Friday, April 3,

Last time we described various types of distinguished linear transformations on an inner product space,

- Self-adjoint transformations
  ($T^* = T$)
- Orthogonal transformations
  ($T^* = T^{-1}$, $F = \mathbb{R}$)
- Unitary transformations
  ($T^* = T^{-1}$, $F = \mathbb{C}$)
- Skew-adjoint transformations
  ($T^* = -T$)

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②

Spectral Theorem If $T: V \to V$ is self adjoint, then V admits an orthonormal basis of eigenvectors.

Example: the general self-adjoint transformation on $\mathbb{R}^2$ is given by the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$, whose characteristic polynomial is

$$f(x) = x^2 - (a+c)x + (ac - b^2)$$

$$\Delta = (a+c)^2 - 4(ac - b^2) = (a-c)^2 + b^2$$

$\Delta \geq 0$ and $\Delta = 0 \Leftrightarrow a = c$ and $b = 0$

③

Checking "by brute force" that every real symmetric 3×3 matrix is diagonalisable is not for the faint of heart!

<u>Lemma 1</u>: If T is self-adjoint, then its eigenvalues are real.

<u>Proof</u> $\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}$

If $Tv = \lambda v$, then we get

$\lambda\langle v,v\rangle = \bar{\lambda}\langle v,v\rangle \Rightarrow \lambda = \bar{\lambda}$. $\boxtimes$

<u>Key lemma</u>: If $W \subseteq V$ is a T-stable subspace, then T sends $W^{\perp}$ to itself.

④

Proof: If $v \in W^{\perp}$, then

$\langle v, w \rangle = 0 \quad \forall\, w \in W$

But then,

$\langle Tv, w \rangle = \langle v, \underset{\overset{\cap}{W}}{Tw} \rangle = 0 \quad \forall\, w \in W$

So $T(v) \in W^{\perp}$.

Proof of Spectral Theorem

(1) Assume first that $F = \mathbb{C}$.

We proceed by induction on n.

If $n = 1$, there is nothing to prove.

Otherwise, $\text{Spectrum}(T) \neq \emptyset$, since F is algebraically closed. Let λ be

⑤

an eigenvalue for T, and let

v. be an associated eigenvector.

$\mathbb{C}v$ is T-stable.

Hence so is $(\mathbb{C}\cdot v)^{\perp} =: W$

Let $T' = T|_{W}$, the restricion

of T to W.

Clearly, $T' : W \to W$ is self

adjoint. Furthermore, $\dim W = \dim V - 1$

By the induction hypotesis, W has

an orthonormal basis of eigenvectors

for T, $(e_2, e_3, \ldots, e_n)$.

Let $e_1 = v / \|v\|$. Then $(e_1, \ldots, e_n)$ is an orthonormal basis of eigenvectors for V. We have therefore proved the spectral theorem for complex inner product spaces.

When $F = \mathbb{R}$, choose an orthonormal basis for V, and let M be the matrix representing T.

⑦

M is a symmetric matrix in $M_n(\mathbb{R})$

Viewing M as a matrix in $M_n(\mathbb{C})$, we see that M is a hermitian matrix (which just happens to have real entries)

Hence all the roots of the characteristic polynomial of M — hence of T — are real. In particular T has an eigenvalue $\lambda \in \mathbb{R}$, and we can proceed by induction, as in the case where $F = \mathbb{C}$. $\boxtimes$

Applications

Definition: An operator T is positive semi definite if $\langle Tv, v\rangle \in \mathbb{R}^{\geq 0}$ for all $v \in V$.

Remark: The condition of being positive semi-definite has an incidence on whether T is self-adjoint.

(Ask poll question 3)

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Indeed, when $F = \mathbb{R}$, the condition $\langle Tv, v\rangle \in \mathbb{R}$ is always satisfied, and does not ensure $T = T^*$.

But when $F = \mathbb{C}$, it does:

$$\left(\langle Tv, v\rangle = \langle T^* v, v\rangle \Rightarrow \langle (T - T^*)v, v\rangle = 0 \quad \forall v \in V \right.$$

$$\left. \Rightarrow \quad T - T^* = 0. \right)$$

$\boxtimes$

<u>Theorem</u>: If T is self adjoint and positive semidefinite, then T has a unique positive semidefinite square root, satisfying $U^2 = T$.

