189-457B: Honors Algebra 4

Assignment 1

Due: Wednesday, January 25

1. Let R be a commutative ring, and let M and N be R-modules. Prove or disprove: the quotient M/IM is isomorphic to N/IN as an R-module (or, equivalently, as an R/I-module) for all non-zero ideals I of R, if and only if M is isomorphic to N as an R-module.

2. Give an example of a $\mathbf{Z}$-module M of infinite cardinality, such that every finitely generated submodule of M is finite.

3. Let R be a (not necessarily commutative) ring. Recall that a left R-module is said to be *simple* if it has no non-zero proper R-submodules. Let M be a simple left R-module. Show that M is isomorphic to R/I as a left R-module for some left ideal I of R. Show that I is a maximal left R-module, i.e., is contained in no larger left ideal. Give an example of a left ideal I in a non-commutative ring R for which R/I is simple and I is not a two-sided ideal (i.e., is not stable under right multiplication by elements of R).

4. Recall that a module is said to be *semisimple* if it is isomorphic to a finite direct sum of simple modules. Using the insights you've gained in exercise 3, show that a commutative ring is semisimple as a module over itself if and only if it is isomorphic to a finite product of fields.

5. Let R be the ring $F[x]$ of polynomials with coefficients in a field F. Describe all the simple modules over R. Show in particular that every such module is finite-dimensional over F.

6. Give an example of a module over $F[x]$ which is two-dimensional over F and not semisimple.

7. Give a complete characterisation of the elements $f \in F[x]$ for which the quotient $F[x]/(f(x))$ is simple. Same question with "simple" replaced by "semisimple".

8. Let G be a finite group and let $R = \mathbf{R}[G]$ be the group ring of G with coefficients in the field $\mathbf{R}$ of real numbers. Let V be an R-module which is finite-dimensional as an $\mathbf{R}$-vector space. Show that V contains a simple R-module. (Note that such a property is not true for an arbitrary ring R; for instance, $\mathbf{Z}$ viewed as a module over itself contains no simple $\mathbf{Z}$-submodules.)

9. Sticking to the notations and assumptions of exercise 8, show that there exists an inner product (in the sense of the theory of inner product spaces over $\mathbf{R}$)

$$\langle\ ,\ \rangle : V \times V \longrightarrow \mathbf{R}$$

which is G-invariant, i.e., satisfies

$$\langle gv, gw\rangle = \langle v, w\rangle, \quad \text{for all } g \in G, \quad v, w \in V.$$

(Hint: you can always construct an inner product $\langle\ ,\ \rangle_0$ on V by identifying V with $\mathbf{R}^n$ and equipping the latter with the standard dot product; now try to construct a G-invariant inner product from $\langle\ ,\ \rangle_0$ by "averaging over G".)

10. Still in the setting of questions 8 and 9, show that if V_0 is an R-submodule of V, there is another R-submodule V_1 of V for which

$$V = V_0 \oplus V_1$$

as R-modules. Conclude that every module over R which is finite-dimensional over $\mathbf{R}$ is a semisimple R-module. (Hint: this question would be rather hard if I asked you to work it out from scratch; it becomes much easier if you use what you constructed in exercise 9.)

The outcome of exercise 10 is a substantial result in the representation theory of finite groups, and deserves to be contrasted with what you learned in exercise 4. This semisimplicity result can be restated as the fact that every finite-dimensional representation of a finite group G over $\mathbf{R}$ can be decomposed as a direct sum *of irreducible representations.*