The work of Granville–Stark on ABC and Siegel zeros

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Math 726: Topics in Number Theory
Modular forms and the theory of complex multiplication

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Notation

- $D < 0$ fundamental discriminant (so $\sqrt{D} \notin \mathbb{R}$, $D = \operatorname{disc} \mathbb{Q}(\sqrt{D})$);
- $C\ell(D)$ = class group of $\mathbb{Q}(\sqrt{D})$;
- $h(D) = |C\ell(D)|$ class number of $\mathbb{Q}(\sqrt{D})$;
- j = j-invariant function:

$$j(\tau) := \frac{\left(1 + 240 \sum_{n\geq 1} \left(\sum_{d|n} d^3\right) q^n\right)^3}{q \prod_{n\geq 1}(1-q^n)^{24}} \qquad \left(\Im(\tau) > 0,\ q = e^{2\pi i \tau}\right)$$

$$\boxed{\begin{array}{c} j(\tau+1) = j(\tau), \\ j(-1/\tau) = j(\tau); \end{array}}$$

$$\boxed{\begin{array}{c} j(i) = 1728, \\ j(e^{2\pi i/3}) = 0, \\ \text{“}j(i\infty) = 1\infty\text{”}; \end{array}}$$

<u>q-expansion of j:</u>

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots$$

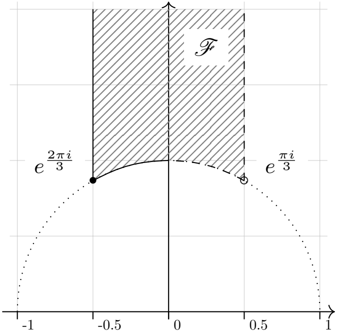
Singular moduli (1/2)

Let $\tau \in \mathfrak{h}$ $(\Im(\tau) > 0)$

- CM-point: $\tau \mid A\tau^2 + B\tau + C = 0$ $\left(\begin{array}{c} A, B, C \in \mathbb{Z},\ A > 0, \\ \gcd(A, B, C) = 1,\ \text{unique} \end{array}\right)$
- Singular modulus: $j(\tau)$ $(j = j\text{-invariant}, \tau \text{ a CM-point})$

CM-points		Quadratic forms
$\tau \in \mathfrak{h}$	$\longleftrightarrow$	$Ax^2 + Bxy + Cy^2$

- $\mathrm{disc}(\tau) := B^2 - 4AC$
- $\tau \sim \tau' \iff (A, B, C) \sim (A', B', C')$
- $\tau \in \mathscr{F} \iff (A, B, C)$ *reduced*



Singular moduli (2/2)

Heegner points Λ_D		Reduced prim. quad. forms of disc. D
$\left\{ \begin{matrix} \text{CM-points } \tau \in \mathscr{F}, \\ \text{disc}(\tau) = D \end{matrix} \right\}$	$\longleftrightarrow$	$\left\{ \begin{matrix} (a,b,c) := ax^2 + bxy + cy^2 \\ \text{s.t. } b^2 - 4ac = D, \text{ and} \\ -a < b \leq a < c \text{ or } 0 \leq b \leq a = c \end{matrix} \right\}$
$\Cup$		$\Cup$
$\tau_D := \underbrace{\frac{\sqrt{D}}{2}}_{D \equiv 0(4)}$ or $\underbrace{\frac{-1+\sqrt{D}}{2}}_{D \equiv 1(4)}$	$\leftrightarrow$	$\underbrace{\text{Principal form}}_{(1,0,-\frac{D}{4}) \text{ or } (1,1,\frac{1-D}{4})}$

$$\mathbb{Z}[\tau_D] = \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$$

Write $\mathrm{H}_D :=$ Hilbert class field of $\mathbb{Q}(\sqrt{D})$.

