values of Rankin-Selberg L-series associated to modular forms and Hecke characters of a quadratic imaginary field K of odd discriminant. We briefly recall the definition of this p-adic L-function that is given in Sec. 5 of [BDP], referring the reader to loc.cit. for a more detailed description.

Let $S_k(\Gamma_0(N),\varepsilon)$ denote the space of cusp forms of weight $k = r+2$ and character ε on $\Gamma_0(N)$. Let $f \in S_k(\Gamma_0(N),\varepsilon)$ be a normalized newform and let E_f denote the subfield of $\mathbb{C}$ generated by its Fourier coefficients.

Definition 2.5. The pair (f, K) is said to satisfy the *Heegner hypothesis* if $\mathcal{O}_K$ contains a cyclic ideal of norm N, i.e., an integral ideal $\mathfrak{N}$ of $\mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} = \mathbb{Z}/N\mathbb{Z}$.

Assume from now on that (f, K) satisfies the Heegner hypothesis, and let $\mathfrak{N}$ be a fixed cyclic $\mathcal{O}_K$-ideal of norm N.

Definition 2.6. A Hecke character χ of K of infinity type (ℓ_1, ℓ_2) is said to be *central critical* for f if

$$\ell_1 + \ell_2 = k \qquad \text{and} \qquad \varepsilon_\chi = \varepsilon.$$

The reason for the terminology of Definition 2.6 is that when χ satisfies these hypotheses, the complex Rankin L-series $L(f, \chi^{-1}, s)$ is self-dual and $s = 0$ is its central (critical) point.

We now pick a rational integer c prime to pN and denote by $\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$ the set of central critical characters χ such that

$$\mathfrak{f}_\chi = \mathfrak{f}_1\mathfrak{f}_2 \qquad \text{with} \qquad \mathfrak{f}_1 \mid c, \quad \mathfrak{f}_2 \mid \mathfrak{N}.$$

The set $\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$ can be expressed as a disjoint union

$$\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon) = \Sigma_{\mathrm{cc}}^{(1)}(c,\mathfrak{N},\varepsilon) \cup \Sigma_{\mathrm{cc}}^{(2)}(c,\mathfrak{N},\varepsilon),$$

where $\Sigma_{\mathrm{cc}}^{(1)}(c,\mathfrak{N},\varepsilon)$ and $\Sigma_{\mathrm{cc}}^{(2)}(c,\mathfrak{N},\varepsilon)$ denote the subsets consisting of characters of infinity type $(k+j,-j)$ with $1-k \le j \le -1$ and $j \ge 0$ respectively. We shall also denote by $\hat{\Sigma}_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$ the completion of $\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$ relative to the p-adic compact open topology on $\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$ which is defined in Section 5.4 of [BDP]. The infinity types of Hecke characters in $\Sigma_{cc}^{(1)}(c,\mathfrak{N},\varepsilon)$ and $\Sigma_{cc}^{(2)}(c,\mathfrak{N},\varepsilon)$ correspond respectively to the white and black dots in the shaded regions in Figure 2. We note that the set $\Sigma_{\mathrm{cc}}^{(2)}(c,\mathfrak{N},\varepsilon)$ of classical central critical characters "of type 2" is dense in $\hat{\Sigma}_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$.

For all $\chi \in \Sigma_{\mathrm{cc}}^{(2)}(c,\mathfrak{N},\varepsilon)$ of infinity type $(k+j,-j)$ with $j \ge 0$, let $E_{f,\chi}$ denote the subfield of $\mathbb{C}$ generated by E_f and the values of χ, and let $E_{f,\chi,\varepsilon}$ be the field generated by $E_{f,\chi}$ and by the abelian extension of $\mathbb{Q}$ cut out by ε. The *algebraic part* of $L(f,\chi^{-1},0)$ is defined by the rule

$$L_{\mathrm{alg}}(f,\chi^{-1},0) := w(f,\chi)^{-1}C(f,\chi,c)\cdot\frac{L(f,\chi^{-1},0)}{\Omega(A)^{2(k+2j)}}, \tag{44}$$

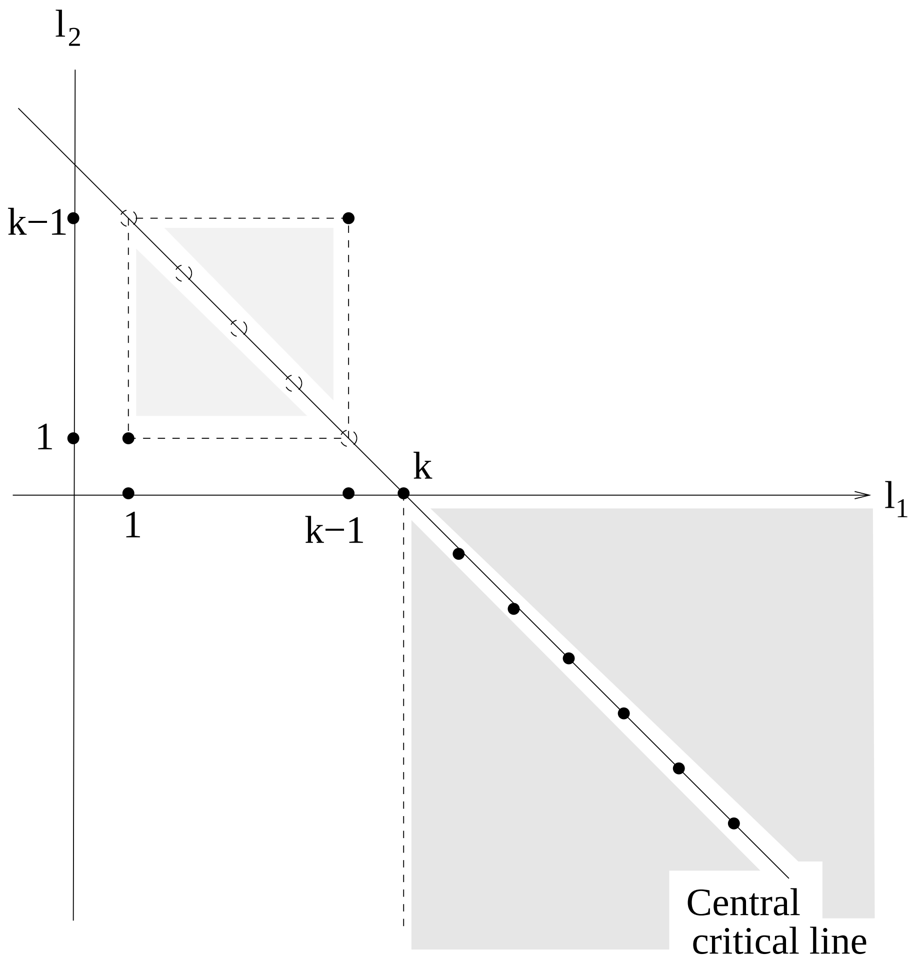

FIGURE 2. Critical infinity types for the p-adic Rankin L-function

where $w(f,\chi)^{-1} \in E_{f,\chi,\varepsilon}$ is the scalar (of complex norm 1) defined in equation (181) of [BDP] and $C(f,\chi,c)$ is the explicit real constant defined in Theorem 5.6 of [BDP]. For primitive χ, i.e. those such that $c \mid \mathfrak{f}_\chi$,

$$C(f,\chi,c) = \frac{\pi^{r+2j+1} j!(r+1+j)! w_K}{\sqrt{D}^{r+1+2j} c^{r+3+2j}} \prod_{q|c} (q - \varepsilon_K(q))^2, \tag{45}$$

where $w_K = \#\mathcal{O}_K^\times$ is the number of roots of unity in K. (Indeed, because χ is assumed to be primitive and D necessarily divides the conductor of ε_f, the set $S(f)$ in the statement of Theorem 5.6. of [BDP] is empty in our setting.)

Theorem 6.5 of [BDP] shows that $L_{\rm alg}(f,\chi^{-1},0)$ belongs to $\overline{\mathbb{Q}}$. By analogy with the definition of the Katz p-adic L-function, it is natural to attempt to p-adically interpolate the special values $L_{\rm alg}(f,\chi^{-1},0)$ as χ ranges over $\Sigma_{\rm cc}^{(2)}(c,\mathfrak{N},\varepsilon)$.

Proposition 2.7. *Let $\chi \mapsto L_p(f,\chi)$ be the function on $\Sigma_{\rm cc}^{(2)}(c,\mathfrak{N},\varepsilon)$ defined by*

$$L_p(f,\chi) := \Omega_p(A)^{2(k+2j)} (1 - \chi^{-1}(\bar{\mathfrak{p}}) a_p(f) + \chi^{-2}(\bar{\mathfrak{p}}) \varepsilon(p) p^{k-1})^2 L_{\rm alg}(f,\chi^{-1},0), \tag{46}$$

for all χ of infinity type $(k+j,-j)$ with $j \geq 0$. This function extends (uniquely) to a p-adically continuous function on $\hat{\Sigma}_{\rm cc}(c,\mathfrak{N},\varepsilon)$.

This statement is proved in Proposition 6.10 of [BDP].

The function $\chi \mapsto L_p(f,\chi)$ on $\hat{\Sigma}_{\rm cc}(c,\mathfrak{N},\varepsilon)$ will be referred to as the *p-adic Rankin L-function* attached to the cusp form f.

2.3. **A p-adic Gross-Zagier formula.** In this section, we specialise to the case where the newform f is of weight $k = 2$, and assume that χ is a finite order Hecke character of K satisfying

$$\chi \mathrm{N}_K \quad \text{belongs to} \quad \Sigma_{cc}^{(1)}(c, \mathfrak{N}, \varepsilon).$$

In particular, this character lies outside the domain $\Sigma_{cc}^{(2)}(c, \mathfrak{N}, \varepsilon)$ of classical interpolation defining $L_p(f, -)$. The p-adic Gross-Zagier formula alluded to in the title of this section relates the special value $L_p(f, \chi\mathrm{N}_K)$ to the formal group logarithm of a Heegner point on the modular abelian variety attached to f.

Recall that E_f and E_χ are the subfields of $\bar{\mathbb{Q}}$ generated, respectively, by the Fourier coefficients of f and the values of χ. Let $E_{f,\chi}$ denote the field generated by both E_f and E_χ, and let $T_f \subset E_f$ and $T_{f,\chi} \subset E_{F,\chi}$ denote their respective integer rings.

Let $\Gamma := \Gamma_\varepsilon(N) \subset \Gamma_0(N)$ be the subgroup attached to f, defined by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ such that } \varepsilon(a) = 1. \right\}, \tag{47}$$

The associated modular curve C has a model over $\mathbb{Q}$ obtained by realising C as the solution to a moduli problem, which we now describe. Given an abelian group G of exponent N, denote by G^* the set of elements of G of order N. This set of "primitive elements" is equipped with a natural free action by $(\mathbb{Z}/N\mathbb{Z})^\times$, which is transitive when G is cyclic.

Definition 2.8. A $\Gamma_\varepsilon(N)$-*level structure* on an elliptic curve E is a pair (C_N, t), where

(1) C_N is a cyclic subgroup scheme of E of order N,
(2) t is an orbit in C_N^* for the action of $\ker \varepsilon$.

If E is an elliptic curve defined over a field L, then the Γ-level structure on E is defined over the field L if C_N is a group scheme over L and t is fixed by the natural action of $\mathrm{Gal}(\bar{L}/L)$.

The curve C coarsely classifies the set of isomorphism classes of triples (E, C_N, t) where E is an elliptic curve and (C_N, t) is a Γ-level structure on E. When Γ is torsion-free (which occurs, for example, when ε is odd and N is divisible by a prime of the form $4n+3$ and a prime of the form $3n+2$) the curve C is even a fine moduli space; for any field L, one then has

$$C(L) = \{\text{Triples } (E, C_N, t) \text{ defined over } L\}/L\text{-isomorphism}.$$

Since the datum of t determines the associated cyclic group C_N, we sometimes drop the latter from the notation, and write (E, t) instead of (E, C_N, t) when convenient. The group scheme $A[\mathfrak{N}]$ of $\mathfrak{N}$-torsion points in A is a cyclic subgroup scheme of A of order N defined over F. A Γ-level structure on A of the form $(A[\mathfrak{N}], t)$ is said to be *of Heegner type* (associated to the ideal $\mathfrak{N}$).

The Eichler-Shimura construction associates to f an abelian variety B_f with endomorphism by an order in T_f, and a surjective morphism

$$\Phi_f : J_1(N) \longrightarrow B_f$$

of abelian varieties over $\mathbb{Q}$, called the *modular parametrisation*, which is well-defined up to a rational isogeny and factors through the natural projection $J_1(N) \longrightarrow J_\varepsilon(N)$. Let

$$\omega_f = 2\pi i f(\tau) d\tau \in \Omega^1(J_1(N)/E_f)$$

be the regular differential on $J_1(N)$ attached to f, and let $\omega_{B_f} \in \Omega^1(B_f/E_f)^{T_f}$ be the unique regular differential satisfying

$$\Phi_f^*(\omega_{B_f}) = \omega_f. \tag{48}$$

Let A be an elliptic curve with endomorphisms by the order $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ of conductor c, defined over the ring class field H_c of conductor c. The finite order Hecke character χ can be viewed as a character

$$\chi : \mathrm{Gal}(H_{c,\mathfrak{N}}/K) \longrightarrow E_\chi,$$

where $H_{c,\mathfrak{N}}$ is the finite abelian extension of the ring class field H_c generated by the $\mathfrak{N}$-torsion points of A. The pair $(A, A[\mathfrak{N}])$ corresponds to a point on $X_0(N)(H_c)$, and the triple $(A, A[\mathfrak{N}], t)$ (for any $(\Gamma_\varepsilon(N)$-level structure attached to $A[\mathfrak{N}]$) corresponds to a point in $X_1(N)(H_{c,\mathfrak{N}})$. Denote these points by $[A, A[\mathfrak{N}]]$ and $[A, A[\mathfrak{N}], t]$ respectively. Fix a cusp ∞ of $X_1(N)$ which is defined over $\mathbb{Q}$, and let

$$\Delta = [A, A[\mathfrak{N}], t] - (\infty) \quad \in J_1(N)(H_{c,\mathfrak{N}}). \tag{49}$$

To the pair (f,χ) we associate the Heegner point by letting $G = \mathrm{Gal}(H_{c,\mathfrak{N}}/K)$ and setting

$$P_f(\chi) := \sum_{\sigma\in G} \chi^{-1}(\sigma)\Phi_f(\Delta^\sigma) \quad \in B_f(H_{c,\mathfrak{N}}) \otimes_{T_f} E_{f,\chi}. \tag{50}$$

The embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$ that was fixed from the outset allows us to consider the formal group logarithm

$$\log_{\omega_{B_f}} : B_f(H_{c,\mathfrak{N}}) \longrightarrow \mathbb{C}_p.$$

We extend this function to $B_f(H_{c,\mathfrak{N}}) \otimes_{T_f} E_{f,\chi}$ by $E_{f,\chi}$-linearity.

Theorem 2.9. *With notations and assumptions as above,*

$$L_p(f,\chi N_K) = (1-\chi^{-1}(\bar{\mathfrak{p}})p^{-1}a_p(f) + \chi^{-2}(\bar{\mathfrak{p}})\varepsilon_f(p)p^{-1})^2 \log^2_{\omega_{B_f}}(P_f(\chi)).$$

Proof. Let

$$\mathscr{E}(f,\chi) := (1-\chi^{-1}(\bar{\mathfrak{p}})p^{-1}a_p(f) + \chi^{-2}(\bar{\mathfrak{p}})\varepsilon_f(p)p^{-1})^2 \quad \in \quad E^\times_{f,\chi}$$

be the Euler factor appearing in the statement of Theorem 2.9. Let F denote the p-adic completion of $H_{c,\mathfrak{N}}$. Theorem 6.13 of [BDP] in the case $k=2$ and $r=j=0$, with χ replaced by $\chi \mathrm{N}_K$, gives

$$L_p(f,\chi\mathrm{N}_K) = \mathscr{E}(f,\chi) \times \left(\sum_{\sigma\in G} \chi^{-1}(\mathfrak{a})\cdot \mathrm{AJ}_F(\Delta^\sigma)(\omega_f)\right)^2. \tag{51}$$

Note that in this context, the p-adic Abel-Jacobi map AJ_F that appears in (51) is related to the formal group logarithm by

$$\mathrm{AJ}_F(\Delta)(\omega_f) = \log_{\omega_f}(\Delta).$$

Therefore,

$$L_p(f,\chi\mathrm{N}_K) = \mathscr{E}(f,\chi) \left(\sum_{\sigma\in G} \chi^{-1}(\sigma)\log_{\omega_f}(\Delta^\sigma)\right)^2. \tag{52}$$

Theorem 2.9 follows from this formula and the fact that, by (48),

$$\log_{\omega_f}(\Delta) = \log_{\Phi_f^*(\omega_{B_f})}(\Delta) = \log_{\omega_{B_f}}(\Phi_f(\Delta)).$$

□

In the special case where f has rational Fourier coefficients and $\chi = 1$ is the trivial character, the abelian variety B_f is an elliptic curve quotient of $J_0(N)$ and the Heegner point $P_f := P_f(1)$ belongs to $B_f(K)$. Theorem 2.9 implies in this case that

$$L_p(f,\mathrm{N}_K) = \left(\frac{p+1-a_p(f)}{p}\right)^2 \log^2(P_f), \tag{53}$$

where $\log : B_f(K_\mathfrak{p}) \longrightarrow K_\mathfrak{p}$ is the formal group logarithm attached to a rational differential on $B_f/\mathbb{Q}$. Equation (53) exhibits a strong analogy with Theorem 1 of the Introduction, although it applies to arbitrary (modular) elliptic curves and not just elliptic curves with complex multiplication.

The remainder of Chapter 2 explains how Theorem 2.9 can in fact be used to prove Theorems 1 and 2 of the Introduction. The key to this proof is a relation between the Katz p-adic L-function of Section 2.1 and the p-adic Rankin L-function $L_p(f,\chi)$ of Section 2.2 in the special case where f is a theta series attached to a Hecke character of the imaginary quadratic field K. This explicit relation is described in the following section.

2.4. **A factorisation of the p-adic Rankin L-series.** This section focuses on the Rankin L-function $L_p(f,\chi)$ of f and K in the special case where f is a theta series associated to a Hecke character of the *same* imaginary quadratic field K.

More precisely, let ψ be a fixed Hecke character of K of infinity type $(k-1,0)$ with $k=r+2\geq 2$. Consider the associated theta series:

$$\theta_\psi := \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}} = \sum_{n=1}^{\infty} a_n(\theta_\psi) q^n,$$

where the first sum is taken over integral ideals of K. The Fourier coefficients of θ_ψ generate a number field E_{θ_ψ} which is clearly contained in E_ψ.

The following classical proposition is due to Hecke and Schoenberg. (Cf. [Ogg] or Sec. 3.2 of [Za]).

