THE EFFICIENT EVALUATION OF p-ADIC THETA FUNCTIONS AND p-ADIC UNIFORMISATION

THE p-ADIC UPPER HALF PLANE

DEF. LET p BE A PRIME NUMBER. DEFINE:

$$\mathcal{H}_p := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$$

- NOW CHOOSE SET OF REPRESENTATIVES P_M FOR $\mathbb{P}^1(\mathbb{Q}_p)$ MODULO p^M, e.g.

$$[a_i, 1], \quad \{a_i\}_{i=0}^{p^M-1}, \quad a_i \in \frac{\mathbb{Z}_p}{p^M\mathbb{Z}_p}$$

$$[1, b_i], \quad \{b_i\}_{i=0}^{p^{M-1}-1}, \quad b_i \in \frac{p\mathbb{Z}_p}{p^M\mathbb{Z}_p}$$

Def $\Omega_M := \mathbb{P}^1(\mathbb{C}_p) - \bigcup_{x \in P_M} B(x, M)$

$$\Omega_M^- := \mathbb{P}^1(\mathbb{C}_p) - \bigcup_{x \in P_M} B^-(x, M-1)$$

- So $\mathcal{H}_p = \bigcup_M \Omega_M = \bigcup_M \Omega_M^-$

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THE BRUHAT-TITS TREE

<u>DEF</u> THE BRUHAT-TITS TREE FOR $PGL_2(\mathbb{Q}_p)$ IS THE GRAPH T WHOSE VERTICES ARE EQUIVALENCE CLASSES OF LATTICES IN $\mathbb{Q}_p^2$.

TWO VERTICES ARE JOINED BY AN EDGE IF $x=[L]$, $x'=[L']$ AND

$$pL \subsetneq L' \subsetneq L .$$

- T IS $p+1$-REGULAR TREE
- VERTICES OF T AT DISTANCE M FROM ANY GIVEN VERTEX ARE IN BIJECTION WITH $\mathbb{P}^1\left(\frac{\mathbb{Z}_p}{p^M\mathbb{Z}_p}\right)$
- $PGL_2(\mathbb{Q}_p)$ ACTS ON T:

$$g[L]:=[gL]$$

<u>DEF</u> LET v_0 THE VERTEX $\begin{pmatrix}1&0\\0&1\end{pmatrix}$. WE WILL CALL IT **STANDARD VERTEX**.

LET v_1 BE THE VERTEX $\begin{pmatrix}1&0\\0&p\end{pmatrix}v_0$. LET e_0 BE THE EDGE JOINING v_0 AND v_1.
WE WILL CALL IT **STANDARD EDGE**.

- $\mathrm{Stab}(v_0) = PGL_2(\mathbb{Z}_p)$
- $\mathrm{Stab}(e_0) = \{\gamma \in PGL_2(\mathbb{Z}_p) \mid \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \bmod p\}$

Ends of T

Def Classes of infinite paths without backtracking are called ends.
Where two infinite sequences of vertices define the same end if they differ by a finite segment.

- Topology on Ends: if $e = \langle [L], [L'] \rangle$ is an edge, define:

$$U(e) := \{ \langle [L], [L'], \dots \rangle \in \mathrm{Ends}(T) \}$$

- $U(e)$ give basis for topology on $\mathrm{Ends}(T)$
- $\mathrm{Ends}(T) \simeq \mathbb{P}^1(\mathbb{Q}_p)$ $PGL_2(\mathbb{Q}_p)$-equivariant homeomorphism.

The reduction map

Proposition The points of T are in bijection with the set of equivalence classes of norms on $\mathbb{Q}_p^2$.

INDEED:

- IF $x = [L]$, LET ℓ_0, ℓ_1 S.T. $L = \langle \ell_0, \ell_1 \rangle$

THEN $x \longmapsto \gamma$ WITH $\gamma(a\ell_0 + b\ell_1) = \inf\{w(a), w(b)\}$

- IF $x = (1-t)[L] + t[L']$, LET ℓ_0, ℓ_1 S.T.

$L = \langle \ell_0, \ell_1 \rangle$, $L' = \langle \ell_0, p\ell_1 \rangle$.

THEN $x \longmapsto \gamma$ WITH

$$\gamma(a\ell_0 + b\ell_1) = \inf\{w(a), w(b) - t\}$$

<u>PROPOSITION</u> THERE IS A $PGL_2(\mathbb{Q}_p)$-EQUIVARIANT MAP $\pi : \mathcal{H}_p \to T$ GIVEN BY:

$z := [x, y] \longmapsto \gamma_z$, WITH $\gamma_z((a,b)) = w(ax + by)$.