- $\mathrm{H}_D = \mathbb{Q}(\sqrt{D}, j(\tau_D))$ $\quad \left([\mathrm{H}_D : \mathbb{Q}(\sqrt{D})] = [\mathbb{Q}(j(\tau_D)) : \mathbb{Q}] = h(D) \right)$
- $\{ j(\tau) \mid \tau \in \Lambda_D \} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{D}))$-conjugates of $j(\tau_D)$
- $j(\tau_D)$ is an algebraic **integer!**

The equation $x^3 - Dy^2 = 1728$

In class, we saw the equation $x^3 - Dy^2 = 1728 \quad (= j(i))$.

- Factorization of differences of singular moduli
- Solutions of the type $(j(\tau_D), j(\tau_D) - 1728) = (x^3, Dy^2)$
- Idea: $ABC \implies$ few such solns $\implies$ **lower bounds for** $h(D)$

ABC : $(a + b = c)$

$$a, b, c \in \mathbb{Z} \text{ coprime. } \forall \varepsilon > 0, \exists C_\varepsilon > 0 \text{ s.t. } \max\{|a|, |b|, |c|\} < C_\varepsilon \Big(\prod_{p|abc} p \Big)^{1+\varepsilon}$$

Example (Class number 1)

We've seen that $h(D) = 1 \implies x, y\sqrt{|D|} \in \mathbb{Z}$. However, in this case,

$$|j(\tau_D)| \leq \max\{|x|^3, |y\sqrt{D}|^2, 1728\} \overset{\text{abc}}{\leq} C_\varepsilon \cdot |x \cdot y\sqrt{D}|^{1+\varepsilon} \leq C_\varepsilon \cdot |j(\tau_D)|^{\frac{5}{6}(1+\varepsilon)},$$

so there can only be finitely many $D < 0$ with $h(D) = 1$. △

How does $h(D)$ grow?

What do we know?

- $h(D) \to \infty$ as $D \to -\infty$ (Heilbronn, 1934)
  - List $h(D)$ (small values of $|D|$)
  - Estimate $h(D)$ (large values of $|D|$)

- $-D \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$ are the *only* D s.t. $h(D) = 1$
  - Solved by Heegner (∼1952) [based on Weber's work]
  - Reworked and clarified by Stark, Birch (1967∼9)
  - Baker's solution (∼1970) [based on Baker's theorem]

- $h(D)$ grows *roughly* like $|D|^{1/2}$ (GRH $\implies$ err $\approx (\log\log|D|)^{\pm 1}$)
  - Hecke (1917): If $\zeta_{\mathbb{Q}(\sqrt{D})}(s) \neq 0$ **near** 1, then $\dfrac{h(D)}{|D|^{1/2}} \gg (\log|D|)^{-1}$
  - Landau (1918): $\max\left\{\dfrac{h(D_1)}{|D_1|^{1/2}}, \dfrac{h(D_2)}{|D_2|^{1/2}}\right\} \gg (\log|D_1 D_2|)^{-1}$
  - Siegel (1935): $\dfrac{h(D)}{|D|^{1/2}} \gg_\varepsilon |D|^{-\varepsilon}$ (unconditional but ineffective)

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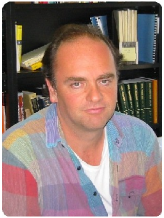
The Granville–Stark Theorem

Theorem (Granville–Stark, 2000)

Uniform abc-conj. $\implies$ *"No Siegel zeros" for* $\zeta_{\mathbb{Q}(\sqrt{D})}(s),\ D < 0$

- Study $x^3 - y^2 = 1728$
- Uniform ABC for number fields (U-*abc*)
- U-*abc* $\implies$ lower bounds for $h(D)$
- Lower bounds $h(D)$ $\iff$ $\frac{1}{2}$ "no Siegel zeros"



A. Granville

H. Stark

The equation $x^3 - y^2 = 1728$

- Solutions of the type $(j(\tau_D), j(\tau_D) - 1728) = (x^3, y^2)$

$$\underline{\mathbf{ABC}} \text{ (for } \mathbb{Q}\text{) :} \quad \log \max\{|a|, |b|, |c|\} < (1+\varepsilon) \underbrace{\boxed{\sum_{p|abc} \log p}}_{\text{log-conductor}} + O_\varepsilon(1)$$