Proposition 2.10. *The theta series θ_ψ belongs to $S_k(\Gamma_0(N),\varepsilon)$, where*

(1) *The level N is equal to DM, with $M = N_{\mathbb{Q}}^K \mathfrak{f}_\psi$,*
(2) *The Nebentypus character ε is equal to $\varepsilon_K \varepsilon_\psi$.*

Recall that an ideal of $\mathcal{O}_K$ is said to be cyclic if the quotient of $\mathcal{O}_K$ by this ideal is cyclic (as a group under addition).

Lemma 2.11. *If the conductor $\mathfrak{f}_\psi$ of ψ is a cyclic ideal of norm M prime to D, then θ_ψ satisfies the Heegner hypothesis relative to K.*

Proof. In this case, the modular form θ_ψ is of level $N=DM$, by Proposition 2.10. But then the ideal

$$\mathfrak{N} := (\sqrt{-D})\mathfrak{f}_\psi. \tag{54}$$

is a cyclic ideal of K of norm N. □

We will assume from now on that the condition in Lemma 2.11 is satisfied. Furthermore, we will always take $\mathfrak{N}$ to be the ideal in (54).

The goal of this section is to factor the p-adic Rankin L-function $L_p(\theta_\psi,\chi)$ as a product of two Katz p-adic L-functions. As a preparation to stating the main result we record the following lemma:

Lemma 2.12. *Let χ be any character in $\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$.*

(1) *If χ belongs to $\Sigma_{\mathrm{cc}}^{(2)}(c,\mathfrak{N},\varepsilon)$, then $\psi^{-1}\chi$ belongs to $\Sigma_{\mathrm{sd}}^{(2)}(c)$ and $\psi^{*-1}\chi$ belong to $\Sigma_{\mathrm{sd}}^{(2)}(cN)$.*
(2) *If χ belongs to $\Sigma_{\mathrm{cc}}^{(1)}(c,\mathfrak{N},\varepsilon)$, then $\psi^{-1}\chi$ belongs to $\Sigma_{\mathrm{sd}}^{(1)}(c)$ and $\psi^{*-1}\chi$ belong to $\Sigma_{\mathrm{sd}}^{(2)}(cN)$.*

Proof. This lemma follows from a direct verification which is left to the reader. It should be noted that when χ is of type $(k+j,-j)$ then $\psi^{-1}\chi$ is of infinity type $(1+j,-j)$ and $\psi^{*-1}\chi$ is of infinity type $(1+(1+r+j),-(1+r+j))$. In particular, if χ lies on the edge of $\Sigma_{\mathrm{cc}}^{(1)}(c,\mathfrak{N},\varepsilon)$, then $\psi^{-1}\chi$ lies on the edge of $\Sigma_{\mathrm{sd}}^{(1)}(c)$, i.e., is of infinity type $(0,1)$. □

Theorem 2.13. *For all $\chi\in\Sigma_{\mathrm{cc}}(c,\mathfrak{N},\varepsilon)$,*

$$L_p(\theta_\psi,\chi) = \frac{w(\theta_\psi,\chi)^{-1} w_K}{2^{r+1+2j} c^{r+3+2j}} \prod_{q|c} (q-\varepsilon_K(q))^2 \times \mathscr{L}_p(\psi^{-1}\chi) \times \mathscr{L}_p(\psi^{*-1}\chi). \tag{55}$$

Proof. Since $\Sigma_{\mathrm{cc}}^{(2)}(c,\mathfrak{N},\varepsilon)$ is dense in $\hat{\Sigma}_{\mathrm{cc}}^{(2)}(c,\mathfrak{N},\varepsilon)$, it suffices to prove the formula for the characters χ in this range, where it follows directly from the interpolation properties defining the respective p-adic L-functions. More precisely, by (46),

$$\frac{L_p(\theta_\psi,\chi)}{\Omega_p(A)^{2(k+2j)}} \quad = \quad \left((1-\psi\chi^{-1}(\bar{\mathfrak{p}}))(1-\psi^*\chi^{-1}(\bar{\mathfrak{p}})\right)^2 L_{\mathrm{alg}}(\theta_\psi,\chi^{-1},0). \tag{56}$$

Let $\delta_c := \prod_{q|c}(q - \varepsilon_K(q))^2$. By the definition of $L_{\mathrm{alg}}(\theta_\psi, \chi^{-1}, 0)$ given in (44) and (45),

$$\begin{aligned} L_{\mathrm{alg}}(\theta_\psi, \chi^{-1}, 0) &= w(\theta_\psi, \chi)^{-1} C(\theta_\psi, \chi, c) \frac{L(\theta_\psi, \chi^{-1}, 0)}{\Omega(A)^{2(k+2j)}} \\ &= w(\theta_\psi, \chi)^{-1} w_K \delta_c \frac{\pi^{r+2j+1} j!(1+r+j)!}{\sqrt{D}^{1+r+2j} c^{3+r+2j}} \times \frac{L(\psi\chi^{-1}, 0) L(\psi^*\chi^{-1}, 0)}{\Omega(A)^{2(k+2j)}} \\ &= \frac{w(\theta_\psi, \chi)^{-1} w_K \delta_c}{c^{r+3+2j}} \left(\frac{j! \pi^j L(\psi\chi^{-1}, 0)}{\sqrt{D}^j \Omega(A)^{1+2j}} \right) \times \left(\frac{(1+r+j)! \pi^{1+r+j} L(\psi^*\chi^{-1}, 0)}{\sqrt{D}^{1+r+j} \Omega(A)^{1+2(1+r+j)}} \right). \end{aligned} \tag{57}$$

Combining (56) and (57) with the interpolation property of the Katz p-adic L-function given in Proposition 2.3, we obtain

$$\frac{L_p(\theta_\psi, \chi)}{\Omega_p(A)^{2(k+2j)}} = \frac{w(\theta_\psi, \chi)^{-1} w_K \delta_c}{2^{r+1+2j} c^{r+3+2j}} \times \frac{\mathscr{L}_p(\psi^{-1}\chi)}{\Omega_p(A)^{1+2j}} \times \frac{\mathscr{L}_p(\psi^{*-1}\chi)}{\Omega_p(A)^{1+2(1+r+j)}}. \tag{58}$$

Clearing the powers of $\Omega_p(A)$ on both sides gives the desired result. □

The Nebentypus character ε can be viewed as a finite order Galois character of $G_{\mathbb{Q}}$. Recall that $E_{\psi,\chi,\varepsilon}$ denotes the smallest extension of $E_{\psi,\chi}$ containing the field through which this character factors.

Corollary 2.14. *For all $\chi \in \Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$, with $\mathfrak{c} = cN$,*

$$L_p(\theta_\psi, \chi) = \mathscr{L}_p(\psi^{-1}\chi) \times \mathscr{L}_p(\psi^{*-1}\chi) \pmod{E^\times_{\psi,\chi,\varepsilon}}.$$

Proof. This follows from Theorem 2.13 in light of the fact that the constant that appears on the right hand side of (55) belongs to $E^\times_{\psi,\chi,\varepsilon}$. □

2.5. **Proof of Rubin's Theorem.** The goal of this section is to prove Theorem 2 of the Introduction. Let $\nu \in \Sigma_{\mathrm{sd}}(\mathfrak{c})$ be a Hecke character of infinity type $(1, 0)$ satisfying conditions (6) and (7) of the Introduction.

Definition 2.15. A pair (ψ, χ) of Hecke characters is said to be *good* for ν if it satisfies the following conditions.

1. The character ψ is of type $(1, 0)$ and satisfies the Heegner hypothesis, so that the associated theta series θ_ψ belongs to $S_2(\Gamma_0(N), \varepsilon)$ where $(N) = \mathfrak{N}\bar{\mathfrak{N}}$ is the norm of a cyclic ideal $\mathfrak{N}$ of $\mathcal{O}_K$ and ε is an appropriate even Dirichlet character of modulus N.
2. The character χ is of finite order, and $\chi \mathrm{N}_K$ belongs to $\Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$.
3. The character $\psi\chi^{-1}$ is equal to ν, i.e., $\psi^{-1}\chi\mathrm{N}_K = \nu^*$.
4. The classical L-value $L(\psi^*\chi^{-1}\mathrm{N}_K^{-1}, 0)$ is non-zero, i.e., $\mathscr{L}_p(\psi^{*-1}\chi\mathrm{N}_K) \neq 0$.

The modular abelian variety B_{θ_ψ} attached to ψ is a CM abelian variety in the sense of Definition 1.5. Hence it is K-isogenous to the CM abelian variety B_ψ constructed in Section 1.2. In particular, the modular parametrisation $\Phi_\psi := \Phi_{\theta_\psi}$ can be viewed as a surjective morphism of abelian varieties over K

$$\Phi_\psi : J_1(N) \longrightarrow B_\psi. \tag{59}$$

Given a good pair (ψ, χ), recall the Heegner divisor $\Delta \in J_1(N)(H_{c,\mathfrak{N}})$ that was constructed in Section 2.3, and the Heegner point

$$P_\psi(\chi) := P_{\theta_\psi}(\chi) = \sum_{\sigma \in G} \chi^{-1}(\sigma) \Phi_\psi(\Delta^\sigma) \quad \in B_\psi(H_\chi) \otimes_{T_\psi} E_{\psi,\chi} \tag{60}$$

that was defined in equation (50) of that section with $f = \theta_\psi$. Recall also that ω_ψ is an E_ψ-vector space generator of $\Omega^1(B_\psi/E_\psi)^{T_\psi}$. Viewing the point $P_\psi(\chi)$ as a formal linear combination of elements of $B_\psi(H_\chi)$ with coefficients in $E_{\psi,\chi}$, we define the expression $\log_{\omega_\psi}(P_\psi(\chi))$ by E_χ-linearity.

In the rest of this section, we will denote by $E'_{\psi,\chi}$ the subfield of $\bar{\mathbb{Q}}$ generated by E_ψ, E_χ, and the abelian extension H'_χ of K cut out by the finite order characters χ and χ^*. The motivation for singling out good pairs for a special definition lies in the following proposition.

Proposition 2.16. *For any pair (ψ, χ) which is good for ν,*

$$\mathscr{L}_p(\nu^*) = \Omega_p(\psi^*\chi^{-1}) \log^2_{\omega_\psi}(P_\psi(\chi)) \pmod{(E'_{\psi,\chi})^\times}, \tag{61}$$

where $\Omega_p(\xi)$ is the p-adic period attached to a critical Hecke character ξ in Definition 1.14.

Proof. By Theorem 2.9 applied to $f = \theta_\psi$,

$$L_p(\theta_\psi, \chi \mathrm{N}_K) = \log^2_{\omega_\psi}(P_\psi(\chi)) \pmod{E^\times_{\psi,\chi}}. \tag{62}$$

On the other hand, since $E'_{\psi,\chi}$ contains $E_{\psi,\chi,\varepsilon}$, Corollary 2.14 implies that

$$\begin{aligned} L_p(\theta_\psi, \chi\mathrm{N}_K) &= \mathscr{L}_p(\psi^{-1}\chi\mathrm{N}_K)\mathscr{L}_p(\psi^{*-1}\chi\mathrm{N}_K) \pmod{(E'_{\psi,\chi})^\times} \\ &= \mathscr{L}_p(\nu^*)\mathscr{L}_p(\psi\chi^{*-1}) \pmod{(E'_{\psi,\chi})^\times}, \end{aligned} \tag{63}$$

where the second equality follows from condition 3 in the definition of a good pair. The character $\psi\chi^{*-1}$ lies in the range $\Sigma^{(2)}_{\mathrm{sd}}(\mathfrak{c})$ of classical interpolation for the Katz L-function, and $\mathscr{L}_p(\psi^{*-1}\chi\mathrm{N}_K) = \mathscr{L}_p(\nu\chi/\chi^*)$ is non-zero by condition 4 in the definition of a good pair. Therefore, by Theorem 2.4,

$$\mathscr{L}_p(\psi\chi^{*-1}) = \Omega_p(\psi^*\chi^{-1}) \pmod{E^\times_{\psi,\chi}}. \tag{64}$$

Proposition 2.16 follows by combining the equalities in (62) and (63) and using (64). □

To go further, we will analyse the expressions occurring in the right hand side of (61) and relate them to quantities depending solely on ν and not on the good pair (ψ, χ).

Lemma 2.17. *For all good pairs (ψ, χ) attached to ν,*

$$\Omega_p(\psi^*\chi^{-1}) = \Omega_p(\nu^*) \pmod{(E'_{\psi,\chi})^\times}.$$

Proof. Condition 3 in the definition of a good pair implies that $\psi^*\chi^{-1} = \nu^*\chi^*/\chi$. The lemma therefore follows from Lemma 1.15, with ψ replaced by ν^* and χ by the finite order character χ^*/χ, which factors through $\mathrm{Gal}(H'_\chi/K)$. □

It will be useful to view the point $P_\psi(\chi)$ appearing in (61) as an element of $B^0_{\psi,\chi}(H_{c,\mathfrak{N}})$ or as a K-rational point on the abelian variety $B_{\psi,\chi}$ that was introduced in Section 1.2. More precisely, after setting

$$\mathbf{P}_\psi(\chi) := \sum_{\sigma \in G} \Phi_\psi(\Delta^\sigma) \otimes \chi^{-1}(\sigma) \quad \in B_\psi(\bar{K}) \otimes_{T_\psi} T_{\psi,\chi} = B^0_{\psi,\chi}(\bar{K}), \tag{65}$$

we observe that, for all $\tau \in \mathrm{Gal}(\bar{K}/K)$,

$$\begin{aligned} \tau *_0 \mathbf{P}_\psi(\chi) &= \sum_{\sigma\in G} \Phi_\psi(\Delta^{\tau\sigma}) \otimes \chi^{-1}(\sigma) \\ &= \sum_{\sigma\in G} \Phi_\psi(\Delta^{\sigma}) \otimes \chi^{-1}(\sigma\tau^{-1}) = \mathbf{P}_\psi(\chi)\chi(\tau). \end{aligned}$$

The point $\mathbf{P}_\psi(\chi)$ therefore belongs to $B_{\psi,\chi}(K)$ by (33).

Recall the differentials ω_ψ and $\omega_{\psi,\chi} \in \Omega^1(B_{\psi,\chi}/E_{\psi,\chi})^{T_{\psi,\chi}}$.

Lemma 2.18. *For all good pairs (ψ, χ) attached to $\nu = \psi\chi^{-1}$,*

$$\log_{\omega_\psi}(P_\psi(\chi)) = \log_{\omega_{\psi,\chi}}(\mathbf{P}_\psi(\chi)).$$

Proof. Let $G = \mathrm{Gal}(H_{c,\mathfrak{N}}/K)$ and let $P = \Phi_\psi(\Delta)$. By definition,

$$\begin{aligned} \log_{\omega_\psi}(P_\psi(\chi)) &= \sum_{\sigma\in G} \chi(\sigma)^{-1} \log_{\omega_\psi}(P^\sigma) = \sum_{\sigma\in G} \chi(\sigma)^{-1} \log_{i^*(\omega_{\psi,\chi})}(P^\sigma) \\ &= \sum_{\sigma\in G} \chi(\sigma)^{-1} \log_{\omega_{\psi,\chi}}(P^\sigma \otimes 1) = \sum_{\sigma\in G} \log_{\chi(\sigma)^{-1}\omega_{\psi,\chi}}(P^\sigma \otimes 1) \\ &= \sum_{\sigma\in G} \log_{\omega_{\psi,\chi}}(P^\sigma \otimes \chi(\sigma)^{-1}) = \log_{\omega_{\psi,\chi}}\left(\sum_{\sigma\in G} P^\sigma \otimes \chi(\sigma)^{-1}\right). \end{aligned}$$

This completes the proof. □

Lemma 2.19. *There exists $P_\nu \in B_\nu(K)$ and $\omega_\nu \in \Omega^1(B_\nu/E_\nu)^{T_\nu}$ such that*

$$\log_{\omega_{\psi,\chi}}(\mathbf{P}_\psi(\chi)) = \log_{\omega_\nu}(P_\nu) \pmod{(E'_{\psi,\chi})^\times}.$$

Proof. Recall from Lemma 1.10 that there is a K-rational isogeny

$$B_\nu \otimes_{T_\nu} T_{\psi,\chi} \longrightarrow B_{\psi,\chi}.$$

Composing it with the natural morphism $B_\nu \longrightarrow B_\nu \otimes_{T_\nu} T_{\psi,\chi}$, we obtain a T_ν-equivariant morphism $j : B_\nu \longrightarrow B_{\psi,\chi}$ defined over K with finite kernel. The fact that $L(\nu, s)$ has a simple zero at $s = 1$ implies that $B_\nu(K) \otimes \mathbb{Q}$ is one-dimensional over E_ν, and therefore that $B_{\psi,\chi}(K) \otimes \mathbb{Q}$ is one-dimensional over $E_{\psi,\chi}$. In particular, if P_ν is any generator of $B_\nu(K) \otimes \mathbb{Q}$, we may write

$$\mathbf{P}_\psi(\chi) = \lambda j(P_\nu)$$

for some non-zero scalar $\lambda \in E_{\psi,\chi}^\times$. But letting

$$\omega_\nu = j^*(\omega_{\psi,\chi}) \in \Omega^1(B_\nu/E'_{\psi,\chi})^{T_\nu},$$

we have

$$\begin{aligned} \log_{\omega_{\psi,\chi}}(\mathbf{P}_\psi(\chi)) &= \log_{\omega_{\psi,\chi}}(\lambda j(P_\nu)) = \log_{\lambda^*\omega_{\psi,\chi}}(j(P_\nu)) = \lambda \log_{\omega_{\psi,\chi}}(j(P_\nu)) \\ &= \lambda \log_{j^*\omega_{\psi,\chi}}(P_\nu) = \lambda \log_{\omega_\nu}(P_\nu). \end{aligned}$$

The lemma now follows after multiplying ω_ν by an appropriate scalar in $(E'_{\psi,\chi})^\times$ so that it belongs to $\Omega^1(B_\nu/E_\nu)^{T_\nu}$. □

Proposition 2.20. *There exists $\omega_\nu \in \Omega^1(B_\nu/E_\nu)^{T_\nu}$ and $P_\nu \in B_\nu(K)$ such that*

$$\mathscr{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1} \log^2_{\omega_\nu}(P_\nu) \pmod{(E'_{\psi,\chi})^\times}, \tag{66}$$

for all good pairs (ψ, χ) attached to ν.

Proof. This follows after invoking Lemmas 2.17, 2.18 and 2.19 to rewrite the expression appearing in the right hand side of Proposition 2.16. □

While Proposition 2.20 brings us close to Theorem 2 of the Introduction, it is somewhat more vague in that both sides of the purported equality may differ *a priori* by a non-zero element of the typically larger field $E'_{\psi,\chi}$. The alert reader will also notice that this proposition is potentially vacuous for now, because the existence of a good pair for ν has not yet been established! The next proposition repairs this omission, and directly implies Theorem 2 of the Introduction.