We CALL IT REDUCTION MAP.

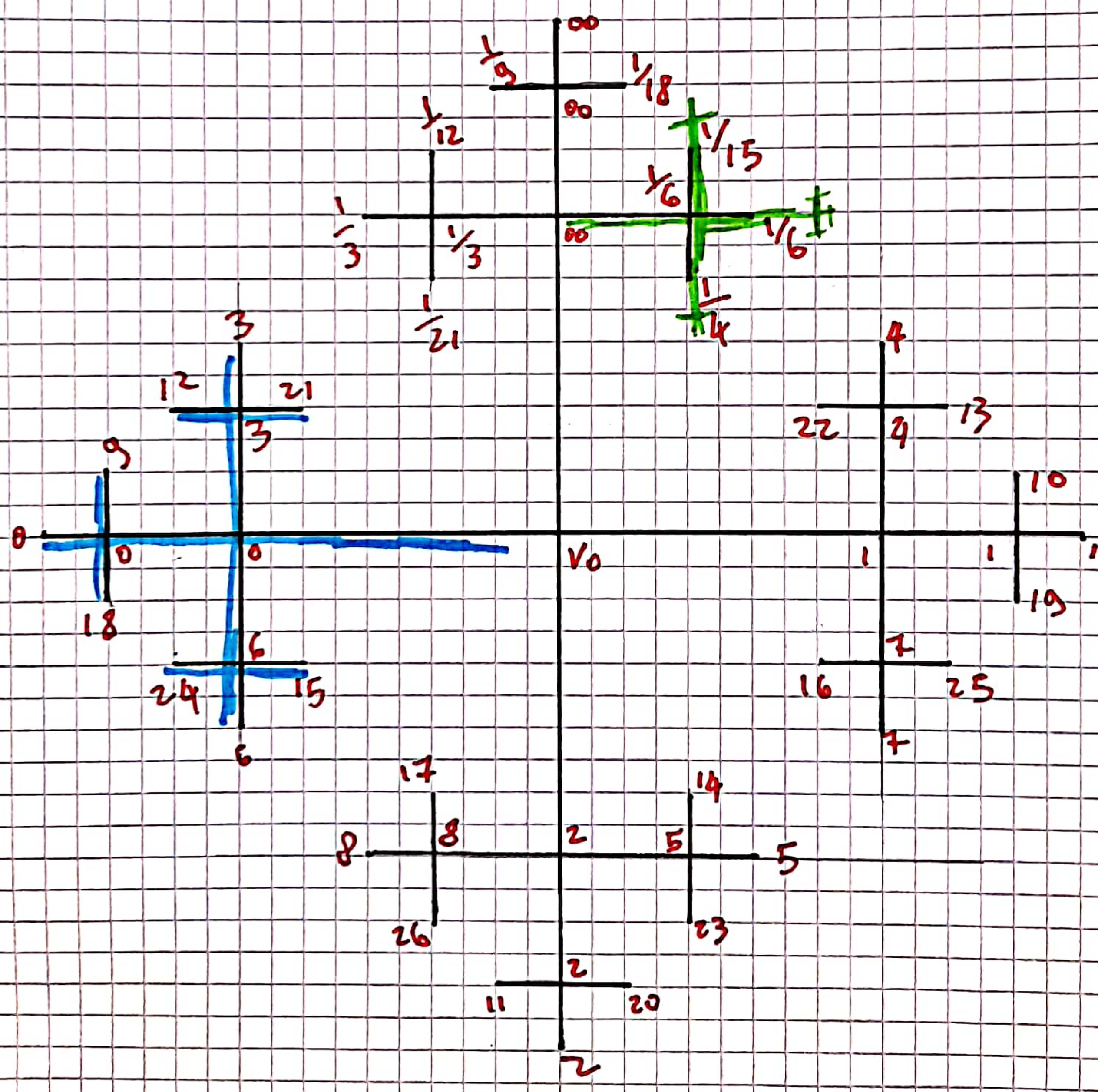
- $\pi^{-1}(v_0) = \{z \in \mathcal{H}_p \mid |z - t| \geq 1$, FOR $t = 0, \dots p-1$, AND $|z| \leq 1\}$

