Wednesday, April 8

Some remarks about the final exam:

Because the exam will be a take-home exam, and you will have four days to write it, the questions will require a bit more thought and creativity than the more routine questions you would expect in a 3 hour in-class exam.

In preparing for it, you should strive for a broad understanding of the concepts, in addition to going over past assignments.

On Monday's lecture, we went over a few applications of the spectral theorem:

#1. Existence and uniqueness of positive definite square roots of positive definite self-adjoint operators.

#2. Polar decomposition: $T = UP$, U unitary, P positive self-adjoint.

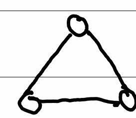
#3. Fourier analysis on finite groups

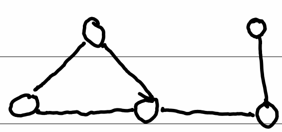
$$L^2(G) = \bigoplus_{\phi \in \hat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)} \mathbb{C}\cdot\phi \qquad \text{(orthogonal direct sum.)}$$

$L^2(G) \simeq L^2(\hat{G})$ as inner product spaces.

Application # 4 Spectral Graph Theory.

Definition: A graph G is a finite set $V(G)$ of vertices together with a set $E \subset V(G)^2$ of "edges," satisfying $(v_1, v_2) \in E \Leftrightarrow (v_2, v_1) \in E$

Examples  ,  , 

 , etc...

Graphs are the mathematical structure that can be used to

Model networks, which are ubiquitous in many applications (the internet, distribution or communication networks, modelling the spread of infectious diseases, etc...)

Two vertices $v, w \in V(G)$ are said to be <u>adjacent</u> if $(v,w) \in E$

We write $v \sim w$ if v and w are adjacent.

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Adjacency operators

One can associate to a graph G the inner product space

$L^2(V) = \mathbb{C}$ - valued functions on the vertices,

together with the Adjacency operator

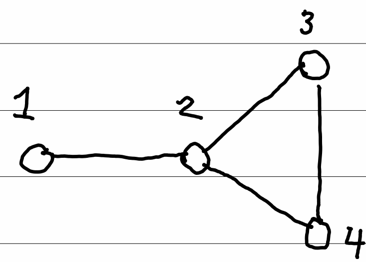
$$A : L^2(V) \longrightarrow L^2(V)$$
$$f \longmapsto Af(v) = \sum_{w \sim v} f(w)$$

. .

⑥

Example

$V(G) = \{1, 2, 3, 4\}$

$G =$ 

$(\delta_1, \delta_2, \delta_3, \delta_4)$ = orthonormal basis of $L^2(V)$

In this basis, A is represented by the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Note: $M = M^t$.

Proposition: The adjacency operator is self-adjoint.

Proof: $\langle Af, g \rangle = \sum_{v \in V(G)} Af(v)\,\overline{g(v)}$

$= \sum_{v \in V(G)} \sum_{w \sim v} f(w)\,\overline{g(v)}$

$= \sum_{(v,w) \in E} f(w)\,\overline{g(v)} = \sum_{w \in V(G)} \sum_{v \sim w} f(w)\,\overline{g(v)}$

$= \sum_{w \in V(G)} f(w)\,\overline{(Ag)(w)} = \langle f, Ag \rangle$. ∎

Corollary: The space $L^2(V(G))$ has an orthonormal basis of eigenvectors for the adjacency operator.

$$\text{Spectrum}(A) = \{ \lambda_1, \lambda_2, \ldots, \lambda_t \}$$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_t \in \mathbb{R}$$

Main questions of spectral graph theory

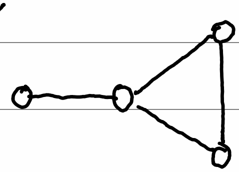
1. What geometric and combinatorial properties of $G = (V(G), E)$ can be read off from the spectrum of A?

2. What is the meaning of the eigenfunctions for A?

A sample result

Definition: The diameter of a graph is the maximum distance between vertices. $\delta(G) :=$ diameter of G

E.g.,

$\delta\Big($  $\Big) = 2$

Theorem: $\#\,\mathrm{spectrum}(A) \geq \delta + 1$

Proof: For $v, w \in V(G)$

$\langle A^k v, w \rangle = \#$ of paths of length k from v to w.

In particular

$$\langle A^k v, w \rangle = 0 \quad \forall \quad k < \mathrm{dist}(v,w)$$

Let (v,w) be a pair of vertices of maximal distance δ apart.

$$\langle A^k v, w \rangle = \begin{cases} 0 & \text{if} \quad k < \delta \\ \geq 1 & \text{if} \quad k = \delta \end{cases}$$

Hence $\mathrm{span}(1, A, A^2, \dots, A^k, \dots)$ has dimension $\geqslant \delta + 1$

On the other hand we have

Lemma Let A be an endomorphism which is diagonalisable, and let $F[A]$ be the F-subalgebra of $\mathrm{End}(V)$ generated by A: $F[A] = \mathrm{span}(1, A, A^2, \ldots)$

Then $\dim_F F[A] \leq \#\,\mathrm{spectrum}(A)$

Proof Let $V = V_1 \oplus \cdots \oplus V_t$ be the decomposition of V into A-eigenspaces, with $t = \#\,\mathrm{spectrum}(A)$. Then the map

$$\begin{aligned} F[A] &\longrightarrow F^t \\ T &\longmapsto (T|_{V_1}, T|_{V_2}, \ldots, T|_{V_t}) \end{aligned}$$

is injective. $\boxtimes$

Regular graphs

The *degree* of a vertex is the number of vertices that are adjacent to it.