Proposition 2.21. *The set S_ν of pairs (ψ, χ) that are good for ν is non-empty. Furthermore,*

$$\bigcap_{(\psi,\chi)\in S_\nu} E'_{\psi,\chi} = E_\nu. \tag{67}$$

The proof of Proposition 2.21 rests crucially on a non-vanishing result of Rohrlich and Greenberg ([Ro], [Gre]) for the central critical values of Hecke L-series. In order to state it, we fix a rational prime ℓ which is split in K and let

$$K_\infty^- = \cup_{n\geq 0} K_n^-$$

be the so-called *anti-cyclotomic $\mathbb{Z}_\ell$ extension* of K; it is the unique $\mathbb{Z}_\ell$-extensions of K which is Galois over $\mathbb{Q}$ and for which $\mathrm{Gal}(K_\infty^-/\mathbb{Q}) = \mathbb{Z}_\ell \rtimes (\mathbb{Z}/2\mathbb{Z})$ is a generalised dihedral group.

Lemma 2.22 (Greenberg, Rohrlich)**.** *Let ψ_0 be a self-dual Hecke character of K of infinity type* $(1, 0)$*. Assume that the sign w_{ψ_0} in the functional equation of $L(\psi_0, s)$ is equal to* 1*. Then there are infinitely many finite-order characters χ of* $\mathrm{Gal}(K_\infty^-/K)$ *for which* $L(\psi_0\chi, 1) \neq 0$*.*

Proof. In light of the hypothesis that $w_{\psi_0} = 1$, Theorem 1 of [Gre] implies that the Katz p-adic L-function (with $p = \ell$) does not vanish identically on any open ℓ-adic neighbourhood of ψ_0 in $\Sigma_{\mathrm{sd}}(c)$, with $c = \mathrm{cond}(\psi_0)$, say. (Cf. the discussion in the first paragraph of the proof of Proposition 1 on p. 93 of [Gre].) If U is any sufficiently small such neighbourhood, then

1. The restriction to U of the Katz p-adic L-function is described by a power series with p-adically bounded cofficients, and therefore admits only finitely many zeros by the Weierstrass preparation theorem.
2. The region U contains contains a dense subset of points of the form $\psi_0\chi$, where χ is a finite order character of $\mathrm{Gal}(K_\infty^-/K)$.

Lemma 2.22 follows directly from these two facts. □

Proof of Proposition 2.21. Let $\bar{S}_\nu \supset S_\nu$ be the set of pairs satisfying conditions 1-3 in the definition of a good pair, but without necessarily requiring the more subtle fourth condition. The proof of Proposition 2.21 will be broken down into four steps.

Step 1. The set $\bar{S}_\nu$ is non-empty.

To see this, let ψ be any Hecke character (of conductor $\mathfrak{m}$ prime to the conductor of ν and ν^*, say) satisfying condition 1 in Definition 2.15. Setting $\chi = \psi\nu^*\mathrm{N}_K^{-1}$, the pair (ψ, χ) satisfies conditions 1 and 3 by construction. Furthermore, the character $\chi\mathrm{N}_K = \psi\nu^*$ is of type $(1, 1)$ and its central character is equal to

$$\varepsilon_\chi = \varepsilon_\psi\varepsilon_{\nu^*} = \mathrm{N}^2\varepsilon_M\varepsilon_K = \mathrm{N}^2\varepsilon,$$

where ε is the nebentype character attached to θ_ψ. It follows that the character $\chi\mathrm{N}_K$ belongs to $\Sigma_{\mathrm{cc}}(cN, \mathfrak{N}, \varepsilon)$ with $\mathfrak{N} = \mathfrak{m} \cdot \sqrt{-D}$. (The integer c is related to the conductor of ν.) Therefore, the pair (ψ, χ) belongs to $\bar{S}_\nu$.

Step 2. Given $(\psi, \chi) \in \bar{S}_\nu$, there exist (ψ_1, χ_1) and $(\psi_2, \chi_2) \in S_\nu$ with $E'_{\psi_1,\chi_1} \cap E'_{\psi_2,\chi_2} \subset E'_{\psi,\chi}$.

To see this, let $\ell = \lambda\bar{\lambda}$ be a rational prime which splits in K and is relatively prime to the class number of K and the conductors of ψ and χ. For such a prime, let

$$K_\infty = \cup_{n\geq 0} K_n, \qquad K'_\infty = \cup_{n\geq 0} K'_n$$

be the unique $\mathbb{Z}_\ell$-extensions of K which are unramified outside of λ and $\bar{\lambda}$ respectively, with $[K_n : K] = \ell^n$ and likewise for K'_n. The condition that ℓ does not divide the class number of K implies that the fields K_n and K'_n are totally ramified at λ and $\bar{\lambda}$ respectively. If α is any character of $\mathrm{Gal}(K_\infty/K)$, the pair $(\psi_1, \chi_1) := (\psi\alpha, \chi\alpha)$ still belongs to $\bar{S}_\nu$. (For instance, $\psi\alpha$ satisfies the first condition in Definition 2.15, with $\mathfrak{N}$ replaced by $\mathfrak{N}\lambda^n$ and N by $N\ell^n$, for a suitable $n \geq 0$.) Furthermore,

$$L(\psi_1^*\chi_1^{-1}\mathrm{N}_K^{-1}, 0) = L(\psi^*\chi^{-1}\mathrm{N}_K^{-1} \cdot (\alpha^*/\alpha), 0). \tag{68}$$

The character α^*/α is an anticyclotomic character of K of ℓ-power order and conductor, and all such characters can be obtained by choosing α appropriately. The fact that (ψ, χ) is a good pair implies that the sign $w_{\psi^*\chi^{-1}}$ is equal to 1. Hence, by Lemma 2.22, there exists a choice of α for which the L-value appearing on the right of (68) is non-vanishing. The corresponding pair (ψ_1, χ_1) belongs to S_ν and satisfies

$$E'_{\psi_1,\chi_1} \subset E'_{\psi,\chi}\mathbb{Q}(\zeta_{\ell^n})K_nK'_n$$

for some n. Repeating the same construction with a different rational prime in the place of ℓ yields a second pair $(\psi_2, \chi_2) \in S_\nu$. Since the extension $E'_{\psi_1,\chi_1}/E_{\psi,\chi}$ is totally ramified at the primes above ℓ, while E'_{ψ_2,χ_2} is unramified at these primes, it follows that $E'_{\psi_1,\chi_1} \cap E'_{\psi_2,\chi_2}$ is contained in $E'_{\psi,\chi}$, as was to be shown.

Thanks to Step 2, we are reduced to showing that

$$\bigcap_{(\psi,\chi)\in\bar{S}_\nu} E'_{\psi,\chi} = E_\nu. \tag{69}$$

The next step shows that the fields $E'_{\psi,\chi}$ can be replaced by $E_{\psi,\chi}$ in this equality.

Step 3. For all $(\psi, \chi) \in \bar{S}_\nu$, there exists a finite order character α of G_K such that the pair $(\psi\alpha, \chi\alpha)$ belongs to $\bar{S}_\nu$ and

$$E'_{\psi,\chi} \cap E'_{\psi\alpha,\chi\alpha} \subset E_{\psi,\chi}.$$

To see this, note that the finite order character χ has cyclic image, isomorphic to $\mathbb{Z}/n\mathbb{Z}$ say. Step 3 is completed by choosing a character α of order n in such a way that $(\psi\alpha, \chi\alpha)$ belongs to $\bar{S}_\nu$ and

$$(H_\chi H_{\chi^*}) \cap (H_{\chi\alpha}H_{\chi\alpha^*}) = K.$$

For instance, one could choose α to be totally ramified at a prime λ of norm a rational prime ℓ which is relatively prime the conductors of ψ and χ.

Step 4. We are now reduced to showing

$$\bigcap_{(\psi,\chi)\in\bar{S}_\nu} E_{\psi,\chi} = E_\nu. \tag{70}$$

We will do this by showing

$$\text{There exists a pair } (\psi,\chi)\in\bar{S}_\nu \text{ such that } E_{\psi,\chi}=E_\nu. \tag{71}$$

We begin by choosing an ideal $\mathfrak{m}_0$ of $\mathcal{O}_K$ with the property that $\mathcal{O}_K/\mathfrak{m}_0=\mathbb{Z}/M\mathbb{Z}$ is cyclic, and an odd quadratic Dirichlet character ε_M of conductor dividing M. Let ψ_0 be any Hecke character satisfying

$$\psi_0((a))=\varepsilon_M(a \bmod \mathfrak{m}_0)a$$

on principal ideals (a) of K. Such a ψ_0 satisfies condition 1 in Definition 2.15, and therefore, after letting χ_0 be the finite order character satisfying

$$\nu^*=\psi_0^{-1}\chi_0\mathrm{N}_K,$$

it follows that (ψ_0,χ_0) belongs to $\bar{S}_\nu$. Furthermore, the restriction of ψ_0 to the group of principal ideals of K takes values in K, and therefore

$$\chi_0(\sigma)\in E_\nu, \qquad \text{for all } \sigma\in G_H:=\mathrm{Gal}(\bar{K}/H). \tag{72}$$

The character ψ_0 itself takes values in a CM field of degree $[H:K]$, denoted E_0, which need not be contained in E_ν in general. To remedy this problem, let H_0 be the abelian extension of the Hilbert class field H cut out by the character χ_0. Next, let H_0' be any abelian extension of K containing H such that

(1) There is an isomorphism $u:\mathrm{Gal}(H_0'/K)\longrightarrow\mathrm{Gal}(H_0/K)$ of abstract groups such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Gal}(H_0'/H) \arrow[r] \arrow[d, dotted] & \mathrm{Gal}(H_0'/K) \arrow[r] \arrow[d, dotted] & \mathrm{Gal}(H/K) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Gal}(H_0/H) \arrow[r] & \mathrm{Gal}(H_0/K) \arrow[r] & \mathrm{Gal}(H/K) \arrow[r] & 0 \end{tikzcd} \tag{73}$$

commutes, where the dotted arrows indicate the isomorphisms induced by u and the other arrows are the canonical maps of Galois theory.

(2) The relative discriminant of H_0' over K is relatively prime to its conjugate (and therefore to the discriminant of K, in particular).

If the bottom exact sequence of groups in (73) is split, then the extension H_0' is readily produced, using class field theory. To handle the general case, we follow an approach that is suggested by the proof of Prop. 2.1.7 in [Se]. Let $\tilde{\Phi}:=\mathrm{Gal}(H_0/K)$ and let $\psi:G_K\longrightarrow\tilde{\Phi}$ be the homomorphism attached to the extension H_0. Since H is everywhere unramified over K, the restriction ψ_v of ψ to a decomposition group at the prime v of K maps the inertia subgroup I_v to $\mathfrak{C}:=\mathrm{Gal}(H_0/H)$. Let S be any finite set of primes of K which generates the class group of K and satisfies $v\in S\Rightarrow\bar{v}\notin S$. We claim that there is a homomorphism $\epsilon:G_K\longrightarrow\mathfrak{C}$ satisfying

$$\epsilon_v=\psi_v \quad \text{on } I_v, \qquad \text{for all } v\notin S.$$

When the ground field K is replaced by $\mathbb{Q}$ and $S=\emptyset$, this is shown in Lemma 2.1.6 of [Se], and the proof given there adapts readily to our situation. The field H_0' can now be obtained as the fixed field of the kernel of the homomorphism $\psi\epsilon^{-1}$. With the extension H_0' in hand, let $\alpha:\mathrm{Gal}(H_0'/K)\longrightarrow E_\chi^\times$ be the finite order Hecke character given by

$$\alpha(\sigma)=\chi_0(u(\sigma))^{-1},$$

and set $(\psi,\chi)=(\psi_0\alpha,\chi_0\alpha)$. By construction, (ψ,χ) belongs to $\bar{S}_\nu$. We claim that χ and ψ take values in E_ν. Since $\nu^*=\psi^{-1}\chi\mathrm{N}_K$, it is enough to prove this statement for χ. But observe that, for all $\mathfrak{a}\in I_\mathfrak{f}$ (where $\mathfrak{f}$ is divisible by the conductor of χ_0, χ, and ψ), we have

$$\chi(\mathfrak{a})=\chi_0(\sigma_\mathfrak{a})/\chi_0(u(\sigma_\mathfrak{a}))=\chi_0(\sigma_\mathfrak{a}u(\sigma_\mathfrak{a})^{-1}).$$

But the element $\sigma_\mathfrak{a}u(\sigma_\mathfrak{a})^{-1}$ belongs to $\mathrm{Gal}(H_0/H)$ by construction, and hence $\chi_0(\sigma_\mathfrak{a}^{-1}u(\sigma_\mathfrak{a}))$ belongs to E_ν by (72). It follows that ψ and χ are E_ν–valued, and therefore $E_{\psi,\chi}=E_\nu$, as claimed in (71). □

2.6. **Elliptic curves with complex multiplication.** Theorem 2 of the Introduction admits an alternate formulation involving algebraic points on elliptic curves with complex multiplication rather than K-rational points on the CM abelian varieties B_ν of Theorem 1.6. The goal of this section is to describe this variant.

We begin by reviewing the explicit construction of B_ν in terms of CM elliptic curves. The reader is referred to §4 of [GS], whose treatment we largely follow, for a more detailed exposition.

Let F be any abelian extension of K for which

$$\nu_F := \nu \circ \mathrm{N}_{F/K} \tag{74}$$

becomes K-valued, and let $\mathfrak{f}$ be its conductor. The ideal $\mathfrak{f}$ is divisible by $\mathfrak{f}_\nu$, and there exists an elliptic curve A/F with complex multiplication by $\mathcal{O}_K$ whose associated Grossencharacter is ν_F. (Cf. Thm. 6 of [Shi] and its corollary on p. 512.) Let

$$B := \mathrm{Res}_{F/K}(A). \tag{75}$$

It is an abelian variety over K of dimension $d := [F : K]$. Let $G := \mathrm{Gal}(F/K) = \hom_K(F, \bar{\mathbb{Q}})$, where the natural identification between these two sets arises from the distinguished embedding of F into $\bar{\mathbb{Q}}$ that was fixed from the outset. By definition of the restriction of scalars functor, there are natural isomorphisms

$$B_{/F} = \prod_{\sigma\in G} A^\sigma, \qquad B(\bar{K}) = A(\bar{K} \otimes_K F) = \prod_{\sigma\in G} A^\sigma(\bar{K})$$

of algebraic groups over F and abelian groups respectively. In particular, a point of $B(\bar{K})$ is described by a d-tuple $(P_\tau)_{\tau\in G}$, with $P_\tau \in A^\tau(\bar{K})$. Relative to this identification, the Galois group G_K acts on $B(\bar{K})$ on the left by the rule

$$\xi(P_\tau)_\tau = (\xi P_\tau)_{\xi\tau}, \quad \text{for all } \xi \in G_K. \tag{76}$$

Consider the "twisted group ring"

$$T := \oplus_{\sigma\in G} \mathrm{Hom}_F(A, A^\sigma) = \left\{ \sum_{\sigma\in G} a_\sigma \sigma, \text{ with } a_\sigma \in \mathrm{Hom}_F(A, A^\sigma) \right\}, \tag{77}$$

with multiplication given by

$$(a_\sigma \sigma)(a_\tau \tau) = a_\sigma a_\tau^\sigma \sigma\tau, \tag{78}$$

where the isogeny a_τ^σ belongs to $\hom_F(A^\sigma, A^{\sigma\tau})$ and the composition of isogenies in (78) is to be taken from left to right. The right action of T on $B(\bar{K})$ defined by

$$(P_\tau)_\tau * (a_\sigma\sigma) := (a_\sigma^\tau(P_\tau))_{\tau\sigma} \tag{79}$$

commutes with the Galois action described in (76), and corresponds to a natural inclusion $T \hookrightarrow \mathrm{End}_K(B)$. The K-algebra $E := T \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to a finite product

$$E = \prod_i E_i$$

of CM fields, and $\dim_K(E) = \dim(B)$. Therefore, the pair (B, E) is a CM abelian variety in the sense of Definition 1.5. The compatible system of ℓ-adic Galois representations attached to (B, E) corresponds to an E-valued algebraic Hecke character $\tilde{\nu}$ in the sense of Definition 1.3, satisfying the relation

$$\sigma_{\mathfrak{a}}(P) = P * \tilde{\nu}(\mathfrak{a}), \quad \text{for all } \mathfrak{a} \in I_{\mathfrak{f}\ell} \text{ and } P \in B(\bar{K})_{\ell^\infty}, \tag{80}$$

where $\sigma_{\mathfrak{a}} \in G_K^{\mathrm{ab}}$ denotes as before the Artin symbol attached to $\mathfrak{a} \in I_{\mathfrak{f}\ell}$.

The element $\tilde{\nu}(\mathfrak{a}) \in T$ is of the form $\varphi_{\mathfrak{a}}\sigma_{\mathfrak{a}}$, where

$$\varphi_{\mathfrak{a}} : A \to A^{\sigma_{\mathfrak{a}}}, \tag{81}$$

is an isogeny of degree $\mathrm{N}\mathfrak{a}$ satisfying

$$\varphi_{\mathfrak{a}}(P) = P^{\sigma_{\mathfrak{a}}}, \tag{82}$$

for any $P \in A[\mathfrak{g}]$ with $(\mathfrak{g}, \mathfrak{a}) = 1$. Note that the isogenies $\varphi_{\mathfrak{a}}$ satisfy the following cocycle condition:

$$\varphi_{\mathfrak{a}\mathfrak{b}} = \varphi_{\mathfrak{b}}^{\sigma_{\mathfrak{a}}} \circ \varphi_{\mathfrak{a}}. \tag{83}$$

The following proposition relates the Hecke characters $\tilde{\nu}$ and ν.

Proposition 2.23. *Given any homomorphism $j \in \mathrm{Hom}_K(E, \mathbb{C})$, let $\nu_j := j \circ \tilde{\nu}$ be the corresponding $\mathbb{C}$-valued Hecke character of K of infinity type $(1,0)$. The assignment $j \mapsto \nu_j$ gives a bijection from $\mathrm{Hom}_K(E, \mathbb{C})$ to the set $\Sigma_{\nu,F}$ of Hecke characters ν' of K (of infinity type $(1,0)$) satisfying*

$$\nu' \circ N_{F/K} = \nu \circ N_{F/K}.$$

Proposition 2.23 implies that there is a unique homomorphism $j_\nu \in \mathrm{Hom}_K(E, \mathbb{C})$ satisfying $j_\nu \circ \tilde{\nu} = \nu$. In particular, j_ν maps E to E_ν and T to a finite index subring of T_ν. The abelian variety B_ν attached to ν in Theorem 1.6 can now be defined as the quotient $B \otimes_{T,j_\nu} T_\nu$. In subsequent constructions, it turns out to be more useful to realise B_ν as a subvariety of B, which can be done by setting

$$B_\nu := B[\ker j_\nu]. \tag{84}$$

The natural action of T on B_ν factors through the quotient $T/\ker(j_\nu)$, an integral domain having E_ν as field of fractions.