- Applying the same logic for $x^3 - y^2 - 1728 = 0$, we would get:

$$\begin{aligned} \log \max\{|x|^3, |y|^2\} &\leq \log \max\{|x|^3, |y|^2, 1728\} \\ &\overset{\text{abc}}{\leq} (1+\varepsilon) \log\text{-cond}(12|xy|) + \mathbf{tax} \\ &\leq \frac{5}{6}(1+\varepsilon) \log \max\{|x|^3, |y|^2\} + \mathbf{tax} \end{aligned}$$

How to make this precise? $\longrightarrow$ $\boxed{\log \max\{|x|^3, |y|^2\} < (6+\varepsilon') \cdot \mathbf{tax}}$

Let $K/\mathbb{Q}$ be a NF, $\mathcal{M}_K$ its places, $\mathcal{M}_K^{\text{non}} \subseteq \mathcal{M}_K$ non-arch. places

For a point $P = [x_0 : \cdots : x_n] \in \mathbb{P}_K^n$, define:

- (naïve, abs, log) height $\text{ht}(P)$

$$\frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} \log\left(\max_i \{\|x_i\|_v\}\right)$$

- (log) conductor $\mathcal{N}_K(P)$

$$\frac{1}{[K:\mathbb{Q}]} \sum_{\substack{v \in \mathcal{M}_K^{\text{non}} \\ \exists i,j \leq n \text{ s.t.} \\ v(x_i) \neq v(x_j)}} f_v \log(p_v)$$

$$\updownarrow$$

$$\begin{cases} v \sim \mathfrak{p} = \mathfrak{p}_v \\ p_v \sim \mathfrak{p}_v \cap \mathbb{Q} \\ f_v := [K_v : \mathbb{Q}_{p_v}] \end{cases}$$

For $a, b, c \in \mathbb{Z}$ coprime,

- $\text{ht}([a:b:c]) = \log\max\{|a|, |b|, |c|\}$
- $\mathcal{N}_{\mathbb{Q}}([a:b:c]) = \log\left(\prod_{p|abc} p\right)$

For $\alpha \in \overline{\mathbb{Q}}$, $\text{ht}(\alpha) := \text{ht}([\alpha : 1])$.

$$\alpha \text{ integral} \Rightarrow \text{ht}(\alpha) = \frac{1}{|\mathcal{A}|} \sum_{\alpha^* \in \mathcal{A}} \log^+ |\alpha^*|$$

$\mathcal{A} = \{\text{conjugates of } \alpha\}$

abc-conjecture for number fields (2/2)

abc-conjecture for number fields

Fix $K/\mathbb{Q}$ a number field. Then, for every $\varepsilon > 0$, there is $\mathcal{C}(K,\varepsilon) \in \mathbb{R}_+$ such that, $\forall a,b,c \in K \mid a+b+c=0$, we have

$$\mathrm{ht}([a:b:c]) < (1+\varepsilon)\Big(\mathcal{N}_K([a:b:c]) + \log(\mathrm{rd}_K)\Big) + \mathcal{C}(K,\varepsilon),$$

where $\mathrm{rd}_K := |\Delta_K|^{1/[K:\mathbb{Q}]}$ *is the* root-discriminant *of* K*.*

Example (a, b, c integral, coprime)

For $a,b,c \in \mathcal{O}_K$ <u>coprime</u> (meaning $v(a) \neq v(b) \neq v(c)$ for $v < \infty$),

$$\max\{|\mathrm{N}_{K/\mathbb{Q}}(a)|^{\frac{1}{[K:\mathbb{Q}]}}, |\mathrm{N}_{K/\mathbb{Q}}(b)|^{\frac{1}{[K:\mathbb{Q}]}}, |\mathrm{N}_{K/\mathbb{Q}}(c)|^{\frac{1}{[K:\mathbb{Q}]}}\}$$
$$< C_{K,\varepsilon}\left(|\Delta_K| \prod_{\mathfrak{p} \mid (abc)} \left|\mathcal{O}_K/\mathfrak{p}\mathcal{O}_K\right|\right)^{1+\varepsilon},$$

with $C_{K,\varepsilon} = \exp([K:\mathbb{Q}]\cdot\mathcal{C}(K,\varepsilon))$.