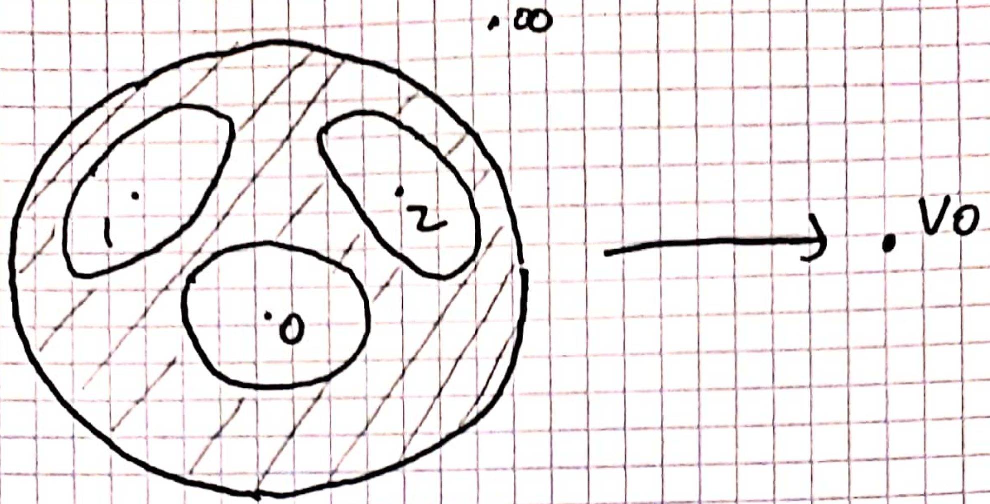
IS THE STANDARD AFFINOID.

- $\pi^{-1}(\ell_0) = \{z \in \mathcal{H}_p \mid \frac{1}{p} < |z| < 1\}$ IS THE STANDARD ANNULUS.
- $\mathcal{H}_M^-$ IS PREIMAGE OF SUBTREES OF T MADE OF POINTS AT DISTANCE AT MOST $M-1$ FROM v_0.
- $\mathcal{H}_p$ IS A TUBULAR NEIGHBOURHOOD OF T.

- AFFINOIDS ARE PREIMAGES of VERTICES
- ANNULI ARE PREIMAGES OF EDGES

Def A RIGID ANALYTIC FUNCTION IS A $\mathbb{C}_p$ - VALUED FUNCTION f ON $\mathcal{H}_p$, SUCH THAT ITS RESTRICTION TO ANY AFFINOID IS A UNIFORM LIMIT, WITH RESPECT TO THE SUP NORM, OF RATIONAL FUNCTIONS ON $\mathbb{P}^1(\mathbb{C}_p)$ HAVING POLES OUTSIDE THE AFFINOID.





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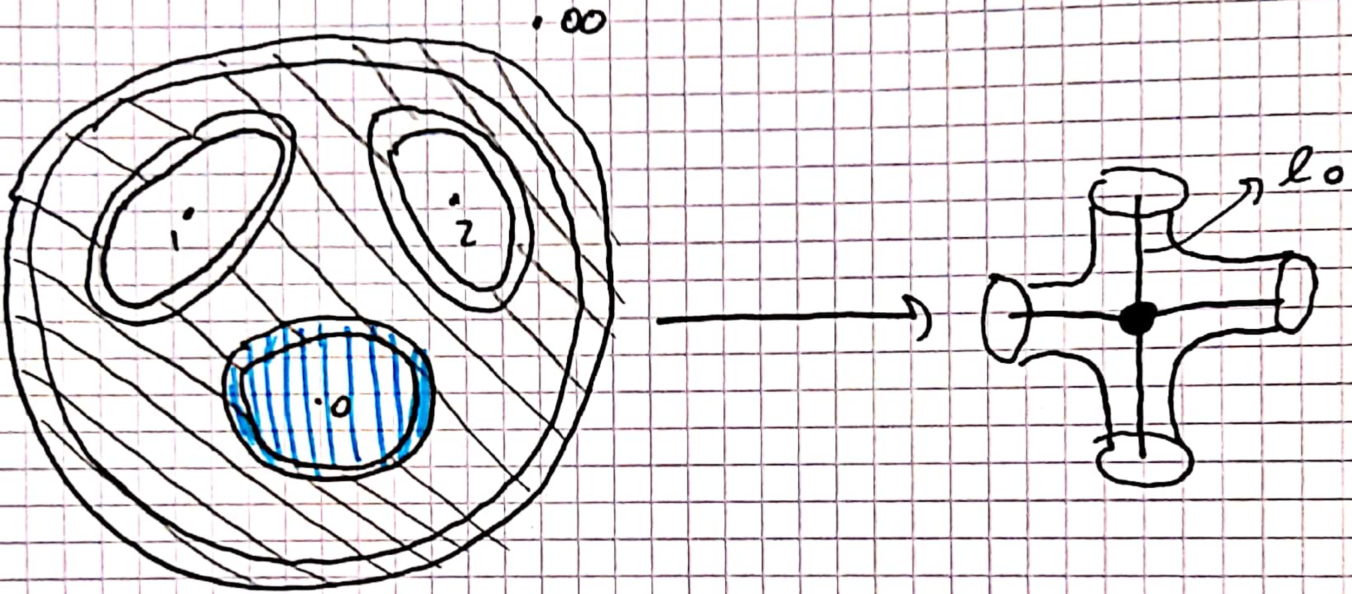