Consider the inclusion

$$i_\nu : B_\nu(K) \hookrightarrow B(K) = A(F), \tag{85}$$

where the last identification arises from the functorial property of the restriction of scalars. The following Proposition gives an explicit description of the image of $(B_\nu(K) \otimes E_\nu)^{T_\nu}$ in $A(F) \otimes_{\mathcal{O}_K} E_\nu$ under the inclusion i_ν obtained from (85).

Proposition 2.24. *Let $\tilde{E}$ be any field containing E_ν. The inclusion i_ν of (85) identifies $(B_\nu(K) \otimes \tilde{E})^{T_\nu}$ with*

$$(A(F) \otimes_{\mathcal{O}_K} \tilde{E})^\nu := \left\{ P \in A(F) \otimes_{\mathcal{O}_K} \tilde{E} \text{ such that } \varphi_{\mathfrak{a}}(P) = \nu(\mathfrak{a}) \times P^{\sigma_{\mathfrak{a}}}, \text{ for all } \mathfrak{a} \in I_{\mathfrak{f}} \right\}.$$

Proof. It follows from the definitions that $i_\nu(B_\nu(K))$ is identified with the set of (P_τ) with $P_\tau \in A^\tau(\bar{K})$ satisfying

$$\xi P_\tau = P_{\xi\tau}, \quad \text{for all } \xi \in G_K. \tag{86}$$

Furthermore, if such a (P_τ) belongs to $(B_\nu(K) \otimes E_\nu)^{T_\nu}$, then after setting $\tilde{\nu}(\mathfrak{a}) = \varphi_{\mathfrak{a}} \sigma_{\mathfrak{a}}$ as in (81), we also have

$$(\varphi_{\mathfrak{a}}^\tau(P_\tau))_{\tau\sigma_{\mathfrak{a}}} = (P_\tau)_\tau * \tilde{\nu}(\mathfrak{a}) = (\nu(\mathfrak{a}) P_\tau)_\tau. \tag{87}$$

Equating the $\sigma_{\mathfrak{a}}$-components of these two vectors gives

$$\varphi_{\mathfrak{a}}(P_1) = \nu(\mathfrak{a}) P_{\sigma_{\mathfrak{a}}} = \nu(\mathfrak{a}) \sigma_{\mathfrak{a}} P_1,$$

where 1 is the identity embedding of F and the last equality follows from (86). The Proposition follows directly from this, after noting that the identification of $B(K)$ with $A(F)$ is simply the one sending $(P_\tau)_\tau$ to P_1. □

Given a global field F as in (74), let F_ν denote the subfield of $\bar{\mathbb{Q}}$ generated by F and E_ν. Recall that $\omega_A \in \Omega^1(A/F)$ is a non-zero regular differential and that $\Omega_p(A)$ is the associated p-adic period.

Theorem 2.25. *There exists a point $P_{A,\nu} \in (A(F) \otimes_{\mathcal{O}_K} E_\nu)^\nu$ such that*

$$\mathscr{L}_p(\nu^*) = \Omega_p(A)^{-1} \log^2_{\omega_A}(P_{A,\nu}) \pmod{F_\nu^\times}.$$

The point $P_{A,\nu}$ is non-zero if and only if $L'(\nu, 1) \neq 0$.

Proof. Theorem 2 of the Introduction asserts that

$$\mathscr{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1} \log^2_{\omega_\nu}(P_\nu), \tag{88}$$

for some point $P_\nu \in B_\nu(K) \otimes \mathbb{Q}$ which is non-trivial if and only if $L'(\nu, 1) \neq 0$. Invoking Lemma 1.15, we find

$$\Omega_p(\nu^*)^{-1} = \Omega_p(A)^{-1} \pmod{F_\nu^\times}. \tag{89}$$

Furthermore, by 2.24, we can view P_ν as a point $P_{A,\nu} \in (A(F) \otimes_{\mathcal{O}_K} E_\nu)^\nu$, and we have

$$\log_{\omega_\nu}(P_\nu) = \log_{\omega_A}(P_{A,\nu}) \pmod{F_\nu^\times}. \tag{90}$$

Theorem 2.28 now follows by rewriting (88) using (89) and (90). □

2.7. **A special case.** This section is devoted to a more detailed and precise treatment of Theorem 2.25 under the following special assumptions:

(1) The quadratic imaginary field K has class number one, odd discriminant, and unit group of order two. This implies that $K = \mathbb{Q}(\sqrt{-D})$ where $D := -\operatorname{Disc}(K)$ belongs to the finite set

$$S := \{7, 11, 19, 43, 67, 163\}.$$

(2) Let ψ_0 be the Hecke character of K of infinity type $(1, 0)$ given by the formula

$$\psi_0((a)) = \varepsilon_K(a \bmod \sqrt{-D})a. \tag{91}$$

The character ψ_0 determines (uniquely, up to an isogeny) an elliptic curve $A/\mathbb{Q}$ satisfying

$$\operatorname{End}_K(A) = \mathcal{O}_K, \qquad L(A/\mathbb{Q}, s) = L(\psi_0, s).$$

After fixing A, we will also write ψ_A instead of ψ_0. It can be checked that the conductor of ψ_A is equal to $\sqrt{-D}$, and that

$$\psi_A^* = \bar{\psi}_A, \qquad \psi_A\psi_A^* = \mathrm{N}_K, \qquad \varepsilon_{\psi_A} = \varepsilon_K.$$

Remark 2.26. The rather stringent assumptions on K that we have imposed exclude the arithmetically interesting, but somewhat idiosyncratic, cases where $K = \mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(i)$, and $\mathbb{Q}(\sqrt{-2})$. The detailed study of Chow-Heegner points in these special cases is the subject of Dong Quan Nguyen Ngoc's forthcoming PhD thesis.

With the above assumptions, the character ψ_A can be used to give an explicit description of the set $\Sigma_{\mathrm{sd}}(c\sqrt{-D})$:

Lemma 2.27. *Let c be an integer prime to D, and let ν be a Hecke character in $\Sigma_{\mathrm{sd}}(c\sqrt{-D})$. Then ν is of the form*

$$\nu = \psi_A\chi^{-1},$$

where χ is a finite order ring class character of K of conductor dividing c.

Proof. The fact that ν and ψ_A both have central character ε_K implies that χ is a ring class character. Its conductor divides $c\sqrt{-D}$, since the same is true for ν and ψ_A. But it is also clear that χ is unramified at $\sqrt{-D}$, since the local components of ν and ψ_A at this prime of K are equal. The result follows. □

Given a ring class character χ of conductor c as above with values in a field E_χ, let

$$(A(H_c) \otimes_{\mathcal{O}_K} E_\chi)^\chi := \{P \in A(H_c) \otimes_{\mathcal{O}_K} E_\chi \text{ such that } \sigma P = \chi(\sigma)P, \quad \forall \sigma \in \operatorname{Gal}(H_c/K)\}. \tag{92}$$

Finally, choose a regular differential $\omega_A \in \Omega^1(A/K)$, and write $\Omega_p(A)$ for the p-adic period attached to this choice as in Section 2.1. Since $A = B_\psi$ is the abelian variety attached to ψ, it follows that $\Omega_p(\psi_A^*) = \Omega_p(A)$.

The following theorem is a more precise variant of Theorem 2.25.

Theorem 2.28. *Let χ be a ring class character of K of conductor prime to $\sqrt{-D}$. Then there exists a point $P_A(\chi) \in (A(H_\chi) \otimes_{\mathcal{O}_K} E_\chi)^\chi$ such that*

$$\mathscr{L}_p(\psi_A^*\chi) = \Omega_p(A)^{-1}\mathfrak{g}(\chi)\log^2_{\omega_A}(P_A(\chi)) \pmod{E_\chi^\times}.$$

The point $P_A(\chi)$ is non-zero if and only if $L'(\psi_A\chi^{-1}, 1) \neq 0$.

Proof. By Theorem 2 of the Introduction,

$$\mathscr{L}_p(\psi_A^*\chi) = \mathscr{L}_p(\nu^*) = \Omega_p(\nu^*)^{-1}\log^2_{\omega_\nu}(P_\nu), \tag{93}$$

for some point $P_\nu \in B_\nu(K) \otimes \mathbb{Q}$ which is non-trivial if and only if $L'(\psi_A\chi^{-1}, 1) \neq 0$. Using the fact that $\chi^{*-1} = \chi$ and invoking Lemma 1.15, we find

$$\Omega_p(\nu^*)^{-1} = \Omega_p(\psi_A^*\chi^{*-1})^{-1} = \Omega_p(A)^{-1}\mathfrak{g}(\chi)^{-1} \pmod{E_\chi^\times}. \tag{94}$$

After noting that (as in equation (31)) $B_\nu = B_{\psi,\chi} = (A \otimes_{\mathcal{O}_K} T_\chi)^{\chi^{-1}}$ as abelian varieties over K, we observe that $\omega_\nu = \omega_{\psi,\chi}$ and that the point $P_\nu \in B_\nu(K)$ can be written as

$$P_\nu = \sum_{\sigma \in G} P^\sigma \otimes \chi^{-1}(\sigma),$$

for some $P \in A(H_c) \otimes \mathbb{Q}$. Letting $P_{A,\chi}$ be the corresponding element in $A(H_c) \otimes_{\mathcal{O}_K} E_\chi$ given by

$$P_{A,\chi} = \sum_{\sigma \in G} \chi^{-1}(\sigma) P^\sigma,$$

we have

$$\log^2_{\omega_\nu}(P_\nu) = \log^2_{\omega_{\psi,\chi}}(P_\nu) = \mathfrak{g}(\chi)^2 \log^2_{\omega^0_{\psi,\chi}}(P_\nu) = \mathfrak{g}(\chi)^2 \log^2_{\omega_A}(P_{A,\chi}) \pmod{E_\chi^\times}, \tag{95}$$

where the second equality follows from Lemma 1.9 and the last from Lemma 2.18. Theorem 2.28 now follows by rewriting (93) using (94) and (95). □

In the special case where χ is a quadratic ring class character of K, cutting out an extension $L = K(\sqrt{a})$ of K, we obtain

$$\mathscr{L}_p(\psi_A^* \chi) = \Omega_p(A)^{-1} \sqrt{a} \log^2_{\omega_A}(P^-_{A,L}) \pmod{K^\times}, \tag{96}$$

where $P^-_{A,L}$ is a K-vector space generator of the trace 0 elements in $A(L) \otimes \mathbb{Q}$. Since in this case $\psi_A \chi$ is the Hecke character attached to a CM elliptic curve over $\mathbb{Q}$, one recovers from (96) Rubin's Theorem 1 of the Introduction.

3. Chow-Heegner points

The goal of the first three sections of this chapter is to recall the construction of the motives attached to Hecke characters and to modular forms. The remaining three sections are devoted to the definition of Chow-Heegner points on CM elliptic curves, as the image of generalised Heegner cycles by modular parametrisations attached to CM forms.

3.1. **Motives for rational and homological equivalence.** We begin by laying down our conventions regarding motives, following [Del]. We will work with either Chow motives or Grothendieck motives. For X a nonsingular variety over a number field F, let $C^m(X)$ denote the group of algebraic cycles of codimension m on X defined over F. Let $\sim$ denote rational equivalence in $C^m(X)$, and set

$$\mathcal{C}^m(X) := C^m(X)/\sim .$$

Given two nonsingular varieties X and Y over F, and E any number field, we define the groups of correspondences

$$\mathrm{Corr}^m(X,Y) := \mathcal{C}^{\dim X + m}(X \times Y) \qquad \mathrm{Corr}^m(X,Y)_E := \mathrm{Corr}^m(X,Y) \otimes_{\mathbb{Z}} E.$$

Definition 3.1. A *motive* over F with coefficients in E is a triple (X, e, m) where X/F is a nonsingular projective variety, $e \in \mathrm{Corr}^0(X,X)_E$ is an idempotent, and m is an integer.

Definition 3.2. The category $\mathcal{M}_{F,E}$ of *Chow motives* is the category whose objects are motives over F with coefficients in E, with morphisms defined by

$$\mathrm{Hom}_{\mathcal{M}_{F,E}}((X,e,m),(Y,f,n)) = f \circ \mathrm{Corr}^{n-m}(X,Y)_{\mathbb{Q}} \circ e.$$

The category $\mathcal{M}^{\mathrm{hom}}_{F,E}$ of *Grothendieck motives* is defined in exactly the same way, but with homological equivalence replacing rational equivalence. We will denote the corresponding groups of cycle classes by $\mathcal{C}^r_0(X)$, $\mathrm{Corr}^m_0(X,Y)$, $\mathrm{Corr}^m_0(X,Y)_E$ etc.

Since rational equivalence is finer than homological equivalence, there is a natural functor

$$\mathcal{M}_{F,E} \to \mathcal{M}^{\mathrm{hom}}_{F,E},$$

so that every Chow motive gives rise to a Grothendieck motive. Further, the category of Grothendieck motives is equipped with natural *realisation functors* arising from any cohomology theory satisfying the Weil axioms. We now recall the description of the image of a motive $M = (X, e, m)$ over F with coefficients in E under the most important realizations:

The Betti realisation: Recall that our conventions about number fields supply us with an embedding $F \longrightarrow \mathbb{C}$. The Betti realisation is defined in terms of this embedding by

$$M_B := e \cdot (H^*(X(\mathbb{C}), \mathbb{Q})(m) \otimes E).$$

It is a finite-dimensional E-vector space with a natural E-Hodge structure arising from the comparison isomorphism between the singular cohomology and the de Rham cohomology over $\mathbb{C}$.

The ℓ-adic realisation: Let $\bar{X}$ denote the extension of X to $\bar{\mathbb{Q}}$. The ℓ-adic cohomology of $\bar{X}$ gives rise to the ℓ-adic étale realisation of M:

$$M_\ell := e \cdot \left(H^*_{\mathrm{et}}(\bar{X}, \mathbb{Q}_\ell(m)) \otimes E\right).$$

It is a free $E \otimes \mathbb{Q}_\ell$-module of finite rank equipped with a continuous linear G_F-action.

The de Rham realisation: The de Rham realisation of M is defined by

$$M_{\mathrm{dR}} := e \cdot (H^*_{\mathrm{dR}}(X/F)(m) \otimes_{\mathbb{Q}} E),$$

where $H^*_{\mathrm{dR}}(X/F)$ denotes the hypercohomology of the deRham complex of sheaves over X. The module M_{dR} is a free $E \otimes F$-module of finite rank equipped with a decreasing, separated and exhaustive Hodge filtration.

Moreover, there are natural comparison isomorphisms

$$M_B \otimes_{\mathbb{Q}} \mathbb{C} \simeq M_{\mathrm{dR}} \otimes_F \mathbb{C}, \tag{97}$$

$$M_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq M_\ell, \tag{98}$$

which are $E \otimes \mathbb{C}$-linear and $E \otimes \mathbb{Q}_\ell$-linear respectively. Thus

$$\mathrm{rank}_E\, M_B = \mathrm{rank}_{E\otimes F}\, M_{\mathrm{dR}} = \mathrm{rank}_{E\otimes\mathbb{Q}_\ell}(M_\ell),$$

and this common integer is called the E-rank of the motive M.

We define the period $\Omega(M)$ to be the determinant of the comparison isomorphism (97) relative to any choice of E-basis for M_B and EF basis of $M_{\mathrm{dR}} \otimes_F EF$. Thus the period $\Omega(M)$ is a well-defined element of

$$\mathbb{C}^\times/(EF)^\times.$$

Remark 3.3. If F is a p-adic field, one also has a comparison isomorphism

$$M_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR},p} \simeq M_{\mathrm{dR}} \otimes_F B_{\mathrm{dR},p}, \tag{99}$$

where $B_{\mathrm{dR},p}$ is Fontaine's ring of p-adic periods, which is endowed with a decreasing, exhaustive filtration and a continuous G_F-action. This comparison isomorphism is compatible with natural filtrations and G_F-actions on both sides.

Remark 3.4. Our definition of motives with coefficients coincides with *Language B* of Deligne [Del]. There is an equivalent way of defining motives with coefficients (the *Language A*) where the objects are motives M in $\mathcal{M}_{F,\mathbb{Q}}$ equipped with the structure of an E-module: $E \to \mathrm{End}(M)$, and morphisms are those that commute with the E-action. We refer the reader to Sec. 2.1. of loc. cit. for the translation between these points of view.

3.2. **The motive of a Hecke character.** For more general algebraic Hecke characters which are not of type $(1, 0)$, one no longer has an associated abelian variety. Nevertheless, such a character still gives rise to a motive over K with coefficients in the field generated by its values. In the following two sections, we will quickly recall some facts about the motive of a Hecke character ψ of K.

By taking Tate twists and duals, it suffices to construct this motive when ψ has infinity type $(r, 0)$, with r a positive integer. So let us suppose that $\psi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ is such a Hecke character of conductor $\mathfrak{f}_\psi$, and let E_ψ be the field generated over K by the values of ψ on the finite idèles. Pick an abelian extension F of K containing the ray class field $K(\mathfrak{f}_\psi)$ such that $\psi_F := \psi \circ \mathrm{N}_{F/K}$ satisfies the equation

$$\psi_F = \psi_A^r.$$

Here ψ_A is the Hecke character of F with values in K associated to an elliptic curve A/F with complex multiplication by $\mathcal{O}_K$.