What do we get?

$$(j(\tau_D),\, j(\tau_D) - 1728) = (x^3, y^2) \qquad \boxed{x^3 - y^2 = 1728}$$

$$j(\tau_D) \in \mathrm{H}_D \qquad \widetilde{\mathrm{H}}_D := \mathrm{H}_D(x,y) \quad ([\widetilde{\mathrm{H}}_D : \mathrm{H}_D] \le 6)$$

- "$\log \max\{|x|^3, |y|^2\}$" $< (6+\varepsilon) \cdot \mathbf{tax}(\widetilde{\mathrm{H}}_D, \varepsilon)$

More precisely:

$$\mathrm{ht}(j(\tau_D)) < 6\left((1+\varepsilon)\log(\mathrm{rd}_{\widetilde{\mathrm{H}}_D}) + \mathcal{C}(\widetilde{\mathrm{H}}_D, \varepsilon)\right)$$

- "Factorization": $\mathrm{rd}_{\widetilde{\mathrm{H}}_D} \le 6\sqrt{|D|}$ $\qquad (\mathrm{rd}_{\mathrm{H}_D} = \mathrm{rd}_{\mathbb{Q}(\sqrt{D})} = \sqrt{|D|})$
- Uniform *ABC*-conj.: $\mathcal{C}(K,\varepsilon) = \mathcal{C}(\varepsilon)$ $\qquad \boxed{\Longrightarrow\ \mathbf{tax}(\varepsilon) \lesssim \frac{1}{2}\log|D|}$

Lower bounds for $h(D)$

Lemma [G–S]

Unif. ABC-conj. $\implies \mathrm{ht}(j(\tau_D)) \leq (3 + o(1)) \log |D|$

However,

$$\mathrm{ht}(j(\tau_D)) = \frac{1}{h(D)} \sum_{(a,b,c)} \log^+ |j(\tau_{(a,b,c)})|$$

$$= \frac{1}{h(D)} \sum_{(a,b,c)} \frac{\pi\sqrt{|D|}}{a} + O(1),$$

and thus:

$ax^2 + bxy + cy^2$ reduced, $b^2 - 4ac = D$

$$\tau_{(a,b,c)} = \frac{-b + \sqrt{D}}{2a}$$

Theorem [G–S]

$$h(D) \overset{\text{U-}abc}{\geq} \left(\frac{\pi}{3} + o(1)\right) \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)} \frac{1}{a}$$

q-expansion!

$$\boxed{j(\tau) = \frac{1}{q} + O(1)}$$

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Dedekind zeta function of $\mathbb{Q}(\sqrt{D})$

$$\zeta_K(s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_K} \frac{1}{\mathfrak{N}(\mathfrak{a})^s} = \frac{c_{-1}}{s-1} + c_0 + O(s-1) \quad (s \to 1)$$

Conjecture (“no Siegel zeros” for $D < 0$)

There is $\delta > 0$ such that $\zeta_{\mathbb{Q}(\sqrt{D})}(\beta) \neq 0$ for $1 - \frac{\delta}{\log |D|} \leq \beta < 1$.

- Dirichlet’s class number formula: $c_{-1}(D) = \frac{\pi\, h(D)}{\sqrt{|D|}} \quad (D < -4)$

$$\zeta_{\mathbb{Q}(\sqrt{D})}(\beta) \leq -\frac{1}{\delta} \frac{\pi\, h(D) \log |D|}{\sqrt{|D|}} + c_0(D) + O\big((\log |D|)^{-1}\big)$$

$$\overset{\text{U-}abc}{\leq} -\left(\frac{1}{\delta} + o(1)\right) \sum_{(a,b,c)} \frac{1}{a} + \boxed{c_0(D)} + o(1)$$

Kronecker's limit formula (KLF)

We just need to show that $\boxed{c_0(D) \ll \sum_{(a,b,c)} \frac{1}{a}}$ (as $D \to -\infty$).

$$\zeta_{\mathbb{Q}(\sqrt{D})}(s) = \frac{1}{2} \sum_{(a,b,c)} \Bigg(\underbrace{\sum_{\substack{m,n=-\infty \\ m,n \neq 0}}^{+\infty} \frac{1}{(am^2 + bmn + cn^2)^s}}_{\text{Epstein zeta function: } Z_{(a,b,c)}(s)} \Bigg)$$

KLF [Kronecker 1863, Weber 1881, . . ., Chowla–Selberg 1967, . . .]