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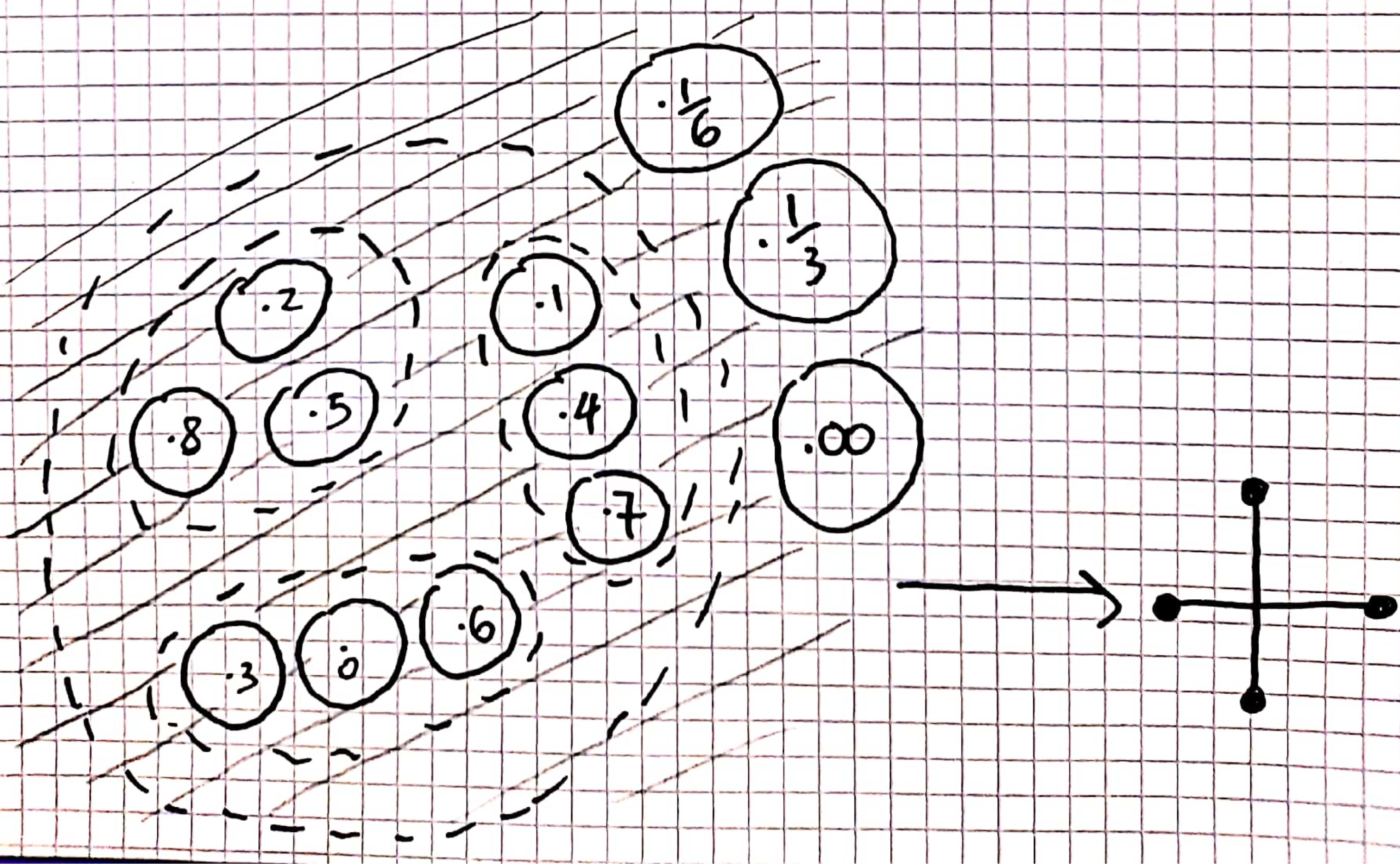