We construct the motive $M(\psi_F) \in \mathcal{M}_{F,K}$ associated to ψ_F by considering an appropriate piece of the middle cohomology of the variety A^r over F. As in the Introduction, write $[\alpha]$ for the element of $\mathrm{End}_F(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to an element $\alpha \in K$. Define an idempotent $e_r \in \mathrm{Corr}^0(A^r, A^r)_{\mathbb{Q}}$ by

$$e_r := \left(\frac{\sqrt{-D} + [\sqrt{-D}]}{2\sqrt{-D}}\right)^{\otimes r} + \left(\frac{\sqrt{-D} - [\sqrt{-D}]}{2\sqrt{-D}}\right)^{\otimes r}.$$

Then $M(\psi_F)$ is the motive in $\mathcal{M}_{F,K}$ defined (in language A) by

$$M(\psi_F) := (A^r, e_r, 0),$$

where K acts diagonally on A^r. The ℓ-adic étale realisation $M(\psi_F)_\ell$ of this motive is free of rank one over $K \otimes \mathbb{Q}_\ell$, and G_F acts on it via ψ_F, viewed as a $(K \otimes \mathbb{Q}_\ell)^\times$-valued Galois character:

$$M(\psi_F)_\ell = e_r H^r_{\mathrm{et}}(\overline{A^r}, \mathbb{Q}_\ell) = (K \otimes \mathbb{Q}_\ell)(\psi_F).$$

The de Rham realisation $M(\psi_F)_{\mathrm{dR}}$ is a free one-dimensional $F \otimes_{\mathbb{Q}} K$-vector space, generated as an F-vector space by the classes of

$$\omega_A^r := e_r(\omega_A \wedge \cdots \wedge \omega_A) \qquad \text{and} \qquad \eta_A^r := e_r(\eta_A \wedge \cdots \wedge \eta_A),$$

where η_A is the unique class in $H^1_{\mathrm{dR}}(A/F)$ satisfying

$$[\alpha]^*\eta_A = \bar{\alpha}\eta_A \text{ for all } \alpha \in K, \qquad \text{and} \qquad \langle \omega_A, \eta_A \rangle = 1.$$

The Hodge filtration on $M(\psi_F)_{\mathrm{dR}}$ is given by

$$\begin{aligned} \mathrm{Fil}^0 M(\psi_F)_{\mathrm{dR}} = M(\psi_F)_{\mathrm{dR}} &= F \cdot \omega_A^r + F \cdot \eta_A^r, \\ \mathrm{Fil}^1 M(\psi_F)_{\mathrm{dR}} = \cdots = \mathrm{Fil}^r M(\psi_F)_{\mathrm{dR}} &= F \cdot \omega_A^r, \\ \mathrm{Fil}^{r+1} M(\psi_F)_{\mathrm{dR}} &= 0. \end{aligned}$$

It can be shown that $M(\psi_F)$ descends to a motive $M(\psi) \in \mathcal{M}_{K,E_\psi}$, whose ℓ-adic realisation is a free rank one module over $E\psi \otimes \mathbb{Q}_\ell$ on which G_K acts via the character ψ.

3.3. **Deligne-Scholl motives.** In this section, we will let ψ be a Hecke character of K of infinity type $(r+1, 0)$. This Hecke character gives rise to a theta-series

$$\theta_\psi = \sum_{n=1}^{\infty} a_n(\theta_\psi) q^n \quad \in \quad S_{r+2}(\Gamma_0(N), \varepsilon),$$

as in Proposition 2.10. Observe that the subfield of $\bar{\mathbb{Q}}$ generated by the Fourier coefficients $a_n(\theta_\psi)$ is always contained (albeit sometimes properly) in E_ψ. Henceforth, we will view θ_ψ as a modular form with Fourier coefficients in E_ψ.

Deligne has attached to θ_ψ a compatible system $\{V_\ell(\theta_\psi)\}$ of two-dimensional ℓ-adic representations of $G_{\mathbb{Q}}$ with coefficients in $E_\psi \otimes \mathbb{Q}_\ell$, such that for any prime $p \nmid N\ell$, the characteristic polynomial of the Frobenius element at p is given by

$$X^2 - a_p(\theta_\psi)X + \varepsilon(p)p^{r+1}.$$

This representation is realised in the middle ℓ-adic cohomology of a variety which is fibered over a modular curve. More precisely, recall the group $\Gamma = \Gamma_\varepsilon(N)$ defined in equation (47) and the modular curve C whose complex points are identified with $\Gamma \backslash \mathcal{H}^*$. Let W_r be the r-th Kuga-Sato variety over C. It is a canonical compactification and desingularisation of the r-fold self-product of the universal elliptic curve over C. (See for example [BDP], Chapter 2 for more details on this definition.)

Theorem 3.5. *(Scholl) There is a projector $e_{\theta_\psi} \in \mathrm{Corr}^0(W_r, W_r) \otimes E_\psi$ whose associated Grothendieck motive $M(\theta_\psi) := (W_r, e_{\theta_\psi}, 0)$ satisfies*

$$M(\theta_\psi)_\ell \simeq V_\ell(\theta_\psi)$$

as $E_\psi[G_{\mathbb{Q}}]$-modules.

We remark that $M(\theta_\psi)$ is a motive over $\mathbb{Q}$ with coefficients in E_ψ, and that its ℓ-adic realisation $M(\theta_\psi)_\ell$ is identified with $e_{\theta_\psi}(H^{r+1}_{\mathrm{et}}(\bar{W}_r, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} E_\psi)$.

The de Rham realisation

$$M(\theta_\psi)_{\mathrm{dR}} = e_{\theta_\psi} H^{r+1}_{\mathrm{dR}}(W_r/E_\psi)$$

is a two-dimensional E_ψ-vector space equipped with a canonical decreasing, exhaustive and separated Hodge filtration. This vector space and its associated filtration can be described concretely in terms of the cusp form θ_ψ as follows.

Let C^0 denote the complement in C of the subscheme formed by the cusps. Setting $W_r^0 := W_r \times_C C^0$, there is a natural analytic uniformization

$$W_r^0(\mathbb{C}) = (\mathbb{Z}^{2r} \rtimes \Gamma) \backslash (\mathbb{C}^r \times \mathcal{H}),$$

where the action of $\mathbb{Z}^{2r}$ on $\mathbb{C}^r \times \mathcal{H}$ is given by

$$(m_1, n_1, \ldots, m_r, n_r)(w_1, \ldots, w_r, \tau) := (w_1 + m_1 + n_1\tau, \ldots, w_r + m_r + n_r\tau, \tau), \tag{100}$$

and Γ acts by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w_1, \ldots, w_r, \tau) = \left(\frac{w_1}{c\tau + d}, \ldots, \frac{w_r}{c\tau+d}, \frac{a\tau + b}{c\tau + d}\right). \tag{101}$$

The holomorphic $(r+1)$ form

$$\omega_{\theta_\psi} := (2\pi i)^{r+1}\theta_\psi(\tau) dw_1 \cdots dw_r d\tau \tag{102}$$

on $W_r^0(\mathbb{C})$ extends to regular differentials on W_r. This differential is defined over the field E_ψ, by the q-expansion principle, hence lies in $H_{\mathrm{dR}}^{r+1}(W_r) \otimes_{\mathbb{Q}} E_\psi$. Its class generates the $(r+1)$-st step in the Hodge filtration:

$$\mathrm{Fil}^{r+1}(M(\theta_\psi)_{\mathrm{dR}}) = e_{\theta_\psi} \,\mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{r+1}(W_r) = E_\psi \cdot \omega_{\theta_\psi}. \tag{103}$$

3.4. **Modular parametrisations attached to CM forms.** In this section, we will explain how the Tate conjectures imply the existence of algebraic cycle classes generalising those in Conjecture 7 of the Introduction. We first recall the statement of the Tate conjecture.

Conjecture 3.6 (Tate)**.** *Let V be a smooth projective variety over a number field F. Then the ℓ-adic étale cycle class map*

$$\mathrm{cl}_\ell : \mathrm{CH}^j(V)(F) \otimes \mathbb{Q}_\ell \longrightarrow H_{\mathrm{et}}^{2j}(\bar{V}, \mathbb{Q}_\ell)(j)^{G_F} \tag{104}$$

is surjective.

A class in the target of (104) is called an *ℓ-adic Tate cycle*. The Tate conjecture will be used in our constructions through the following simple consequence. Recall the Chow groups $\mathrm{CH}^d(V)(F)$ defined in the Introduction.

Lemma 3.7. *Let V_1 and V_2 be smooth projective varieties of dimension d over a number field F, and let $e_j \in \mathrm{Corr}^0(V_j, V_j) \otimes E$ (for $j = 1, 2$) be idempotents satisfying*

$$e_j H_{\mathrm{et}}^*(\bar{V}_j, \mathbb{Q}_\ell) \otimes E = e_j H_{\mathrm{et}}^d(\bar{V}_j, \mathbb{Q}_\ell) \otimes E, \qquad j = 1, 2.$$

Let $M_j := (V_j, e_j, 0)$, be the associated motives over F with coefficients in E, and suppose that the ℓ-adic realisations of M_1 and M_2 are isomorphic as $(E \otimes \mathbb{Q}_\ell)[G_F]$-modules. If Conjecture 3.6 is true for $V_1 \times V_2$, then there exists a correspondence $\Pi \in \mathrm{CH}^d(V_1 \times V_2)(F) \otimes E$ for which

1. *the induced morphism*
$$\Pi_\ell : (M_1)_\ell \longrightarrow (M_2)_\ell \tag{105}$$
of ℓ-adic realisations is an isomorphism of $E \otimes \mathbb{Q}_\ell[G_F]$-modules;
2. *the induced morphism*
$$\Pi_{\mathrm{dR}} : (M_1)_{\mathrm{dR}} \longrightarrow (M_2)_{\mathrm{dR}} \tag{106}$$
is an isomorphism of $E \otimes F$-vector spaces.

Proof. Let

$$h : e_1 H_{\mathrm{et}}^d(\bar{V}_1, E \otimes \mathbb{Q}_\ell) \simeq e_2 H_{\mathrm{et}}^d(\bar{V}_2, E \otimes \mathbb{Q}_\ell)$$

be any isomorphism of $(E \otimes \mathbb{Q}_\ell)[G_F]$-modules. It corresponds to a Tate cycle

$$\begin{aligned} Z_f &\in \left(H_{\mathrm{et}}^d(\bar{V}_1, E \otimes \mathbb{Q}_\ell)^\vee \otimes H_{\mathrm{et}}^d(\bar{V}_2, E \otimes \mathbb{Q}_\ell)\right)^{G_F} \\ &= \left(H_{\mathrm{et}}^d(\bar{V}_1, E \otimes \mathbb{Q}_\ell(d)) \otimes H_{\mathrm{et}}^d(\bar{V}_2, E \otimes \mathbb{Q}_\ell)\right)^{G_F} \\ &\subset \left(H_{\mathrm{et}}^{2d}(\overline{V_1 \times V_2}, E \otimes \mathbb{Q}_\ell(d))\right)^{G_F}, \end{aligned}$$

where the superscript $^\vee$ in the first line denotes the E-linear dual, the second line follows from the Poincaré duality, and the third from the Künneth formula. By Conjecture 3.6, there are elements $\alpha_1, \ldots, \alpha_t \in E \otimes \mathbb{Q}_\ell$ and cycles $\Pi_1, \ldots, \Pi_t \in \mathrm{CH}^d(V_1 \times V_2)(F)$ satisfying

$$Z_f = \sum_{j=1}^{t} \alpha_j \, \mathrm{cl}_\ell(\Pi_j).$$

After multiplying Z_f by a suitable power of ℓ, we may assume without loss of generality that the coefficients α_j belong to $\mathcal{O}_E \otimes \mathbb{Z}_\ell$. If $(\beta_1, \ldots, \beta_t) \in \mathcal{O}_E^t$ is any vector which is sufficiently close to $(\alpha_1, \ldots, \alpha_t)$ in the ℓ-adic topology, then the corresponding algebraic cycle

$$\Pi := \sum_{j=1}^{t} \beta_j \cdot \Pi_j \quad \in \mathrm{CH}^d(V_1 \times V_2)(F) \otimes E$$

satisfies condition 1 in the statement of Lemma 3.7. Condition 2 is verified by embedding F into one of its ℓ-adic completions F_λ and applying Fontaine's comparison functor to (105) in which source and targets are deRham representations of G_{F_λ}. This shows that Π_{dR} induces an isomorphism on the deRham cohomology over $F_\lambda \otimes E$, and part 2 follows. □

The following proposition (in which, to ease notations, we identify differential forms with their image in de Rham cohomology) justifies Conjecture 7 of the Introduction. Notations are as in Section 3.2 and 3.3, with ψ a Hecke character of infinity type $(r+1, 0)$.

Proposition 3.8. *If the Tate conjecture is true for $W_r \times A^{r+1}$, then there is an algebraic cycle class $\Pi^? \in \mathrm{CH}^{r+1}(W_r \times A^{r+1})(F) \otimes E_\psi$ satisfying*

$$\Pi^{?*}_{\mathrm{dR}}(\omega_A^{r+1}) = \omega_{\theta_\psi}. \tag{107}$$

Proof. Let $M_{1,F} := (A^{r+1}, e_{r+1}, 0)$ be the motive over F with coefficients in K attached to the Hecke character $\psi_F = \psi_A^{r+1}$ in Section 3.2, and let $M_{2,F} := (W_r, e_{\theta_\psi}, 0)$ be the Scholl motive over F attached to the theta-series θ_ψ in Section 3.3. Since the ℓ-adic realisations $(M_{1,F})_\ell \otimes E_\psi$ and $(M_{2,F})_\ell$ are isomorphic, Lemma 3.7 implies the existence of a correspondence $\Pi^?$ in $\mathrm{CH}^{r+1}(W_r \times A^{r+1})(F) \otimes E_\psi$ which induces an isomorphism on the de Rham realisations. This isomorphism respects the Hodge filtrations and therefore sends the class $[\omega_A^{r+1}]$ to a non-zero E_ψ-rational multiple of $[\omega_{\theta_\psi}]$. After suitably rescaling $\Pi^?$, one can therefore assume that it satisfies (107). □

Note that the ambient F-variety $Z := W_r \times A^{r+1} = W_r \times A^r \times A$ in which the correspondence $\Pi^?$ is contained is equipped with three obvious projection maps

$$\begin{tikzcd} & Z \arrow[dl, "\pi_0"'] \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\ W_r & A^r & A. \end{tikzcd}$$

Let X_r be the F-variety

$$X_r = W_r \times A^r.$$

After setting

$$\pi_{01} = \pi_0 \times \pi_1 : Z \longrightarrow X_r, \qquad \pi_{12} = \pi_1 \times \pi_2 : Z \longrightarrow A^r \times A,$$

we recall the simple (but key!) observation already made in the Introduction that $\Pi^?$ can be viewed as a correspondence in two different ways, via the diagrams:

$$\begin{tikzcd} & Z \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_{12}"] & \\ W_r & & A^r \times A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Z \arrow[dl, "\pi_{01}"'] \arrow[dr, "\pi_2"] & \\ X_r & & A. \end{tikzcd}$$

In order to maintain a notational distinction between these two ways of viewing $\Pi^?$, the correspondence from X_r to A attached to the cycle $\Pi^?$ is denoted by $\Phi^?$ instead of $\Pi^?$. It induces a natural transformation of functors on F-algebras:

$$\Phi^? : \mathrm{CH}^{r+1}(X_r)_0 \otimes E_\psi \longrightarrow \mathrm{CH}^1(A)_0 \otimes E_\psi = A \otimes E_\psi, \tag{108}$$

where $A \otimes E_\psi$ is the functor from the category of F-algebras to the category of E_ψ-vector spaces which to L associates $A(L) \otimes E_\psi$. The natural transformation $\Phi^?$ is referred to as the *modular parametrisation* attached to the correspondence $\Phi^?$. For any F-algebra L, we will also write

$$\Phi^?_L : \mathrm{CH}^{r+1}(X_r)_0(L) \otimes E_\psi \longrightarrow A(L) \otimes E_\psi \tag{109}$$

for the associated homomorphism on L-rational points (modulo torsion).

Like the class $\Pi^?$, the correspondence $\Phi^?$ also induces a functorial FE_ψ-linear map on de Rham cohomology, denoted

$$\Phi^{?*}_{\mathrm{dR}} : H^1_{\mathrm{dR}}(A/FE_\psi) \longrightarrow H^{2r+1}_{\mathrm{dR}}(X_r/FE_\psi).$$

Recall that $\eta_A \in H^1_{\mathrm{dR}}(A/FE_\psi)$ denotes the unique class in $H^{0,1}_{\mathrm{dR}}(A/FE_\psi)$ satisfying

$$\langle \omega_A, \eta_A \rangle = 1.$$

Proposition 3.9. *The image of the class $\omega_A \in \Omega^1(A/F) \subset H^1_{\mathrm{dR}}(A/F)$ under $\Phi^?_{\mathrm{dR}}$ is given by*

$$\Phi^{?*}_{\mathrm{dR}}(\omega_A) = (\omega_{\theta_\psi} \wedge \eta_A^r).$$

Proof. After writing the cycle

$$\Pi^? = \sum_j m_j Z_j$$

as a E_ψ-linear combination of codimension $(r+1)$ subvarieties of Z, we can define the cycle class map

$$\mathrm{cl}_{\Pi^?} : H^{2r+2}_{\mathrm{dR}}(Z/FE_\psi) \longrightarrow FE_\psi$$

by setting

$$\mathrm{cl}_{\Pi^?}(\omega) = \sum_j m_j \, \mathrm{cl}_{Z_j}(\omega).$$

By Proposition 3.8 and the construction of $\Pi^?_{\mathrm{dR}}$, we have

$$\Pi^?_{\mathrm{dR}}(\omega_A^{r+1}) = \omega_\theta, \tag{110}$$

and

$$\Pi^?_{\mathrm{dR}}(\eta_A^j \omega_A^{r+1-j}) = 0, \qquad \text{for } 1 \le j \le r. \tag{111}$$

By definition of $\Pi^?_{\mathrm{dR}}$, equation (110) can be rewritten as

$$\mathrm{cl}_{\Pi^?}(\pi_0^*(\alpha) \wedge \pi_{12}^*(\lambda \omega_A^{r+1})) = \langle \alpha, \lambda \omega_\theta \rangle_{W_r}, \qquad \text{for all } \alpha \in H^{r+1}_{\mathrm{dR}}(W_r/K), \tag{112}$$

while (111) shows that

$$\mathrm{cl}_{\Pi^?}(\pi_0^*(\alpha) \wedge \pi_{12}^*(\lambda \eta_A^j \omega_A^{r+1-j})) = 0, \qquad \text{when } 1 \le j \le r. \tag{113}$$

Equation (112) can also be rewritten as

$$\mathrm{cl}_{\Phi^?}(\pi_{01}^*(\lambda\alpha \wedge \omega_A^r) \wedge \pi_2^*(\omega_A)) = \langle \lambda\alpha \wedge \omega_A^r, \omega_\theta \wedge \eta_A^r \rangle_{X_r}, \tag{114}$$

while equation (113) implies that, for all $\alpha \in H^{r+1}_{\mathrm{dR}}(W_r/K)$ and all $1 \le j \le r$,

$$\mathrm{cl}_{\Phi^?}(\pi_{01}^*(\lambda\alpha \wedge \eta_A^j \omega_A^{r-j}) \wedge \pi_2^*(\omega_A)) = 0 = \langle \lambda\alpha \wedge \eta_A^j \omega_A^{r-j}, \omega_\theta \wedge \eta_A^r \rangle_{X_r}. \tag{115}$$

In light of the definition of the map $\Phi^?_{\mathrm{dR}}$, equations (114) and (115) imply that

$$\Phi^?_{\mathrm{dR}}(\omega_A) = \omega_\theta \wedge \eta_A^r.$$

The proposition follows. $\square$

Remark 3.10. In the special case that was treated in more detail in Section 2.7, the conjectural modular parametrisation $\Phi^?$ admits a somewhat simpler description. This is because, in that case, we may choose

$$F = E_\psi = K.$$

We may then further assume (without real loss of generality, as it turns out) that the Hecke character ψ is given by

$$\psi = \psi_A^{r+1}, \tag{116}$$

where $\psi_A = \psi_0$ is the K-valued Hecke character of infinity type $(1,0)$ that was defined in equation (91), corresponding to an elliptic curve A/K with complex multiplication by $\mathcal{O}_K$. Under these conditions, the modular parametrisation $\Phi^?$ arises from a class in $\mathrm{CH}^{r+1}(X_r \times A)(K) \otimes K$ and induces a natural transformation of functors on K-algebras:

$$\Phi^? : \mathrm{CH}^{r+1}(X_r)_0 \longrightarrow A. \tag{117}$$

3.5. **Generalised Heegner cycles and Chow-Heegner points.** Recall the notation $\Gamma := \Gamma_\varepsilon(N) \subset \Gamma_0(N)$ in (47), and the associated modular curve C classifying (generalised) elliptic curves with Γ-level structure in the sense of Definition 2.8.

