Some past and present collaborations

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Outline

The general direction of my work has been quite different from those described in previous talks in this seminar, and (I think) also from those of people yet to speak.

Much of my work is on the arithmetic of special surfaces and threefolds, and makes heavy use of symbolic computation.

In this talk I will describe several different joint projects, all of which arose from being asked for computational assistance by someone who works in a somewhat different field.

My approach to doing mathematics

I am interested in a lot of things I know nothing about.

On the other hand, my time is generally quite limited, because I have a full-time job that (under normal circumstances) does not leave me very much time for research.

So I like to work on things in a way that means I can make a little bit of progress in most sessions. This has driven me toward computational work.

Usually I like to start with some small observation and try to build on and around it until I have worked out enough of the pattern to have something to publish.

Many people prefer to go in the opposite direction.

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Setting the stage

I first met Henri in the early nineties, when he was a postdoc and I was an undergraduate. In 2002 I came to CICMA as a postdoc.

ATR extensions of real quadratic fields

At the time Henri was at a fairly early stage in the development of his ideas for constructing rational points on elliptic curves over number fields. He had formulated some precise conjectures and described some philosophical underpinnings, but much still lay in the future.

He generously invited me to help him test his conjectures in some of the smallest cases: in particular, for elliptic curves over real quadratic fields, where the points that are constructed lie in quartic fields of signature $(2, 1)$. We focused on the three smallest curves with everywhere good reduction, defined over $\mathbb{Q}(\sqrt{29}), \mathbb{Q}(\sqrt{37}), \mathbb{Q}(\sqrt{41})$.

In this situation there is no Jacquet-Langlands correspondence to quaternion algebras, and anyway he wanted to get points over non-CM extensions.

What did we do?

He patiently explained to me how to set up the necessary integrals, and I wrote some code to compute them, which involved counting points on elliptic curves over non-prime finite fields (not so easily available back then).

It also required writing down continued fractions for certain elements of the quadratic fields, which (thanks to the efforts of a couple of undergraduates over the summer) I think I could do better now.

In any case, the conjecture claims that certain "CM points" map by an analogue of the modular parametrization to points on the elliptic curve defined over the quartic fields.

What happened?

It all worked out beautifully.

D_K	x	y	P'
-7	β^2+3	$-5\beta^3/2-3\beta^2-8\beta-19/2$	
-16	$\beta^2/2$	$-5\beta^3/4-11\beta^2/4-\beta/4-1/2$	
-23	$(11\beta^2+5)/8$	$-13\beta^3/8-\beta^2-7\beta/8-1/2$	
-35_1	$(2\beta^2+1)/5$	$-59\beta^3/225-43\beta^2/90-89\beta/450-29/90$	P_{-7}
-35_2	$(-4\beta^2-11)/15$	$(-17\beta^3-105\beta^2-43\beta-270)/150$	P_{-7}
-59	$-1/9$	$-11\beta^3/1512-5\beta^2/56-\beta/1512+1/504$	
-63	$7\beta^2/9+5$	$-59\beta^3/225-43\beta^2/90-89\beta/450-29/90$	
-64	$-1/4$	$-3\beta^3/8-5\beta^2/4-\beta/4-3/8$	
-80	$(43\beta^2+51)/10$	$-517/50\beta^3-93/20\beta^2-1233/100\beta$ $-111/20$	P_{-16}
-91	$(98\beta^2+387)/13$	$-18939\beta^3/845-111\beta^2/26$ $-150109\beta/1690-439/26$	
-175	$(-3\beta^2-13)/5$	$-\beta^3/10-11\beta^2/25-37\beta/25-67/10$	See †

Table 29.2. Generators of $E(K)$ modulo torsion

And what happened next?

As previously mentioned, Henri continued to develop these ideas in many deep and fascinating directions.

I moved to Liverpool, where I was strongly influenced by Victor Flynn and Slava Nikulin. I learned to use Magma and started doing computations on del Pezzo and K3 surfaces instead.

But it would be interesting to get back into this.

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A postmodern experience

Never having met Brandon Williams, this is how I picture him. However, I wrote an appendix to a paper of his.

The work of Williams

In 2018 Brandon Williams released a paper on the arXiv in which he determined the rings of Hilbert modular forms for $\mathbb{Q}(\sqrt{29})$ and $\mathbb{Q}(\sqrt{37})$. I was quite impressed by his work, but felt that there was something missing.

His work can be interpreted as describing the Baily-Borel compactification of the Hilbert modular varieties for these fields. On the other hand, Elkies and Kumar have given nice models for such varieties by parametrizing elliptic fibrations that appear on the Kummer surfaces of the quotients of abelian surfaces with real multiplication.