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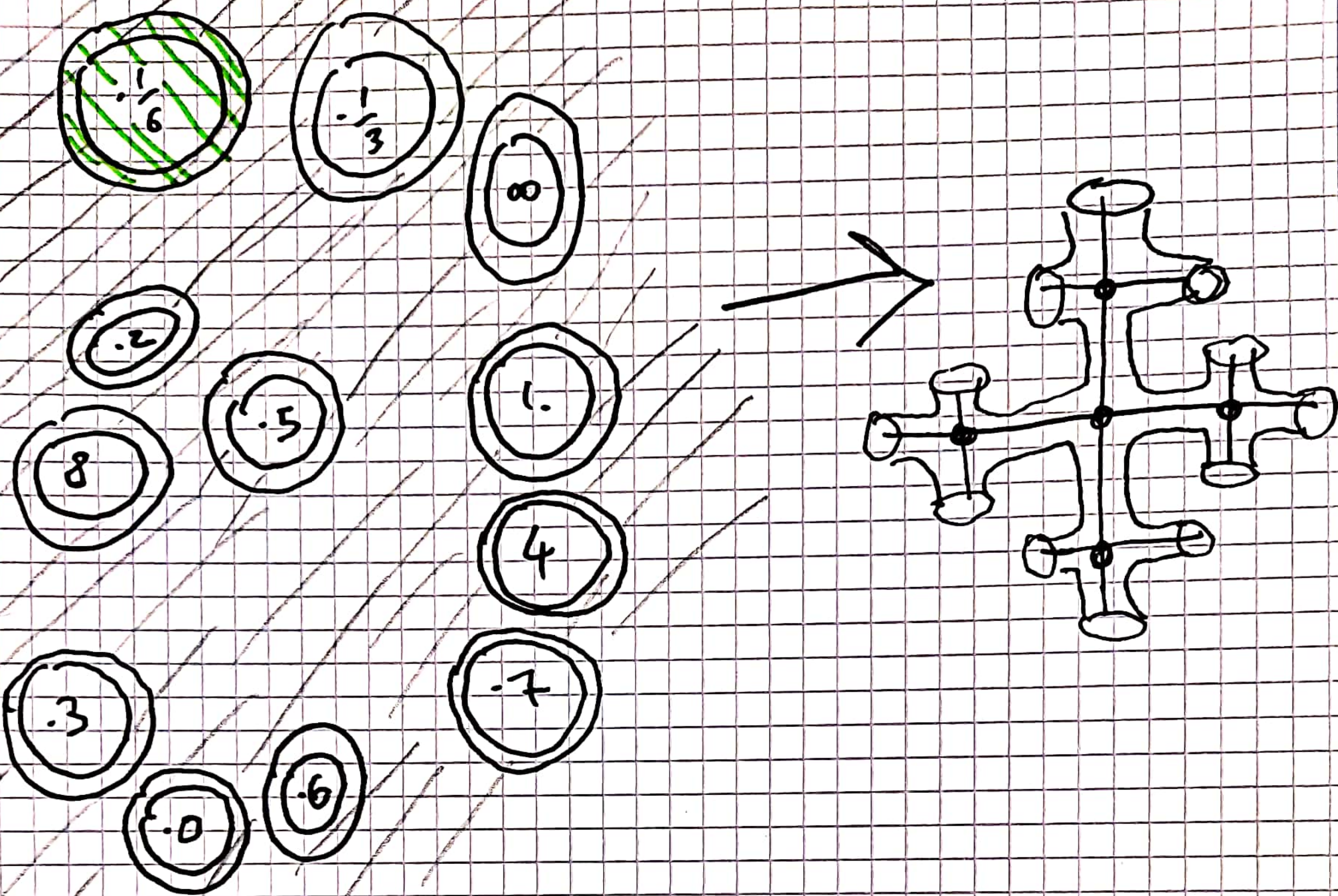

<u>THM (MUMFORD)</u> LET Γ BE A DISCRETE SUBGROUP OF $SL_2(\mathbb{Q}_p)$; ASSUME $\mathcal{H}_p/\Gamma$ CPT. THEN $\mathcal{H}_p/\Gamma$ IS AN ALGEBRAIC CURVE OVER $\mathbb{Q}_p$.

