189-251A: Algebra 2

Final Exam

Monday, April 23, 2012

This exam has 10 questions, worth 10 points each. Calculators and class notes are not allowed.

1. a) Let F be a field, let S be a finite set and let V be the vector space of functions from S to F. Compute the dimension of V by exhibiting an *explicit basis* for V.

b) Suppose S is infinite, but countable. Does V then have a countable basis? Justify your answer.

2. Let $p(x)$ be a non-zero polynomial of degree n with coefficients in a field F and let V be the quotient $F[x]/(p(x))$. Compute the dimension of V by exhibiting an explicit basis for V. (You should include a proof that the set of vectors you come up with is indeed a basis...)

3. a) Let V be the vector space of problem 2, and let $T : V \longrightarrow V$ be the linear transformation given by $T(p(x)) = xp(x)$.

a) Compute the minimal polynomial of T.

b) Compute the characteristic polynomial of T.

c) Give a necessary and sufficient condition on $p(x)$ for T to be diagonalisable over F.

d) Give a necessary and sufficient conditon on $p(x)$ for T to be invertible.

4. Give an example of a non-zero vector space V and a linear transformation $T : V \longrightarrow V$ satisfying $\ker(T) = \text{image}(T)$. Show that such a linear transformation is *never* diagonalisable.

5. Let V be the vector space of 2×2 matrices with entries in the field $\mathbf{R}$ of real numbers, and let $T : V \longrightarrow V$ be the linear transformation given by

$$T(M) = AMA^{-1}, \qquad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Write down a basis for V and the matrix of T relative to this basis.

6. Let $T : V \longrightarrow V$ be a diagonalisable linear transformation, let $\lambda_1, \ldots, \lambda_t$ be the distinct eigenvalues for T and let

$$V = \oplus_{i=1}^{t} V_{\lambda_i}$$

be the associated decomposition of V into a direct sum of eigenspaces. Show that a linear transformation $U : V \longrightarrow V$ commutes with T *if and only if* all the eigenspaces V_{λ_i} are stable under U. (I.e., if and only if U maps V_{λ_i} to itself, for each i.)

7. Define the following terms:

a) The *dual space* V^* of a vector space V;

b) The *dual linear map* T^* attached to a linear transformation $T : V \longrightarrow W$. Be sure to specify what the domain and target of T^* are, and to write down the formula defining T^*.

c) Show that $(T_1T_2)^* = T_2^*T_1^*$ for all $T_1 : V \longrightarrow W$ and $T_2 : U \longrightarrow V$.

8. State and prove the Cauchy-Scwartz inequality for real inner product spaces.

9. A linear transformation T on a finite-dimensional Hermitan inner product space is said to be skew-adjoint if it satisfies the relation $T^* = -T$.

a) Show that a skew-adjoint operator is normal.

b) Show that all the eigenvalues of a skew-adjoint operator are purely imaginary.

c) Show that every normal operator T can be written as a sum $T_1 + T_2$ where T_1 is self-adjoint, T_2 is skew-adjoint, and $T_1T_2 = T_2T_1$.

10. Let $V = \mathbf{R}^n$ equipped with the standard dot product and resulting distance function, and let W be the hyperplane (i.e., subspace of dimension $n-1$) defined by the equation

$$W = \{(x_1, \ldots, x_n) \text{ with } x_1 + \cdots + x_n = 0\}.$$

Show that the vector in W which is closest to the vector $(x_1, \ldots, x_n)$ is the vector $(x_1 - \mu, \ldots, x_n - \mu)$, where $\mu := \frac{x_1 + \cdots + x_n}{n}$ is the *mean* of $x_1, \ldots, x_n$.