Fixing a choice t of Γ-level structure on A attached to $\mathfrak{N}$, the datum of (A,t) determines a point P_A on $C(\tilde{F})$ for some abelian extension $\tilde{F}$ of K, and a canonical embedding ι_A of A^r into the fiber in W_r above P_A. More generally, if $\varphi : A \longrightarrow A'$ is an isogeny defined over F whose kernel intersects $A[\mathfrak{N}]$ trivially (i.e., an isogeny of elliptic curves with Γ-level structure), then the pair $(A', \varphi(t))$ determines a point $P_{A'} \in C(\tilde{F})$ and an embedding $\iota_\varphi : (A')^r \longrightarrow W_r$ which is defined over $\tilde{F}$. We associate to such an isogeny φ a codimension $r+1$ cycle Υ_φ on the variety X_r by letting $\mathrm{Graph}(\varphi) \subset A \times A'$ denote the graph of φ, and setting

$$\Upsilon_\varphi \quad := \quad \mathrm{Graph}(\varphi)^r \subset (A \times A')^r \xrightarrow{\simeq} (A')^r \times A^r \subset W_r \times A^r,$$

where the last inclusion is induced from the pair $(\iota_{A'}, \mathrm{id}_A^r)$. We then set

$$\Delta_\varphi := \epsilon_X \Upsilon_\varphi \in \mathrm{CH}^{r+1}(X_r)_0(\tilde{F}), \tag{118}$$

where ϵ_X is the idempotent given in equation (51) of [BDP], viewed as an element of the ring $\mathrm{Corr}^0(X_r, X_r)$ of algebraic correspondences from X_r to itself.

We will now assume that the field F has been chosen large enough so that it contains $\tilde{F}$ as well as the field of definition of A.

Definition 3.11. The *Chow-Heegner point* attached to the data (ψ, φ) is the point

$$P_\psi^?(\varphi) := \Phi_F^?(\Delta_\varphi) \in A(F) \otimes E_\psi.$$

Note that this definition is only a conjectural one, since the existence of the homomorphism $\Phi_F^?$ depends on the existence of the algebraic cycle $\Pi^?$.

We now discuss some specific examples of φ that will be relevant to us. Let c be a positive integer as in Section 2.2. An isogeny $\varphi_0 : A \longrightarrow A_0$ is said to be a *primitive isogeny of conductor* c if it is of degree c and if the endomorphism ring $\mathrm{End}(A_0)$ is isomorphic to the order $\mathcal{O}_c$ in K of conductor c. The kernel of a primitive isogeny necessarily intersects $A[\mathfrak{N}]$ trivially, i.e., such a φ_0 is an isogeny of elliptic curves with Γ-level structure. The corresponding Chow-Heegner point $P_\psi^?(\varphi_0)$ is said to be of *conductor* c.

Once φ_0 is fixed, one can also consider an infinite collection of Chow-Heegner points indexed by certain projective $\mathcal{O}_c$-submodules of $\mathcal{O}_c$. More precisely, let $\mathfrak{a}$ be such a projective module for which

$$A_0[\mathfrak{a}] \cap \varphi_0(A[\mathfrak{N}]) = 0,$$

and let

$$\varphi_\mathfrak{a} : A_0 \longrightarrow A_\mathfrak{a} := A_0/A_0[\mathfrak{a}]$$

denote the canonical isogeny of elliptic curves with Γ-level structure given by the theory of complex multiplication. Since the isogeny $\varphi_\mathfrak{a}$ is defined over F, the Chow-Heegner point

$$P_\psi^?(\mathfrak{a}) := P_\psi^?(\varphi_\mathfrak{a}\varphi_0) = \Phi_F^?(\Delta_\mathfrak{a}), \qquad \text{where } \Delta_\mathfrak{a} = \Delta_{\varphi_\mathfrak{a}\varphi_0},$$

belongs to $A(F) \otimes E_\psi$ as well.

Lemma 3.12. *For all elements $\lambda \in \mathcal{O}_c$ which are prime to $\mathfrak{N}$, we have*

$$P_\psi^?(\lambda\mathfrak{a}) = \varepsilon(\lambda \bmod \mathfrak{N})\lambda^r P_\psi^?(\mathfrak{a}) \qquad \text{in } A(F) \otimes E_\psi.$$

More generally, for any $\mathfrak{b}$,

$$\varphi_{\mathfrak{a}}(P_\psi^?(\mathfrak{a}\mathfrak{b})) = \psi(\mathfrak{a}) P_\psi^?(\mathfrak{b})^{\sigma_{\mathfrak{a}}}.$$

Proof. Let $P_{\mathfrak{a}}$ be the point of $C(F)$ attached to the elliptic curve $A_{\mathfrak{a}}$ with Γ-level structure, and recall that $\pi^{-1}(P_{\mathfrak{a}})$ is the fiber above $P_{\mathfrak{a}}$ for the natural projection $\pi : X_r \longrightarrow C$. The algebraic cycle

$$\Delta_{\lambda\mathfrak{a}} - \varepsilon(\lambda)\lambda^r \Delta_{\mathfrak{a}}$$

is entirely supported in the fiber $\pi^{-1}(P_{\mathfrak{a}})$, and its image in the homology of this fiber under the cycle class map is 0. The result follows from this using the fact that the image of a cycle Δ supported on a fiber $\pi^{-1}(P)$ depends only on the point P and on the image of Δ in the homology of the fiber. The proof of the general case is similar. □

Let χ be a Hecke character of K of infinity type $(r, 0)$ such that $\chi \mathrm{N}_K$ belongs to $\Sigma_{cc}^{(1)}(c, \mathfrak{N}, \varepsilon)$. As before, let $E_{\psi,\chi}$ denote the field generated by the values of ψ and χ. By Lemma 3.12, the expression

$$\chi(\mathfrak{a})^{-1} P_\psi^?(\mathfrak{a}) \in A(F) \otimes E_{\psi,\chi}$$

depends only on the image of $\mathfrak{a}$ in the class group $G_c := \mathrm{Pic}(\mathcal{O}_c)$. Hence we can define the Chow-Heegner point attached to the theta-series θ_ψ and the character χ by summing over this class group:

$$P_\psi^?(\chi) := \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a}) P_\psi^?(\mathfrak{a}) \in A(F) \otimes E_{\psi,\chi}. \tag{119}$$

The Chow-Heegner point $P_\psi^?(\chi)$ thus defined belongs (conjecturally) to $A(F) \otimes E_{\psi,\chi}$.

3.6. **A special case.** We now specialise the Chow-Heegner point construction to a simple but illustrative case, in which the hypotheses introduced in Section 2.7 are imposed. We further assume

(1) The character ψ is of the form ψ_A^{r+1}, as in (116), so that the modular parametrisation $\Phi^?$ gives a homomorphism from $\mathrm{CH}^{r+1}(X_r)(K)$ to $A(K) \otimes \mathbb{Q}$, as in (117).
(2) The integer r is odd. This implies that ψ is an unramified Hecke character of infinity type $(r+1, 0)$ with values in K, and that its associated theta series θ_ψ belongs to $S_{r+2}(\Gamma_0(D), \varepsilon_K)$.
(3) The character χ is a Hecke character of infinity type $(r, 0)$, and

$$\chi N_K \text{ belongs to } \Sigma_{cc}^{(1)}(c, \sqrt{-D}, \varepsilon_K),$$

with c prime to D. A direct modification of the proof of Lemma 2.27 shows that any such χ can be written as

$$\chi = \psi_A^r \chi_0^{-1},$$

where χ_0 is a ring class character of K of conductor dividing c.

Under these conditions, we have

$$\Gamma = \Gamma_{\varepsilon_K}(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D) \text{ such that } \varepsilon_K(a) = 1 \right\}.$$

Furthermore, the action of G_K on the cyclic group $A[\sqrt{-D}](\bar{K})$ is via the D-th cyclotomic character, and therefore a Γ-level structure of Heegner type on the curve A is necessarily defined over K. The corresponding Γ-level structures on A_0 and on $A_{\mathfrak{a}}$ are therefore defined over the ring class field H_c. It follows that the generalised Heegner cycles Δ_φ belong to $\mathrm{CH}^{r+1}(X_r)_0(H_c)$, for any isogeny φ of conductor c, and therefore–assuming the existence of $\Phi^?$–that

$$P_\psi^?(\mathfrak{a}) \text{ belongs to } A(H_c) \otimes \mathbb{Q}, \qquad P_\psi^?(\chi) \text{ belongs to } (A(H_c) \otimes_{\mathcal{O}_K} E_\chi)^{\chi_0}.$$

4. Chow-Heegner points over $\mathbb{C}_p$

4.1. **The p-adic Abel-Jacobi map.** The construction of the point $P_\psi^?(\chi)$ is only conjectural since it depends on the existence of the cycle $\Pi^?$ and the corresponding map $\Phi^?$. In order to obtain unconditional results, we will replace the conjectural map $\Phi_F^?$ by its analogue $\Phi_F^{\rm et}$ in p-adic étale cohomology mentioned in equation (22) of the Introduction, and studied in detail in Chapter 4 of [BDP].

Fix a rational prime p which does not divide the level N of θ_ψ. The global cohomology class

$$\kappa_\psi(\varphi) := \Phi_F^{\rm et}(\Delta_\varphi) \in H^1(F, V_p(A)\otimes E_\psi)$$

belongs to the pro-p Selmer group of A over F (tensored with E_ψ), and is defined independently of any conjectures. Furthermore, if the correspondence $\Phi_F^?$ exists, then (23) implies that

$$\kappa_\psi(\varphi) = \delta(P_\psi^?(\varphi)),$$

where

$$\delta : A(F)\otimes E_\psi \longrightarrow H^1(F, V_p(A)\otimes E_\psi)$$

is the connecting homomorphism of Kummer theory mentioned in (24).

The topological closure of F in $\mathbb{C}_p$ (relative to the fixed embedding $F\longrightarrow\mathbb{C}_p$) is a finite extension of $\mathbb{Q}_p$, and shall be denoted F_p. Since $\kappa_\psi(\varphi)$ belongs to the Selmer group of A over F, there is a local point in $A(F_p)\otimes E_\psi$, denoted $P_\psi^{(p)}(\varphi)$, such that

$$\kappa_\psi(\varphi)_{|G_{F_p}} = \delta(P_\psi^{(p)}(\varphi)).$$

More generally, the image of $\Phi_F^{\rm et}$ is contained in the Selmer group of A over F, and hence there exists a map

$$\Phi_F^{(p)} : \mathrm{CH}^{r+1}(X_r)_0(F_p)\longrightarrow A(F_p)\otimes E_\psi$$

such that

$$\Phi_F^{(p)}(\Delta_\varphi) = P_\psi^{(p)}(\varphi).$$

The map $\Phi_F^{(p)}$ is the p-adic counterpart of the conjectural map $\Phi_F^?$.

In light of Proposition 3.8 and of the construction of Chow-Heegner points given in Definition 3.11, the following conjecture is a concrete consequence of the Tate (or Hodge) conjecture for the variety $X_r\times A$.

Conjecture 4.1. *The local elements $P_\psi^{(p)}(\varphi)\in A(F_p)\otimes E_\psi$ belong to $A(F)\otimes_{\mathcal{O}_K} E_\psi$.*

The goal of this chapter is to exploit the connection between the local points $P_\psi^{(p)}(\varphi)$ and special values of p-adic L-functions to supply some evidence for Conjecture 4.1.

We begin by relating $P_\psi^{(p)}(\varphi)$ to p-adic Abel-Jacobi maps. The *p-adic Abel-Jacobi map* attached to the elliptic curve A/F_p is a homomorphism

$$\mathrm{AJ}_A : \mathrm{CH}^1(A)_0(F_p)\longrightarrow \Omega^1(A/F_p)^\vee, \tag{120}$$

where the superscript $\vee$ denotes the F_p-linear dual. Under the identification of the Chow group $\mathrm{CH}^1(A)_0(F_p)$ with $A(F_p)$, it is determined by the relation

$$\mathrm{AJ}_A(P)(\omega) = \log_\omega(P), \tag{121}$$

where ω is any regular differential on A over F_p and

$$\log_\omega : A(F_p)\otimes\mathbb{Q}\longrightarrow F_p$$

denotes the formal group logarithm on A attached to this choice of regular differential. It can be extended by E_ψ-linearity to a map from $A(F_p)\otimes E_\psi$ to F_pE_ψ.

There is also a p-adic Abel-Jacobi map on null-homologous algebraic cycles

$$\mathrm{AJ}_{X_r} : \mathrm{CH}^{r+1}(X_r)_0(F_p)\longrightarrow \mathrm{Fil}^{r+1} H^{2r+1}_{\rm dR}(X_r/F_p)^\vee$$

attached to the variety X_r, where Fil^j refers to the j-th step in the Hodge filtration on algebraic de Rham cohomology. Details on the definition of AJ_{X_r} are recalled in Section 4.4 of [BDP], where it is explained how AJ_{X_r} can be calculated via p-adic integration.

The functoriality of the Abel-Jacobi maps under correspondences is expressed in the following commutative diagram relating AJ_A and AJ_{X_r}:

$$\begin{array}{ccc} \mathrm{CH}^{r+1}(X_r)_0(F_p) & \xrightarrow{\mathrm{AJ}_{X_r}} & \mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/F_p)^\vee \\ \downarrow \Phi_F^{(p)} & & \downarrow (\Phi_{\mathrm{dR}}^{?*})^\vee \\ A(F_p)\otimes E_\psi & \xrightarrow{\mathrm{AJ}_A} & \Omega^1(A/F_p)^\vee \otimes E_\psi. \end{array} \tag{122}$$

Proposition 4.2. *For all isogenies $\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$ of elliptic curves with Γ-level structure,*

$$\log_{\omega_A}(P_\psi^{(p)}(\varphi)) = \mathrm{AJ}_{X_r}(\Delta_\varphi)(\omega_\theta \wedge \eta_A^r).$$

Proof. By equation (121) and the definition of $P_\psi^{(p)}(\varphi)$,

$$\log_{\omega_A}(P_\psi^{(p)}(\varphi)) = \mathrm{AJ}_A(P_\psi^{(p)}(\varphi))(\omega_A) = \mathrm{AJ}_A(\Phi_F^{(p)}(\Delta_\varphi))(\omega_A). \tag{123}$$

The commutative diagram (122) combined with Proposition 3.9 shows that

$$\mathrm{AJ}_A(\Phi_F^{(p)}\Delta_\varphi)(\omega_A) = \mathrm{AJ}_{X_r}(\Delta_\varphi)(\Phi_{\mathrm{dR}}^{?*}\omega_A) = \mathrm{AJ}_{X_r}(\Delta_\varphi)(\omega_{\theta_\psi} \wedge \eta_A^r). \tag{124}$$

Proposition 4.2 now follows from (123) and (124). □

We will study the local points $P_\psi^{(p)}(\varphi)$ via the formula of Proposition 4.2, whose terms do not depend on the conjectural existence of $\Pi^?$.

4.2. **Rationality of Chow-Heegner points over $\mathbb{C}_p$.** Recall that $F_{\psi,\chi}$ is the subfield of $\bar{\mathbb{Q}}$ generated over K by F and $E_{\psi,\chi}$, and that $\nu = \psi\chi^{-1}$ is the self-dual Hecke character of K of infinity type $(1, 0)$ attached to (ψ, χ). The main result of this section is

Theorem 4.3. *There exists a global element $P_\psi(\chi) \in A(F) \otimes E_{\psi,\chi}$ such that*

$$\log^2_{\omega_A}(P_\psi^{(p)}(\chi)) = \log^2_{\omega_A}(P_\psi(\chi)) \pmod{F^\times_{\psi,\chi}},$$

for all regular differentials $\omega_A \in \Omega^1(A/F)$. This element is non-zero if and only if

$$L'(\nu, 1) \neq 0 \qquad \textit{and} \qquad L(\psi\chi^{*-1}, 1) \neq 0.$$

Proof. The proof proceeds along the same lines as (but is simpler than) the proof of Theorem 4.4 below. This proof applies to a more special setting but derives a more precise result, in which it becomes necessary to keep a more careful track of the fields of scalars involved. To prove Theorem 4.3, it is therefore enough to rewrite the proof of Theorem 4.4 with E_χ^* replaced by $F^\times_{\psi,\chi}$ and $(\psi_A^{r+1}, \psi_A^r\chi_0)$ replaced by (ψ, χ). Note that equations (128) and (129) hold modulo the larger group $F^\times_{\psi,\chi}$ without the Gauss sum factors which can therefore be ignored. □

4.3. **A special case.** We now place ourselves in the setting of Section 3.6, in which

$$\psi = \psi_A^{r+1}, \qquad \chi = \psi_A^r\chi_0$$

where χ_0 is a ring class character of K of conductor c, and we set

$$P_{A,r}^{(p)}(\chi_0) := P^{(p)}_{\psi_A^{r+1}}(\psi_A^r\chi_0) = P_\psi^{(p)}(\chi).$$

Recall the definition of the χ_0-component $(A(H_c) \otimes E_\chi)^{\chi_0}$ of the Mordell-Weil group over the ring class field H_c that was given in (92).

