189-251B: Algebra 2

Practice Midterm Exam

1. Give the definitions of:
   a) A vector space over a field F.
   b) Linear independence of vectors.
   c) A basis of a vector space.
   d) The dimension of a vector space.

2. Let V be a vector space over a field F, and let $(v_1, \ldots, v_n)$ be a list of vectors in V. Let $T : F^n \to V$ be the linear transformation defined by

$$T(x_1, \ldots, x_n) = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

a) Give a necessary and sufficient condition involving the list $(v_1, \ldots, v_n)$ guaranteeing that T is injective.

b) Give a necessary and sufficient condition involving the list $(v_1, \ldots, v_n)$ guaranteeing that T is surjective.

3. Let T be a linear transformation on a finite-dimensional vector space and let $m(x)$ be its minimal polynomial. Show that T is invertible if and only if $m(0) \neq 0$.

4. Let V denote the vector space of 2×2 matrices with entries in F, and let T be the linear transformation that sends a matrix M to its transpose:

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Suppose that $2 \neq 0$ in F. Show that T is diagonalisable, by producing a basis of eigenvectors for T. List the eigenvalues of T, and the dimensions of the associated eigenspaces.

Bonus question 5. Let T be a linear transformation and let $m(x) \in F[x]$ be its minimal polynomial. Show that $g(T)$ is invertible if and only if $\gcd(m(x), g(x)) = 1$.