The constant term (c_0) of $Z_{(a,b,c)}(s)$ as $s \to 1$ is given by:

$$\frac{\pi}{\sqrt{|D|}} \left(2\gamma - \log(2) - \frac{1}{2} \log |D| - \log \left(\frac{\sqrt{|D|}}{2a} \left| \eta\left(\tfrac{-b+\sqrt{D}}{2a}\right) \right|^4 \right) \right)$$

Estimating $c_0(D)$

$$c_0(D) = \frac{\pi^2}{6} \sum_{(a,b,c)} \frac{1}{a} - \frac{\pi}{\sqrt{|D|}} \sum_{(a,b,c)} \log\left(\frac{\sqrt{|D|}}{2a}\right) + O\left(\frac{h(D)}{\sqrt{|D|}}\right)$$

- Because $a \leq \sqrt{|D|/3}$, we have

$$\frac{h(D)}{\sqrt{|D|}} = \sum_{(a,b,c)} \frac{1}{\sqrt{|D|}} \ll \sum_{(a,b,c)} \frac{1}{a}$$

- For the "log" term,

$$\frac{\pi}{\sqrt{|D|}} \sum_{(a,b,c)} \log\left(\frac{\sqrt{|D|}}{2a}\right) = \frac{\pi\, h(D)}{\sqrt{|D|}} \left(\frac{1}{h(D)} \sum_{(a,b,c)} \log \Im(\tau_{(a,b,c)})\right) + o(1)$$

where $\tau_{(a,b,c)} = (-b + \sqrt{D})/2a$. **How to estimate this?**

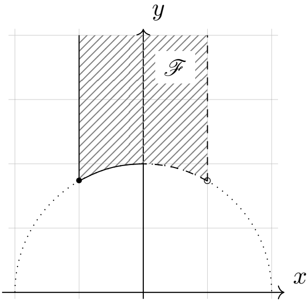
Duke's equidistribution theorem

Theorem (Duke, 1988)

$\Lambda_D = \{\tau_{(a,b,c)} \mid b^2 - 4ac = D\}$ *is equidistributed in* $\mathscr{F}$.

If $f : \mathscr{F} \to \mathbb{C}$ is Riemann-integrable, then:

$$\lim_{D\to-\infty} \frac{1}{h(D)} \sum_{(a,b,c)} f(\tau_{(a,b,c)}) = \int_{\mathscr{F}} f(z)\,\mathrm{d}\mu$$



$$z = x + iy$$

$$\mathrm{d}\mu = \frac{3}{\pi}\frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

(Normalized hyperbolic area element)

W. Duke

Conclusion

Using Duke's theorem, the "log" term becomes:

$$\frac{\pi\, h(D)}{\sqrt{|D|}} \left(\frac{1}{h(D)} \sum_{(a,b,c)} \log \Im(\tau_{(a,b,c)}) \right) \sim \frac{\pi\, h(D)}{\sqrt{|D|}} \left(\frac{3}{\pi} \int_{\mathscr{F}} \frac{\log y}{y^2}\, \mathrm{d}x\mathrm{d}y \right)$$

$$= \frac{\pi\, h(D)}{\sqrt{|D|}} \cdot 0.952934\ldots$$

Thus,

$$\text{“log” term} = O\left(\frac{h(D)}{\sqrt{|D|}} \right) = O\left(\sum_{(a,b,c)} \frac{1}{a} \right). \qquad \square$$

$$h(D) \gg \frac{\sqrt{|D|}}{\log |D|} \sum_{(a,b,c)} \frac{1}{a} \iff \text{“no Siegel zeros” for } D < 0$$

Thank you!

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