Definition: A graph is said to be *regular* if all its vertices have the same degree d.

G is then called a d-regular graph.

Proposition If G is a connected d-regular graph, then d is the largest eigenvalue of A, and the eigenspace for d is one-dimensional, spanned by the constant functions on G.

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Proof Let f be an eigenfunction for A, and let v be the vertex where $|f|$ attains its maximum

Then $$|(Af)(v)| = |f(v_1) + \cdots + f(v_d)| \quad (*)$$
$$\leq |f(v_1)| + \cdots + |f(v_d)| \leq d\,|f(v)|,$$

where $v_1, \ldots, v_d$ are adjacent to v.

The last inequality uses the maximality assumption on v.

Hence $|\lambda_i| \leq d$. Furthermore, If $Af = df$, then we have equality in $(*)$ hence
$$|f(v_1)| + \cdots + |f(v_d)| = d|f(v)|$$
$$\Rightarrow \quad |f(v_1)| = \cdots = |f(v_d)| = |f(v)|$$

Furthermore, $|f(v_1) + \cdots + f(v_d)| = |f(v_1)| + \cdots + |f(v_d)|$
$$\Rightarrow \quad f(v_1) = f(v_2) = \cdots = f(v_d) = f(v).$$

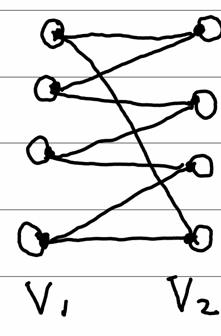
It follows that any eigenfunction for A with eigenvalue d is proportional to the constant function $\mathbb{1}$. $\boxtimes$

When is $-d \in \mathrm{Spec}(A)$?

<u>Definition</u>: A graph G is bipartite if $V(G) = V_1 \sqcup V_2$ and

$$E(G) \subseteq V_1 \times V_2$$

<u>Eg</u>:



Theorem: If G is bipartite, then $-d \in \text{Spectrum}(A)$

Proof: Let $f(v) = \begin{cases} 1 & \text{if } v \in V_1 \\ -1 & \text{if } v \in V_2 \end{cases}$

Clearly $Af = -df$. $\square$

Fact: The converse is also true (We will not prove this.)

The averaging operator

On a d-regular graph, it is

$T = \frac{1}{d} A \qquad Tf(v) = \frac{1}{d} \sum_{w \sim v} f(w)$

For example, $V(G)$ could represent the members of a population, with edges describing the interactions between them. A function $f \in L^2(V(G))$ could represent the distribution of a certain ressource (or virus!) being shared across the population.

$$T^k f \quad (k = 1, 2, 3, \ldots)$$

is a simple model for how this distribution might evolve over time.

Theorem: Let G be a connected d-regular graph which is not bipartite. If $f \in L^2(V(G))$

$$\lim_{k\to\infty} T^k f(v) = \frac{1}{N}\Big(\sum_{w \in V(G)} f(w)\Big)$$

where $N = \# V(G)$.

Proof: Let $\phi_1, \ldots, \phi_N$ be the eigenfunctions of $T = \frac{1}{d} A$, with eigenvalues $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N > -1$

By the spectral theorem, the ϕ_j are pairwise orthogonal and can be chosen to be an orthonormal system for $L^2(G)$. Furthermore,

$$\phi_1(v) = \frac{1}{\sqrt{N}} \quad (\forall v \in V(G))$$

(i.e., ϕ_1 is the appropriate multiple of the constant function, normalised so that $\langle \phi_1, \phi_1 \rangle = 1$.)

Now, write

$$f = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \cdots + \alpha_N \phi_N$$

with $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{C}$.

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$$T^k f = \alpha_1 \lambda_1^k \phi_1 + \alpha_2 \lambda_2^k \phi_2 + \cdots + \alpha_N \lambda_N^k \phi_N$$

but $\lambda_1 = 1$, while $|\lambda_2|, \ldots, |\lambda_N| < 1$

$$\Rightarrow \quad T^k f \longrightarrow \alpha_1 \phi_1 \quad \text{as } k \to \infty$$

$\alpha_1 \phi_1$ is the orthogonal projection of f onto $\mathbb{C} \cdot \phi_1$. Hence

$$\alpha_1 = \langle f, \phi_1 \rangle = \frac{1}{\sqrt{N}} \sum_{v \in V(G)} f(v)$$

$$\alpha_1 \phi_1 = \frac{1}{N} \left(\sum_{v \in V(G)} f(v) \right) \times \mathbb{1}_G$$

constant function 1 on $V(G)$

$$\alpha_1 \phi_1 (v) = \frac{1}{N} \sum_{v \in V(G)} f(v) \qquad \boxtimes$$

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The size of the eigenvalues $\lambda_2, \ldots, \lambda_N$ controls the rate at which $T^k f$ converges to the uniform distribution $\alpha_1 \phi_1 \ldots$