Theorem 4.4. *There exists a global point $P_{A,r}(\chi_0) \in (A(H_c) \otimes E_\chi)^{\chi_0}$ satisfying*

$$\log^2_{\omega_A}(P_{A,r}^{(p)}(\chi_0)) = \log^2_{\omega_A}(P_{A,r}(\chi_0)) \pmod{E_\chi^\times}.$$

Furthermore, the point $P_{A,r}(\chi_0)$ is of infinite order if and only if

$$L'(\psi_A\chi_0^{-1}, 1) \neq 0, \qquad L(\psi_A^{2r+1}\chi_0, r+1) \neq 0.$$

Proof. By Proposition 4.2,

$$\log_{\omega_A}(P^{(p)}_{A,r}(\chi)) = \mathrm{AJ}_{X_r}(\Delta_\psi(\chi))(\omega_{\theta_\psi} \wedge \eta^r_A). \tag{125}$$

Theorem 6.13 of [BDP] with $f = \theta_\psi$ and $j = 0$ gives

$$\mathrm{AJ}_{X_r}(\Delta_\psi(\chi))(\omega_{\theta_\psi} \wedge \eta^r_A)^2 = \frac{L_p(\theta_\psi, \chi \mathrm{N}_K)}{\Omega_p(A)^{2r}} \pmod{E_\chi^\times}. \tag{126}$$

The fact that θ_ψ has Fourier coefficients in K and that its Nebentype character ε_K is trivial when restricted to K implies that the field $E_{\psi,\chi,\varepsilon_K}$ occurring in Corollary 2.14 is equal to E_χ. Therefore, this corollary implies that

$$\begin{aligned} \frac{L_p(\theta_\psi, \chi \mathrm{N}_K)}{\Omega_p(A)^{2r}} &= \mathscr{L}_p(\psi^{-1}\chi \mathrm{N}_K) \times \frac{\mathscr{L}_p(\psi^{*-1}\chi \mathrm{N}_K)}{\Omega_p(A)^{2r}} \pmod{E_\chi^\times} \\ &= \frac{\mathscr{L}_p(\nu^*)}{\Omega_p(A)^{-1}} \times \frac{\mathscr{L}_p(\psi_A^{2r+1}\chi_0 \mathrm{N}_K^{-r})}{\Omega_p(A)^{2r+1}} \pmod{E_\chi^\times}. \end{aligned} \tag{127}$$

The character $\nu^* = \psi^*_A\chi_0$ lies in $\Sigma^{(1)}_{\mathrm{sd}}(c)$ and is of type $(0,1)$. Hence, Theorem 2.28 can be invoked. This theorem gives a global element $P_A(\chi_0) \in (A(H_c) \otimes E_\chi)^{\chi_0}$ which is of infinite order if and only if $L'(\psi_A\chi_0^{-1}, 1) \neq 0$, and satisfies

$$\mathscr{L}_p(\psi^*_A\chi_0) = \Omega_p(A)^{-1}\mathfrak{g}(\chi_0)\log^2_{\omega_A}(P_A(\chi_0)) \pmod{E_\chi^\times}. \tag{128}$$

Furthermore, the character $\psi_A^{2r+1}\chi_0\mathrm{N}_K^{-r}$ belongs to the domain $\Sigma^{(2)}_{\mathrm{sd}}(c)$ of classical interpolation for the Katz p-adic L-function. Proposition 1.16 and Lemma 1.15 show that the p-adic period attached to this central critical character is given by

$$\Omega_p((\psi_A^{2r+1}\chi_0\mathrm{N}_K^{-r})^*) = \Omega_p(A)^{2r+1}\mathfrak{g}(\chi_0)^{-1} \pmod{E_\chi^*}. \tag{129}$$

Therefore, Theorem 2.4 implies that, up to multiplication by a non-zero element of E_χ,

$$\mathscr{L}_p(\psi_A^{2r+1}\chi_0\mathrm{N}_K^{-r}) = \begin{cases} 0 & \text{if } L(\psi_A^{2r+1}\chi_0, r+1) = 0, \\ \Omega_p(A)^{2r+1}\mathfrak{g}(\chi_0)^{-1} & \text{otherwise.} \end{cases} \tag{130}$$

After setting

$$P_{A,r}(\chi_0) = \begin{cases} 0 & \text{if } L(\psi_A^{2r+1}\chi_0, r+1) = 0, \\ P_A(\chi_0) & \text{otherwise,} \end{cases} \tag{131}$$

equations (128) and (130) can be used to rewrite (127) as

$$\frac{L_p(\theta_\psi, \chi\mathrm{N}_K)}{\Omega_p(A)^{2r}} = \log^2_{\omega_A}(P_{A,r}(\chi_0)) \pmod{E_\chi^\times}. \tag{132}$$

Theorem 4.4 now follows by combining (125), (126) and (132). □

We now specialise the setting even further by assuming that $\chi_0 = 1$ is the trivial character, so that $\psi = \psi_A^{r+1}$ and $\chi = \psi_A^r$, and set

$$P^{(p)}_{A,r} := P^{(p)}_{\psi_A^{r+1}}(\psi^r_A).$$

In this case, the coefficient field E_χ is equal to K, and Theorem 4.4 asserts the existence of a point $P_{A,r} \in A(K)$ such that

$$\log^2_{\omega_A}(P^{(p)}_{A,r}) = \log^2_{\omega_A}(P_{A,r}) \pmod{K^\times}.$$

It is instructive to refine the argument used in the proof of Theorem 4.4 to resolve the ambiguity by the non-zero scalar in $K^\times$, in order to examine the dependence on r of the local point $P^{(p)}_{A,r}$. This is the content of the next result.

Theorem 4.5. *For all odd $r \geq 1$, the Chow-Heegner point $P^{(p)}_{A,r}$ belongs to $A(K) \otimes K$ and is given by the formula*

$$\log^2_{\omega_A}(P^{(p)}_{A,r}) = \ell(r) \cdot \log^2_{\omega_A}(P_A), \tag{133}$$

where $\ell(r) \in \mathbb{Z}$ satisfies

$$\ell(r) = \pm \frac{r!(2\pi)^r}{(2\sqrt{D})^r \Omega(A)^{2r+1}} L(\psi_A^{2r+1}, r+1),$$

and P_A is a generator of $A(K) \otimes K$ depending only on A but not on r.

Proof. As in the proof of Theorem 4.4, we combine (125) and Theorem 6.13 of [BDP] with $(f, j) = (\theta_{\psi_A^{r+1}}, 0)$ and $\chi \mathrm{N}_K = \psi_A^r \mathrm{N}_K$ playing the role of χ, to obtain

$$\log^2_{\omega_A}(P_\psi(\chi)) = (1 - (p\chi(\bar{\mathfrak{p}}))^{-1} a_p(\theta_\psi) + (p\chi(\bar{\mathfrak{p}}))^{-2} p^{r+1})^{-2} \frac{L_p(\theta_\psi, \chi \mathrm{N}_K)}{\Omega_p(A)^{2r}}. \tag{134}$$

Since $\chi(\bar{\mathfrak{p}}) = \psi_A(\bar{\mathfrak{p}})^r$ and $a_p(\theta_\psi) = \psi_A^{r+1}(\bar{\mathfrak{p}}) + \psi_A^{r+1}(\mathfrak{p})$, the Euler factor appearing in (134) is given by

$$(1 - \psi_A^{-1}(\mathfrak{p}))^{-2}(1 - \psi_A^{2r+1}(\mathfrak{p}) p^{-r-1})^{-2}.$$

Therefore,

$$\log^2_{\omega_A}(P_\psi(\chi)) = (1 - \psi_A^{-1}(\mathfrak{p}))^{-2}(1 - \psi_A^{2r+1}(\mathfrak{p}) p^{-r-1})^{-2} \frac{L_p(\theta_\psi, \chi \mathrm{N}_K)}{\Omega_p(A)^{2r}}. \tag{135}$$

On the other hand, by Theorem 2.13 with $c = 1$ and $j = 0$

$$\frac{L_p(\theta_\psi, \chi \mathrm{N}_K)}{\Omega_p(A)^{2r}} = \frac{w(\theta_\psi, \chi)^{-1}}{2^r} \times \mathscr{L}_p(\psi_A^*) \times \frac{\mathscr{L}_p(\psi_A^{2r+1} \mathrm{N}_K^{-r})}{\Omega_p(A)^{2r}}. \tag{136}$$

By Lemma 6.3 of [BDP], the norm 1 scalar $w(\theta_\psi, \chi)$ belongs to K, and is only divisible by the primes above $\sqrt{-D}$. Therefore it is a unit in $\mathcal{O}_K$, and hence is equal to ± 1. Therefore

$$\frac{L_p(\theta_\psi, \chi \mathrm{N}_K)}{\Omega_p(A)^{2r}} = \frac{\pm 1}{2^r} \times \frac{\mathscr{L}_p(\psi_A^*)}{\Omega_p(A)^{-1}} \times \frac{\mathscr{L}_p(\psi_A^{2r+1} \mathrm{N}_K^{-r})}{\Omega_p(A)^{2r+1}}. \tag{137}$$

Let $P_A = P_A(1) \in A(K) \otimes K$ be as in (128), but chosen specifically so that

$$\frac{\mathscr{L}_p(\psi_A^*)}{\Omega_p(A)^{-1}} = (1 - \psi_A^{-1}(\mathfrak{p}))^2 \log^2_{\omega_A}(P_A). \tag{138}$$

By the interpolation property for the Katz L-function given in Proposition 2.3 with $j = r$ and $\nu = \psi_A^{2r+1} \mathrm{N}_K^{-r} = \psi_A^{r+1} \psi_A^{*-r}$,

$$\frac{\mathscr{L}_p(\psi_A^{2r+1} \mathrm{N}_K^{-r})}{\Omega_p(A)^{2r+1}} = (1 - \psi_A(\mathfrak{p})^{2r+1} p^{-r-1})^2 \times \frac{r!(2\pi)^r L((\psi_A^*)^{2r+1} \mathrm{N}_K^{-r-1}, 0)}{\sqrt{D}^r \Omega(A)^{2r+1}}. \tag{139}$$

After substituting equations (138) and (139) into (137), and using the fact that

$$L((\psi_A^*)^{2r+1} \mathrm{N}_K^{-r-1}, 0) = L(\psi_A^{2r+1}, r+1),$$

we find

$$\begin{aligned} (1 - \psi_A^{-1}(\mathfrak{p}))^{-2}(1 - \psi_A(\mathfrak{p})^{2r+1} p^{-r-1})^{-2} \quad &\times \quad \frac{L_p(\theta_\psi, \chi \mathrm{N}_K)}{\Omega_p(A)^{2r}} \\ &= \quad \frac{\pm 1}{2^r} \log^2_{\omega_A}(P_A) \times \frac{r!(2\pi)^r L(\psi_A^{2r+1}, r+1)}{\sqrt{D}^r \Omega(A)^{2r+1}}. \end{aligned}$$

Hence, by (135), we obtain

$$\log^2_{\omega_A}(P_\psi(\chi)) \quad = \quad \pm \frac{r!(2\pi)^r}{(2\sqrt{D})^r \Omega(A)^{2r+1}} \times L(\psi_A^{2r+1}, r+1) \times \log^2_{\omega_A}(P_A)$$

The result follows. $\square$

5. Chow-Heegner points over $\mathbb{C}$

5.1. **The complex Abel-Jacobi map.** For simplicity, we will confine ourselves in this section to working under the hypotheses that were made in Section 2.7 where K is assumed in particular to have discriminant $-D$, with

$$D \in S := \{7, 11, 19, 43, 67, 163\}.$$

The difficulty in computing the modular parametrisation $\Phi^?$ and the resulting Chow-Heegner points arises from the fact that it is hard in general to explicitly produce the correspondence $\Phi^?$, or even to ascertain its existence. However, when $F = \mathbb{C}$, the Chow-Heegner points $P^?_\psi(\varphi)$ can be evaluated numerically in practice via integration of smooth differential forms on $X_r(\mathbb{C})$. More precisely, let

$$\mathrm{AJ}^\infty_A : \mathrm{CH}^1(A)_0(\mathbb{C}) \longrightarrow \frac{\mathrm{Fil}^1 H^1_{\mathrm{dR}}(A/\mathbb{C})^\vee}{\mathrm{Im}\, H_1(A(\mathbb{C}), \mathbb{Z})} \tag{140}$$

be the classical complex Abel-Jacobi map attached to A, where the superscript $\vee$ now denotes the complex linear dual. The map AJ^∞_A is defined by the rule

$$\mathrm{AJ}^\infty_A(\Delta)(\omega) = \int_{\partial^{-1}\Delta} \omega, \tag{141}$$

the integral on the right being taken over any one-chain on $A(\mathbb{C})$ having the degree zero divisor Δ as boundary. This classical Abel-Jacobi map admits a higher dimensional generalisation for null-homologous cycles on X_r introduced by Griffiths and Weil:

$$\mathrm{AJ}^\infty_{X_r} : \mathrm{CH}^{r+1}(X_r)_0(\mathbb{C}) \longrightarrow \frac{\mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/\mathbb{C})^\vee}{\mathrm{Im}\, H_{2r+1}(X_r(\mathbb{C}), \mathbb{Z})}, \tag{142}$$

defined just as in (141), but where $\mathrm{AJ}^\infty_{X_r}(\Delta)(\omega)$ is now defined by integrating any smooth representative of the de Rham cohomology class ω against a $(2r+1)$-chain on $X_r(\mathbb{C})$ having Δ as boundary. (Cf. the description in Chapter 3 of [BDP] for example.) The map $\mathrm{AJ}^\infty_{X_r}$ is the complex analogue of the p-adic Abel-Jacobi map AJ_{X_r} that was introduced and studied in Section 4.

The functoriality of the Abel-Jacobi maps under correspondences is expressed in the following commutative diagram which is the complex counterpart of (122):

$$\begin{tikzcd}
\mathrm{CH}^{r+1}(X_r)_0(\mathbb{C}) \arrow[r, "\mathrm{AJ}^\infty_{X_r}"] \arrow[d, dotted, "\Phi^?_{\mathbb{C}}"] & \frac{\mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/\mathbb{C})^\vee}{\mathrm{Im}\, H_{2r+1}(X_r(\mathbb{C}),\mathbb{Z})} \arrow[d, "(\Phi^{?*}_{\mathrm{dR}})^\vee"'] \\
\mathrm{CH}^1(A)_0(\mathbb{C}) \arrow[r, "\mathrm{AJ}^\infty_A"] & \frac{\Omega^1(A/\mathbb{C})^\vee}{\mathrm{Im}\, H_1(A(\mathbb{C}),\mathbb{Z})}.
\end{tikzcd} \tag{143}$$

Since AJ^∞_A is an isomorphism, we can simply define the complex analogue $\Phi_{\mathbb{C}}$ of $\Phi^{(p)}_F$ as the unique map from $\mathrm{CH}^{r+1}(X_r)_0(\mathbb{C})$ to $A(\mathbb{C})$ for which the diagram above (with $\Phi^?_{\mathbb{C}}$ replaced by $\Phi_{\mathbb{C}}$) commutes.

Recall the distinguished element ω_A of $\Omega^1(A/\mathbb{C})$ and let

$$\Lambda_A := \left\{ \int_\gamma \omega_A, \quad \gamma \in H_1(A(\mathbb{C}), \mathbb{Z}) \right\} \subset \mathbb{C}$$

be the associated period lattice. Recall that $\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$ is an isogeny of elliptic curves with Γ-level structure if

$$\varphi(t_A) = t' \quad \text{and} \quad \varphi^*(\omega') = \omega_A.$$

The following proposition, which is the complex counterpart of Proposition 4.2, expresses the Abel-Jacobi image of $P^?_\psi(\varphi)$ in terms of the Griffiths higher Abel-Jacobi map.

Proposition 5.1. *For all isogenies $\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$ of elliptic curves with Γ-level structure,*

$$\mathrm{AJ}^\infty_A(P^?_\psi(\varphi))(\omega_A) = \mathrm{AJ}^\infty_{X_r}(\Delta_\varphi)(\omega_{\theta_\psi} \wedge \eta^r_A) \pmod{\Lambda_{\omega_A}}.$$

Proof. The proof is the same as for Proposition 4.2: By definition of $P_\psi(\varphi)$ combined with the commutative diagram (143),

$$\mathrm{AJ}_A^\infty(P_\psi(\varphi))(\omega_A) = \mathrm{AJ}_A^\infty(\Phi_{\mathbb{C}}^?\Delta_\varphi)(\omega_A) = \mathrm{AJ}_{X_r}^\infty(\Delta_\varphi)(\Phi_{\mathrm{dR}}^{?*}\omega_A).$$

On the other hand, by Proposition 3.9,

$$\Phi_{\mathrm{dR}}^{?*}\omega_A = \omega_{\theta_\psi} \wedge \eta_A^r.$$

Proposition 5.1 follows. □

We now turn to giving an explicit formula for the right hand side of the equation in Proposition 5.1. To do this, let $\Lambda_{\omega'} \subset \mathbb{C}$ be the period lattice associated to the regular differential ω' on A'. Note that Λ_{ω_A} is contained in $\Lambda_{\omega'}$ with index $\deg(\varphi)$.

Definition 5.2. A basis (ω_1, ω_2) of $\Lambda_{\omega'}$ is said to be *admissible* relative to (A', t') if

1. The ratio $\tau := \omega_1/\omega_2$ has positive imaginary part;
2. via the identification $\frac{1}{N}\Lambda_{\omega'}/\Lambda_{\omega'} = A'(\mathbb{C})[N]$, the N-torsion point ω_2/N belongs to the orbit t'.

Given an arbitrary cusp form $f \in S_{r+2}(\Gamma_0(N), \varepsilon)$, consider the cohomology class

$$\omega_f \wedge \eta_A^r = (2\pi i)^{r+1} f(z)dzdw^r \wedge \eta_A^r \in \mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r/\mathbb{C}).$$

Proposition 5.3. *Let Δ_φ be the generalised Heegner cycle corresponding to the isogeny*

$$\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$$

of elliptic curves with Γ-level structure, let (ω_1, ω_2) be an admissible basis for $\Lambda_{\omega'}$, and let $\tau = \omega_1/\omega_2$. Then

$$\mathrm{AJ}_{X_r}^\infty(\Delta_\varphi)(\omega_f \wedge \eta_A^r) = \omega_2^{-r}\frac{(2\pi i)^{r+1}}{(\tau - \bar\tau)^r}\int_{i\infty}^{\tau}(z - \bar\tau)^r f(z)dz. \qquad (144)$$

Proof. We begin by observing that replacing ω_A by a scalar multiple $\lambda\omega_A$ multiplies both the left and right hand sides of (144) by λ^{-r}. Hence we may assume, after possibly rescaling $\Lambda_{\omega'}$, that the admissible basis (ω_1, ω_2) is of the form $(2\pi i\tau, 2\pi i)$ with $\tau \in \mathcal{H}$. The case $j = 0$ in Theorem 3.15 of [BDP] then implies that

$$\begin{aligned}\mathrm{AJ}_{X_r}^\infty(\Delta_\varphi)(\omega_f \wedge \eta_A^r) &= \frac{2\pi i}{(\tau - \bar\tau)^r}\int_{i\infty}^{\tau}(z - \bar\tau)^r f(z)dz \\ &= \omega_2^{-r}\frac{(2\pi i)^{r+1}}{(\tau - \bar\tau)^r}\int_{i\infty}^{\tau}(z - \bar\tau)^r f(z)dz.\end{aligned}$$

The Proposition follows. □

Theorem 5.4. *Let $P_\psi(\varphi)$ be the Chow-Heegner point corresponding to the generalised Heegner cycle Δ_φ. With notations as in Proposition 5.3,*

$$\mathrm{AJ}_A^\infty(P_\psi(\varphi))(\omega_A) = \omega_2^{-r}\frac{(2\pi i)^{r+1}}{(\tau - \bar\tau)^r}\int_{i\infty}^{\tau}(z - \bar\tau)^r \theta_\psi(z)dz \pmod{\Lambda_{\omega_A}}. \qquad (145)$$

Proof. This is an immediate corollary of Propositions 5.1 and 5.3. □

5.2. **Numerical experiments.** We now describe some numerical evaluations of Chow-Heegner points. As it stands, the elliptic curve A of conductor D^2 attached to the canonical Hecke character $\psi_A = \psi_0$ is only determined up to isogeny, and we pin it down by specifying that A is described by the minimal Weierstrass equation

$$A : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where the coefficients $a_1, \ldots, a_6$ are given in Table 1 below.