Hilbert modular varieties and Kummer surfaces

Let K be a totally real number field of degree d. The *Hilbert modular variety* for K is the quotient $\mathcal{H}^d/SL_2(\mathcal{O}_K)$, where $\mathcal{H}$ is the upper half-plane and $SL_2(\mathcal{O}_K)$ acts by its image in $SL_2(\mathbb{R})^d$ given by the d embeddings.

Points of this variety correspond to abelian varieties of dimension d with an action of $\mathcal{O}_K$.

If A is an abelian surface, then $A/\pm 1$ is a surface with 16 singular points. The minimal resolution of this is called the *Kummer surface* of A. It is a K3 surface whose Picard rank is 16 more than that of A.

My contribution

I thought it would be interesting to find an explicit birational equivalence between the model given by Elkies-Kumar and that of Williams. So I started on this and told Williams what I was doing.

He agreed that it would be a useful addition to his paper. We eventually decided that his paper would not change and that I would write an appendix to it.

Fortunately the Journal of Algebra was willing to go along.

How did I do this?

Let's take $\mathbb{Q}(\sqrt{29})$ (the other one is similar but takes more steps).

The model given by Williams is a highly singular surface in a complicated weighted projective space.

First I simplified it by removing some variables that appear with degree 1 in one of the relations. This gave me a model that was easy to embed into $\mathbb{P}^6$ as a surface of degree 15.

The singularities of this surface are quite far from anything one would want to work with.

Improving the model

The basic step is to project away from the worst singularities (a sort of poor person's blowup, for those who can't afford a product of projective spaces).

Eventually the ambient space becomes too small and we need to apply the Veronese embedding to give ourselves more room.

After a few iterations of these steps we get a model with canonical singularities, whose equations look like a familiar K3 surface.

Matching it to Elkies-Kumar

The singular points give rational curves on the desingularization of the K3 surface. There are other rational curves that were not hard to find.

Once you have some rational curves, you can make elliptic fibrations. This gives you more rational curves, etc.

After a short time, I found an elliptic fibration whose general fibre is birationally equivalent to that of a fibration found by Elkies and Kumar.

Future work, and an apology

A few months later, Henri asked me a somewhat related question. Roughly, he wanted to know whether such an identification can be made more naturally for the three fields $\mathbb{Q}(\sqrt{29}), \mathbb{Q}(\sqrt{37}), \mathbb{Q}(\sqrt{41})$ of our paper.

Since Williams' methods give a description of some of the Hirzebruch-Zagier cycles, this is a meaningful thing to do, and (Henri thinks) it might lead to a proof that some of the points we computed are algebraic.

Unfortunately I haven't been working on this lately. Perhaps this fall . . . .

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Inside and out

I first met Colin Ingalls (left) in graduate school. A few years ago he moved to Carleton.

Owen Patashnick and I have been working together for about 10 years, but it would be illegal to talk about most of our work here.

Our slogan

"Owen wants to construct categories of mixed motives; Adam just wants to play with K3 surfaces."

Let us pretend that we understand the motives of curves.

For some people this requires more of an effort than for others. For example, when I hear "motive" I think "compatible system of ℓ-adic representations".

What other motives can we describe? More generally, which varieties are covered by products of curves?

All motives of abelian varieties are exterior powers of quotients of curve motives.

Surfaces

Well, after curves the next thing is surfaces.

Rational and ruled surfaces: no problem.

Surfaces of general type: too hard! (Serre showed that certain surfaces contained in abelian varieties have motives not described by curve motives.)

Enriques surfaces: they're covered by K3 surfaces, so let's think about those first. Also, they don't have interesting motives.

K3 surfaces: maybe just right.

The Kuga-Satake construction

Given a Hodge structure of type $(1, n, 1)$, there is an associated abelian variety A of dimension 2^n, at least up to isogeny. This can be described in terms of a Clifford algebra.

If the Hodge structure comes from a variety V (for example, the transcendental part of H^2 of a K3 surface), then we expect that there is a correspondence between V and A.

This would follow from familiar but unattackable conjectures, so it is interesting to prove it in special cases.

What is known?

Let X be a K3 surface. If X has Picard rank ≥ 19, and sometimes when the rank is 17 or 18, there are finite maps between X and a Kummer surface. (This is the starting point for Calegari's program to prove the potential modularity of high rank K3 surfaces.)

A Kummer surface is covered by an abelian surface: done.

Paranjape showed that a K3 surface given by an equation of the form $t^2 = \prod_{i=1}^{6}(a_i x + b_i y + c_i z)$ is covered by the square of a curve of genus 5 with an automorphism of order 8. These are the K3 surfaces of degree 2 with 15 nodes.

Another case

We thought about Paranjape's construction. Eventually we found a generalization that shows that (among other families) the motives of K3 surfaces of degree 6 with 15 nodes are also described by curves, this time of genus 7.