CONVERSELY IF X IS AN ALGEBRAIC CURVE OVER $\mathbb{Q}_p$ WITH TOTALLY DEGENERATE REDUCTION, THEN THERE IS A DISCRETE GROUP $\Gamma \subset PSL_2(\mathbb{Q}_p)$ SUCH THAT X IS ISOMORPHIC TO $\mathcal{H}_p/\Gamma$.

COMPUTATION OF P-ADIC THETA FUNCTIONS ARISING FROM THE HURWITZ QUATERNIONS

- Let $B = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$
- Let $R = \mathbb{Z}\left[i, j, k, \frac{1+i+j+k}{2}\right]$
- Let $R[\frac{1}{p}]^\times_1 := \{ \gamma \in R[\frac{1}{p}] : Nm(\gamma) = 1 \}$
- $B \otimes \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$ VIA THE MAP:

$$a + bi + cj + dk \stackrel{\iota}{\longmapsto} \begin{pmatrix} a + b\alpha & -c - d\alpha \\ c - d\alpha & a - b\alpha \end{pmatrix}, \text{ WHERE } \alpha^2 = -1.$$

$p \equiv 1 \pmod 4$

Def $\Gamma := \iota\left(R[\frac{1}{p}]^\times_1\right) \subseteq SL_2(\mathbb{Q}_p)$

- RECALL: IF $\tau = [x, y] \in \mathcal{H}_p$ AND $\gamma = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \in \Gamma$, THEN $\gamma\tau = [qx + ry, sx + ty]$

DEF Let $a, b, z \in \mathcal{H}_p$. THE THETA FUNCTION $\Theta(a, b; z)$ IS DEFINED AS:

$$\Theta(a, b; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma a}{z - \gamma b}$$

Def Let $\Gamma_M := \left\{ \iota\left(\frac{x}{p^M}\right) \middle| x \in R, Nm(x) = p^{2M} \right\}$ WITH $M \geq 0$, AND:

$$\phi_0(a,b;z) := \prod_{\gamma \in \Gamma_0} \frac{z-\gamma a}{z-\gamma b}$$

$$\phi_M(a,b;z) := \prod_{\gamma \in \Gamma_M - \Gamma_{M-1}} \frac{z-\gamma a}{z-\gamma b}, \quad M \geq 1.$$

PROPOSITION $\Theta(a,b;z)$ converges for any $a,b,z \in \mathcal{H}_p$ AND IT IS MEROMORPHIC ON $\mathcal{H}_p$ WITH ZEROES AT $\{\gamma a \mid \gamma \in \Gamma\}$ AND POLES AT $\{\gamma b \mid \gamma \in \Gamma\}$.

SKETCH OF PROOF

- $\frac{z-\gamma a}{z-\gamma b} = 1 + \frac{gb-ga}{z-gb}$, $g = i(x)$, $\gamma = i\left(\frac{x}{p^M}\right)$
- IF $w(\det(g)) = m \Rightarrow d(v_0, g v_0) = m$
- $|gb-ga| \to 0$ WHILE $|z-gb|$ IS BIGGER THAN A CONSTANT. $\square$

A FIRST METHOD TO COMPUTE $\Theta(a,b;z)$

- APPROXIMATE $\Theta(a,b;z)$ WITH $\prod_{i=0}^{M} \phi_i(a,b;z)$
- NOT EFFICIENT AS $|\Gamma_M - \Gamma_{M-1}| = 24(1+p+\dots+p^{2M})$
- THIS CAN BE SEEN BY COMPUTING THE p^{2M}-TH Fourier coefficient OF $\sum_{v \in R} e^{2\pi i \tau \langle v,v \rangle}$.

ANOTHER METHOD TO COMPUTE $\vartheta(a,b;z)$

IDEA: WRITE RECURSIVELY QUATERNIONS OF NORM p^M FROM QUATERNIONS OF NM p^{M-1}.

PROPOSITION A PRIMITIVE QUATERNION OF NORM p^M HAS A FACTORIZATION IN QUATERNIONS OF NORM p, UNIQUELY UPTO MULTIPLICATION BY UNITS.

IDEA: USE REPRESENTATIVES FOR EQUIVALENCE RELATION ON QUATERNIONS OF NORM p GIVEN BY MULTIPLICATION BY UNITS.

ALSO: SEPARATE QUATERNIONS OF NORM p^M. ACCORDING TO WHERE THEY SEND STANDARD AFFINOID (OR V_0).

- ϑ WILL BE GIVEN BY A COLLECTION OF POWER SERIES WITH DIFFERENT CENTERS.