<u>Proof</u> By the spectral theorem,

V has an orthonormal basis of eigenvectors for T $(e_1, \dots, e_n)$ with eigenvalues $(\lambda_1, \dots, \lambda_n)$, $\lambda_j \geq 0$.

Let $\sqrt{T}$ be the operator defined by $\sqrt{T}(e_j) = \sqrt{\lambda_j}\, e_j$, $\sqrt{\lambda_j} \geq 0$.

Then $\sqrt{T}$ has the required property.

<u>Collections</u> <u>of</u> <u>self-adjoint</u> <u>transformations</u>

Suppose that $T_1, \ldots, T_r$ is a collection of commuting self-adjoint operators. $T_i T_j = T_j T_i \quad \forall\, i,j \geq 1$
$i,j \leq r$

<u>Theorem</u> The space V has an orthonormal basis ~~of~~ $(e_1, \ldots, e_n)$ of simultaneous eigenvectors for $(T_1, \ldots, T_n)$.

<u>Proof:</u> Induction on $n = \dim V$

Let λ be an eigenvalue for T_1,
$W = \lambda$-eigenspace for T_1
$W^\perp =$ orthogonal complement.

$$V = W \oplus W^\perp$$

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for $j = 1, \dots, r$, T_j preserves W and $W^\perp$

If $v \in W$, $T_j v \overset{?}{\in} W$

Well, $T_1 (T_j v) = T_j T_1 v$
$= T_j \lambda v$
$= \lambda (T_j v)$

So $T_j v$ is a λ-eigenvector for T_1
$\Rightarrow$ $T_j v \in W$

Likewise, $W^\perp$ is preserved by $T_1, \dots, T_r$

because these operators are self-adjoint.

But $\dim W, \dim W^\perp < \dim V$

By strong induction, W has an ON

basis $(e_1, \dots, e_s)$ of simultaneous eigenvectors, and $W^\perp$ has a basis $(e_{s+1}, \ldots, e_n)$ of simultaneous eigenvectors. The n-tuple $(e_1, e_2, \ldots, e_n)$ is the desired orthonormal system of simultaneous eigenvectors for $T_1, \ldots, T_r$. $\boxtimes$

Normal Operators

Definition: An operator is called <u>normal</u> if it commutes with its adjoint.

<u>Examples</u> (1) Self-adjoint, skew adjoint, orthogonal, and unitary operators are normal.

(2) The operator $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ is not normal.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Theorem If $F = \mathbb{C}$, and T is normal, then V has an orthonormal basis of eigenvectors for T. $\square$

Proof $\frac{1}{2}(T + T^*)$ and $\frac{1}{2i}(T - T^*)$ are both self adjoint operators:

$$\frac{1}{2}(T + T^*)^* = \frac{1}{2}(T^* + T^{**}) = \frac{1}{2}(T^* + T)$$

$$\left(\frac{1}{2i}(T + T^*)\right)^* = -\frac{1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*)$$

If we write $T^+ = \frac{1}{2}(T + T^*)$

$T^- = \frac{1}{2i}(T - T^*)$, then

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we have

$$T = T^{+} + iT^{-}$$

where both T^{+} and T^{-} are self adjoint.

One can think of T^{+} and T^{-} as the "real" and "imaginary" parts of T.

If T is normal, then T^{+} and T^{-} commute, since they are both linear combinations of T and T^{*}.

Let $(e_1, \ldots, e_n)$ be a simultaneous orthonormal eigenbasis for T^+, T^-

$$T^+ e_j = \lambda_j^+ e_j \qquad \lambda_j^+ \in \mathbb{R}$$

$$T^- e_j = \lambda_j^- e_j \qquad \lambda_j^- \in \mathbb{R}$$

$$\boxed{T e_j = (\lambda_j^+ + i\lambda_j^-) e_j}$$

<u>Corollary:</u> Every unitary operator $V \to V$ admits an orthonormal system of eigenvectors.

<u>Proof</u> T unitary $\Rightarrow T^* = T^{-1}$
$\Rightarrow T, T^*$ commute $\Rightarrow T$ normal $\square$

<u>Remark</u> When $F = \mathbb{R}$, this is of course false. E.g. T a rotation by angle of θ in $\mathbb{R}^2$ $T^* = T^{-1}$, but T is not diagonalisable unless $\theta \in \mathbb{Z}\cdot\pi$.

<u>Next time</u>: we will describe the polar decomposition of a linear transformation on an IPS $\boxtimes$