In fact, we showed a birational equivalence between the moduli space of a certain family of K3 surfaces of Picard rank 16 and of curves of genus 1 with 4 marked points and some additional data (both moduli spaces are rational).

This let us clarify some issues not discussed by Paranjape: for example, the fact that the map of moduli is of degree 1 is quite helpful for Calegari's program.

What goes into such a result?

Given the curve of genus 1 and the additional data, you can construct the cover of genus 7. Its square has a large automorphism group; a quotient is an elliptic surface over an elliptic curve, which in turn has the desired K3 surface as a quotient. That gives you the map of moduli spaces in one direction.

To go the other way, we look at a fibration on the K3 surface that comes out of our construction. This gives us a map to a third moduli space, which is a subvariety of $\overline{\mathcal{M}}_{0,10}$. In turn, that one gives us a map to $\mathbb{P}^1$ which gives us four marked points: we then get an elliptic curve and (by looking at the differences of some other points there) the rest of the needed data.

The future of this project

Our work produces the desired correspondence (as does Paranjape's), proving the relevant case of the Hodge conjecture. It also shows that the motives of our family of K3 surfaces are described by motives of curves.

Once we get this through prepublication review, it will go on the arXiv.

We also have a construction for another (unrelated) family of K3 surfaces of rank 16. In principle there should be one for every imaginary quadratic field, but since $\mathbb{Q}(\sqrt{-n})$ is not usually a cyclotomic field, you need curves with correspondences, rather than just automorphisms. This is hard.

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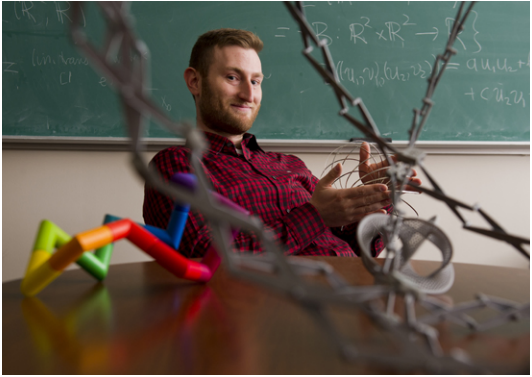
How this got started



Last month Jared Weinstein gave a very interesting seminar, which he concluded by asking for help from people with experience calculating with K3 surfaces. Naturally I volunteered.

What is the problem?

The BSD conjecture predicts the existence of rational points on elliptic curves.

If the base field is a function field of a curve, then this can be interpreted in terms of curves on surfaces.

We have another conjecture that predicts those, namely the Tate conjecture. And in fact the Tate conjecture implies BSD for elliptic curves over the function fields of curves.

What is a shtuka?

I don't know.

Very roughly, if we have two maps $P, Q : S \to X$, an X-shtuka over S is a vector bundle $\mathcal{F}$ on $X \times S$ together with a rational map $(\mathrm{id} \times \mathrm{Fr}_S)^* \mathcal{F} \dashrightarrow \mathcal{F}$. There are also shtukas with level structure.

If $X = \mathbb{P}^1$, then we understand vector bundles, and a rational map of vector bundles is basically just a matrix of rational functions.

What do shtukas have to do with BSD?

Let K be the function field of $X = \mathbb{P}^1$ and let E be an elliptic curve over K of conductor N, so that we get a surface $\mathcal{E} \to X$, and so a relative surface $\mathcal{E} \times \mathcal{E} \to X \times X$.

There is a moduli space of shtukas of conductor N and two "legs" (places where the map is not defined), which maps to $X \times X$ by remembering only the legs. It is expected that there is a correspondence between this moduli space and $\mathcal{E} \times \mathcal{E}$. This is known at the level of cohomology (Drinfeld).

If so, E is called "2-modular".

What is 2-modularity good for?

There is a *Heegner-Drinfeld cycle* on the moduli space of shtukas, which the correspondence relates to one on the square of the elliptic curve. This in turn has a height, which (Yun-Zhang) is essentially the second derivative of the L-function at 1.

So by producing the correspondence, we can prove BSD for some families of elliptic curves of rank 2 (only over function fields—at present I think applying these ideas to number fields is completely out of reach).

Elkies and Weinstein worked out one example in characteristic 2. Weinstein and I are now working on another one in characteristic 3. We also have a candidate for 3-modularity . . . .

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Cards on the table

I will be in Montreal from September 20 to November 29. At present my two main priorities are to work on the project with Jared Weinstein just described and on some ideas of my own to do with modular Calabi-Yau threefolds.

However, I would like to be involved in the work of the special semester and to benefit from your presence more directly.

Especially if you are not very familiar with computational tools, I hope that you will think about how your projects could benefit from my contributions and that you will tell me if you think of something. I would be very interested in forming a new collaboration.

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Thank you

Thank you for your attention.

Are there any questions?