D	a_1	a_2	a_3	a_4	a_6	$\Omega(A)$	P_A
7	1	-1	0	-107	552	1.93331170...	$-$
11	0	-1	1	-7	10	4.80242132...	$(4,5)$
19	0	0	1	-38	90	4.19055001...	$(0,9)$
43	0	0	1	-860	9707	2.89054107...	$(17,0)$
67	0	0	1	-7370	243528	2.10882279...	$(\frac{201}{4}, \frac{-71}{8})$
163	0	0	1	-2174420	1234136692	0.79364722...	$(850,-69)$

Table 1: The canonical elliptic curve A

The penultimate column in Table 1 gives an approximate value for the positive real period $\Omega(A)$ attached to the elliptic curve A and its Néron differential ω_A. In all cases, the Néron lattice Λ_A attached to (A, ω_A) is generated by the periods

$$\omega_1 := \left(\frac{D+\sqrt{-D}}{2D}\right)\Omega(A), \qquad \omega_2 := \Omega(A), \tag{146}$$

and (ω_1, ω_2) is an admissible basis for Λ_A in the sense of Definition 5.2.

The elliptic curve A has Mordell-Weil rank 0 over $\mathbb{Q}$ when $D = 7$ and rank one otherwise. A specific generator P_A for $A(\mathbb{Q}) \otimes \mathbb{Q}$ is given in the last column of Table 1.

Recall that ψ_A is the Hecke character attached to A, satisfying

$$L(\psi_A, s) = L(A, s).$$

As in Theorem 4.5, we let $\psi = \psi_A^{r+1}$ with $r \geq 1$ an *odd* integer, so that the associated theta series θ_ψ belongs to $S_{r+2}(\Gamma_0(D), \varepsilon_K)$ and therefore satisfies the Heegner hypothesis.

After letting $\Gamma = \Gamma_{\varepsilon_K}(D)$, we observe as before that the elliptic curve A has two canonical Γ-level structures defined over K, corresponding to the two orbits of $((\mathbb{Z}/D\mathbb{Z})^\times)^2$ acting on $A[\sqrt{-D}]^*$. Fix such a Γ-level structure t_A once and for all. Let $C := \tilde{X}_0(D)$ be the coarse moduli space classifying elliptic curves with Γ-level structure. The forgetful functor which to (E, t_E) associates $(E, \langle t_E\rangle)$, where $\langle t_E\rangle$ denotes the cyclic subgroup of order D generated by t_E, induces an *isomorphism* from $\tilde{X}_0(D)$ to $X_0(D)$. But the moduli description for $\tilde{X}_0(D)$ is somewhat finer, since an elliptic curve E with Γ-level structure has no non-trivial automorphisms, at least when $\mathrm{End}(E) \neq \mathbb{Z}[i]$ or $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. (In particular, $\tilde{X}_0(D)$ is a fine moduli space when $D \equiv 11 \pmod{12}$.) Denote by W_r the r-th Kuga-Sato variety over the modular curve $\tilde{X}_0(D)$ and let $X_r = W_r \times A^r$.

Assume that an algebraic correspondence $\Pi^?$ from W_r to A^{r+1} as in Proposition 3.8 exists. Assume further that it is defined over $\mathbb{Q}$, and has integral coefficients, i.e., that it belongs to $\mathrm{Corr}^0(W_r, A^{r+1})$ rather than just $\mathrm{Corr}^0(W_r, A^{r+1}) \otimes \mathbb{Q}$. In that case it gives rise to a parametrisation

$$\Phi^? : \mathrm{CH}^{r+1}(X_r)_0 \longrightarrow A,$$

where $X_r = W_r \times A^r$.

Since the space of theta-series attached to the elliptic curve A has dimension $h(-D) = 1$, and since the classes of ω_{θ_ψ} and ω_A are both defined over $\mathbb{Q}$, there is a non-zero rational scalar c satisfying

$$\Pi_{\mathrm{dR}}^{?*}(\omega_A^{r+1}) = c \cdot \omega_{\theta_\psi}.$$

This scalar can be viewed as playing the role of the Manin-constant in the context of the modular parametrisation of A by $\mathrm{CH}^{r+1}(X_r)_0$.

Question 5.5. *When is it possible to choose the integral cycle* $\Pi^?$ *so that* $c = 1$*?*

By allowing $\Pi^?$ to belong to $\mathrm{Corr}^0(W_r, A^{r+1}) \otimes \mathbb{Q}$, we can assume, after suitably rescaling $\Pi^?$, that $c = 1$. We will assume for the rest of this chapter that $\Pi^?$ has been rescaled in this way.

5.2.1. *Chow-Heegner points of level* 1. For $D \in S := \{11, 19, 43, 67, 163\}$, the elliptic curve A has rank 1 over $\mathbb{Q}$. Let $r \geq 1$ be an odd integer. By Theorem 5.4, the Chow-Heegner point $P_{A,r}$ attached to the class of the diagonal $\Delta \subset (A \times A)^r$ is given by

$$\mathrm{AJ}_A^\infty(P_{A,r})(\omega_A) = J_r := \omega_2^{-r}\frac{(2\pi i)^{r+1}}{(\tau-\bar{\tau})^r}\int_{i\infty}^{\tau}(z-\bar{\tau})^r\theta_\psi(z)dz, \tag{147}$$

where (ω_1, ω_2) is the admissible basis of Λ_A given in (146), and $\tau = \frac{\omega_1}{\omega_2} = \frac{D+\sqrt{-D}}{2D}$. Hence the complex point $P_{A,r}$ can be computed as the natural image of the complex number J_r under the Weierstrass uniformisation.

We have calculated the complex points $P_{A,r}$ for all $D \in S$ and all $r \leq 15$, to roughly 200 digits of decimal accuracy. The calculations indicate that

$$P_{A,r} \stackrel{?}{=} \sqrt{-D} \cdot m_r \cdot P_A \pmod{A(\mathbb{C})[\iota_r]}, \tag{148}$$

where P_A is the generator of $A(\mathbb{Q}) \otimes \mathbb{Q}$ given in Table 1, and m_r is the rational integer listed in Table 2 below, in which the columns correspond to $D \in S$ and the rows to the odd r between 1 and 15.

	11	19	43	67	163
1	1	1	1	1	1
3	2	6	36	114	2172
5	-8	-16	440	6920	3513800
7	14	-186	-19026	-156282	3347376774
9	304	4176	-8352	-34999056	-238857662304
11	-352	-33984	33708960	3991188960	-3941159174330400
13	76648	545064	-2074549656	46813903656	1904546981028802344
15	274736	40959504	47714214240	-90863536574160	8287437850155973464480

Table 2: The constants m_r for $1 \leq r \leq 15$.

The first 6 lines in this table, corresponding to $1 \leq r \leq 11$, are in perfect agreement with the values that appear in the third table of Section 3.1 of [RV]. This coincidence, combined with Theorem 3.1. of [RV], suggests the following conjecture which is consistent with the p-adic formulae obtained in Theorem 4.5.

Conjecture 5.6. *For all $D \in S$ and all odd $r \geq 1$, the Chow-Heegner point $P_{A,r}$ belongs to $A(K) \otimes \mathbb{Q}$ and is given by the formula*

$$P_{A,r} = \sqrt{-D} \cdot m_r \cdot P_A, \tag{149}$$

where $m_r \in \mathbb{Z}$ satisfies the formula

$$m_r^2 = \frac{2r!(2\pi\sqrt{D})^r}{\Omega(A)^{2r+1}} L(\psi_A^{2r+1}, r+1),$$

and P_A is the generator of $A(\mathbb{Q}) \otimes \mathbb{Q}$ given in Table 1.

The fact that the experimentally observed equality in (148) only holds modulo a subgroup of $A(\mathbb{C})_{\text{tors}}$ may reflect the fact that the correspondence $\Pi^?$ from W_r to A^{r+1} needs to be taken in $\mathrm{Cor}(W_r, A^{r+1}) \otimes \mathbb{Q}$ in general. The optimal values of ι_r that were observed experimentally are recorded in Table 3 below, for $1 \leq r \leq 31$.

r	11	19	43	67	163	r	11	19	43	67	163
1	3	1	1	1	1	17	3^3	7	1	19	1
3	$3 \cdot 5$	5	1	1	1	19	$3 \cdot 5^2$	$5^2 \cdot 11$	11	1	1
5	$2 \cdot 3^2$	$2 \cdot 7$	2	2	2	21	$3 \cdot 23$	23	23	23	1
7	$2 \cdot 7$	5	1	1	1	23	$3^2 \cdot 5$	$5 \cdot 7$	13	1	1
9	3	11	11	1	1	25	3	1	1	1	1
11	$3^2 \cdot 5$	$5 \cdot 7$	13	1	1	27	$3 \cdot 5$	5	1	29	1
13	3	1	1	1	1	29	$3^2 \cdot 31$	$7 \cdot 11$	$11 \cdot 31$	1	1
15	$3 \cdot 5$	$5 \cdot 17$	17	17	1	31	$3 \cdot 5$	$5 \cdot 17$	17	1	1

Table 3: The ambiguity factor ι_r for $1 \leq r \leq 31$.

Remark 5.7. The data in Table 3 suggests that the term ι_r in (148) is only divisible by primes that are less than or equal to $r+2$. One might therefore venture to guess that the primes ℓ dividing $\iota_{D,r}$ are only those for which the mod ℓ Galois representation attached to ψ_A^{r+1} has very small image, or perhaps non-trivial G_K-invariants.

5.2.2. *Chow-Heegner points of prime level.* We may also consider (for a fixed D and a fixed odd integer r) the Chow-Heegner points on A attached to non-trivial isogenies φ. For instance, let $\ell \neq D$ be a prime. There are $\ell + 1$ distinct isogenies $\varphi_j : A \longrightarrow A'_j$ of degree ℓ (with $j = 0, 1, \ldots, \ell - 1, \infty$) attached to the lattices $\Lambda'_0, \ldots, \Lambda'_{\ell-1}, \Lambda'_\infty$ containing Λ_A with index ℓ. These lattices are generated by the admissible bases

$$\Lambda'_j = \mathbb{Z}\left(\frac{\omega_1 + j\omega_2}{\ell}\right) \oplus \mathbb{Z}\omega_2, \qquad \Lambda'_\infty = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\frac{\omega_2}{\ell}.$$

The elliptic curves A'_j and the isogenies φ are defined over the ring class field H_ℓ of K of conductor ℓ. Let

$$\begin{aligned} J_r(\ell,j) &:= \ell^r\omega_2^{-r}\frac{(2\pi i)^{r+1}}{(\tau-\bar\tau)^r}\int_{i\infty}^{\frac{\tau+j}{\ell}}\left(z-\frac{\bar\tau+j}{\ell}\right)^r\theta_{D,r}(z)dz, \quad 0\le j\le \ell-1,\\ J_r(\ell,\infty) &:= \varepsilon_K(\ell)\omega_2^{-r}\frac{(2\pi i)^{r+1}}{(\tau-\bar\tau)^r}\int_{i\infty}^{\ell\tau}(z-\ell\bar\tau)^r\theta_\psi(z)dz\end{aligned}$$

be the associated complex invariants and let $P_{A,r}(\ell,j)$ and $P_{A,r}(\ell,\infty)$ denote the corresponding points in $\mathbb{C}/\Lambda_A = A(\mathbb{C})$.

We have attempted to verify the following conjecture numerically.

Conjecture 5.8. *For all $\ell\neq D$ and all $j\in\mathbf{P}_1(\mathbb{F}_\ell)$, the complex points $P_{A,r}(\ell,j)$ belong to the Mordell-Weil group $A(H_\ell)\otimes\mathbb{Q}$. More precisely,*

(1) *If ℓ is inert in K, then $\mathrm{Gal}(H_\ell/K)$ acts transitively on the set*

$$\{P_{A,r}(\ell,j),\quad j\in\mathbf{P}_1(\mathbb{F}_\ell)\}$$

of Chow-Heegner points of level ℓ.

(2) *If $\ell=\lambda\bar\lambda$ is split in K, then there exist $j_1,j_2\in\mathbf{P}_1(\mathbb{F}_\ell)$ for which*

$$P_{A,r}(\ell,j_1)=\varepsilon_K(\lambda)\lambda^rP_{A,r},\qquad P_{A,r}(\ell,j_2)=\varepsilon_K(\bar\lambda)\bar\lambda^rP_{A,r},$$

and $\mathrm{Gal}(H_\ell/K)$ acts transitively on the remaining set

$$\{P_{A,r}(\ell,j),\quad j\in\mathbf{P}_1(\mathbb{F}_\ell)-\{j_1,j_2\}\}$$

of Chow-Heegner points of level ℓ.

We have tested this prediction numerically for $r=1$ and all

$$D\in S,\qquad \ell=2,3,5,7,11,$$

as well as in a few cases where $r=3$. In all cases the points $P_{A,r}(\ell,j)$ were identified with algebraic points in $A(H_\ell)\otimes\mathbb{Q}$, with a convincing amount of numerical accuracy. Such calculations sometimes required several hundred digits of numerical precision, together with a bit of trial and error. The necessity for this arose because Conjecture 5.8 only predicts that the points $P_{A,r}(\ell,j)$ belong to $A(H_\ell)\otimes\mathbb{Q}$ and not $A(H_\ell)$. So in practice, these complex points need to be multiplied by a (typically small) integer in order to belong to $A(H_\ell)$. Furthermore, the resulting global points appear (as suggested by (149) in the case $\ell=1$) to be divisible by $\sqrt{-D}$, and this causes their heights to be rather large. It is therefore better in practice to divide the $P_{A,r}(\ell,j)$ by $\sqrt{-D}$, which introduces a further ambiguity of $A(\mathbb{C})[\sqrt{-D}]$ in the resulting global point. The conjecture that was eventually tested numerically is the following non-trivial strengthening of Conjecture 5.8:

Conjecture 5.9. *Given integers $n\in\mathbb{Z}^{\ge1}$ and $0\le s\le D-1$, let*

$$\begin{aligned} J'_r(\ell,j) &= n\cdot\frac{J_r(\ell,j)-s\omega_1}{\sqrt{-D}}, \qquad 0\le j\le\ell-1,\\ J'_r(\ell,\infty) &= n\cdot\frac{J_r(\ell,\infty)-s\varepsilon_K(\ell)\ell^r\omega_1}{\sqrt{-D}},\end{aligned}$$

and let $P'_{A,r}(\ell,j)\in A(\mathbb{C})$ be the associated complex points. Then there exist $n=n_{D,r}$ and $s=s_{D,r}$, depending on D and r but not on ℓ and j, for which the points $P'_r(\ell,j)$ belong to $A(H_\ell)$ and satisfy all the conclusions of Conjecture 5.8 with $P_{A,r}(\ell,j)$ replaced by $P'_{A,r}(\ell,j)$.

We now describe a few sample calculations that lend support to Conjecture 5.9.

1. The case $D=7$. Consistent with the fact that the elliptic curve A has rank 0 over $\mathbb{Q}$ (and hence over K as well), the point $P_{A,r}$ appears to be a torsion point in $A(\mathbb{C})$, for all $1\le r\le 31$. For example, the invariant J_1 agrees with $(\omega_1+\omega_2)/8$ to the 200 decimal digits of accuracy that were calculated. When $\ell=2$, it also appears that the quantities $J_1(2,j)$ belong to $\frac{1}{8}\Lambda_7$. There is no reason, however, to expect the Chow-Heegner points $P_{A,r}(\ell,j)$ to be torsion for larger values of ℓ. Experiments suggest that the constants in Conjecture 5.9 are

$$n_{7,1}=4,\qquad s_{7,1}=0.$$

For example, when $\ell = 3$, the ring class field of conductor ℓ is a cyclic quartic extension of K containing $K(\sqrt{21})$ as its quadratic subfield. In that case, the points $P'_{A,1}(3,j)$ satisfy

$$P'_{A,1}(3,0) = P'_{A,1}(3,1) = -P'_{A,1}(3,2) = -P'_{A,1}(3,\infty),$$

and agree to 600 digits of accuracy with a global point in $A(\mathbb{Q}(\sqrt{21}))$ of relatively small height, with x-coordinate given by

$$x = \frac{25947591117510092692083536058220938825 9}{41395589491845015952295204909998656004}.$$

2. The case $D = 19$. To compute the Chow-Heegner points of conductor 3 in the case $D = 19$ and $r = 1$, it appears that one can take

$$n_{19,1} = 1, \qquad s_{19,1} = 1.$$

Perhaps because of the small value of $n_{19,1}$, the points $P'_{A,1}(\ell, j)$ appear to be of relatively small height and can easily be recognized as global points, even for moderately large values of ℓ. For instance, the points $P'_{A,1}(3,j)$ seem to have x-coordinates of the form

$$x = \frac{-19 \pm 3\sqrt{57}}{2},$$

and their y-coordinates satisfying the degree 4 polynomial

$$x^4 + 2x^3 + 8124x^2 + 8123x - 217886$$

whose splitting field is the ring class field H_3 of K of conductor 3.

When $\ell = 7$, which is split in $K/\mathbb{Q}$, the ring class field H_7 is a cyclic extension of K of degree 6. It appears that the points $P'_{A,1}(7,3)$ and $P'_{A,1}(7,5)$ belong to $A(K)$ and are given by

$$P'_{A,1}(7,3) = \frac{3+\sqrt{-19}}{2} P_A, \quad P'_{A,1}(7,5) = \frac{3-\sqrt{-19}}{2} P_A.$$

The 6 remaining points are grouped into three pairs of equal points,

$$P'_{A,1}(7,0) = P'_{A,1}(7,2), \qquad P'_{A,1}(7,1) = P'_{A,1}(7,6), \qquad P'_{A,1}(7,4) = P'_{A,1}(7,\infty),$$

whose x and y coordinates appear to satisfy the cubic polynomials

$$9x^3 + 95x^2 + 19x - 1444, \qquad 27x^3 - 235x^2 + 557x + 1198$$

respectively. The splitting field of both of these polynomials turns out to be the cubic subfield L of the ring class field of K of conductor 7. One obtains as a by-product of this calculation 3 independent points in $A(L)$ which are linearly independent over $\mathcal{O}_K$. We expect that these three points give a K-basis for $A(L) \otimes \mathbb{Q}$ (and therefore that $A(L)$ has rank 6) but have not checked this numerically.

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