189-251B: Honors Algebra 2

Midterm Exam: Corrections and comments

The following was the grade dsitribution in this midterm:

$>= 100$	5	$95-99$	5	$90-94$	13
$80-89$	13	$70-79$	6	$60-69$	7
$50-59$	1				

So overall the grades ended up being on the high side, and the above will give you an idea of where you stand relative to the rest of the class. If you have any concerns, in particular if you are worried that your performance was not up to your expectations, and do not understand why, do not hesitate to bring this up with me during office hours.

1. State the rank-nullity theorem. (Also known, in Goren's notes, as the "theorem about the kernel and the image".)

This one was meant to be a "slow ball" and indeed most of you caught it. If you did not, do look it up (for instance, in the notes of Eyal Goren.)

2. Let W_n be the vector space polynomials of degree $\leq n$ with coefficients in a field F. (For this question, you may assume without proof that W_n is indeed a vector space over F.)

a) What is the dimension of W_n? Write down a basis for W_n. (You do not need to show it is a basis.)

b) Let $T : W_n \longrightarrow W_{n-1}$ be the linear transformation given by $T(f) = f' + f''$, where f' denotes the usual derivative and f'' is the second derivative. In the case where $n = 3$, write down the matrix of T relative to your chosen bases for W_3 and W_2.

c) Show that T is surjective, for any value of n. (Hint: compute the kernel of T.)

The people who lost points usually did so by forgetting that the desired matrix in part a) had to be a 3×4 matrix and not a 4×4 matrix. I also deducted points in part b) if you did not supply a proof of the fact that the kernel of T is the space of constant polynomials. Some of you did this by computing the kernel directly. My favored approach is to note that a polynomial f in the kernel of T has a derivative, f' which satisfies the differential equation $u' = -u$. The only polynomial satisfying this equation is the 0 polynomial since differentiation reduces the degree, hence $f' = 0$ and therefore f must be a constant. Oh dear, I just realised: this last deduction is only true when the characteristic of the field F is zero!! No one caught me on this omission so it did not create difficulties. But note that it is simply **false** *that T is surjective if the characteristic of F is a prime p and $n \geq p$...*

3. A linear transformation is said to be *idempotent*, or a *projection*, if it satisfies $T^2 = T$, i.e., is equal to its square. Show that such a linear transformation is always diagonalisable (over any field F).

One has to observe that since $p(T) = 0$ for the polynomial $p(x) = x^2 - x$, the minimal polynomial $p_T(x)$ necessarily divides $p(x) = x(x-1)$ and hence is equal to either x, $x-1$ or $x(x-1)$. In any case it factors into distinct linear factors hence we are done by the characterisation of diagonalisable linear transformations seen in class. Those who forgot to mention the possibilities $P_T(x) = x$ or $x - 1$ lost a few points.

4. Let N be a nilpotent linear transformation. Show that $T = 1 + N$ is invertible.

Most of you did this by noting that, as seen in an assignment,

$$T^{-1} = 1 - N + N^2 - N^3 + \cdots + (-1)^e N^e,$$

where e is the least integer such that $N^{e+1} = 0$.

5. True or False: (you do not need to provide justifications...)

I was surprised that very few people got this one completely right. I am curious to know why – the drawback of a T-F question being that the grader does not get much feedback in this respect..

a) Two matrices which are conjugate necessarily have the same characteristic polynomial.

True, because the determinant is preserved under conjugation:

$$\det(PAP^{-1}) = \det(A),$$

by the multiplicativity of the determinant.

b) Two matrices which are conjugate necessarily have the same minimial polynomial.

True, because conjugation is compatible with addition and multiplication of matrices; in particular if f is any polynomial and P any invertible matrix then $f(PAP^{-1}) = Pf(A)P^{-1}$. Therefore the set of polynomials that vanish on A is exactly the same *as the seet of polynomials that vanish on PAP^{-1}. Note that this fact, which was seen in class, underlies the statement that a linear transformation T has a minimial polynomial (independently of its representation as a matrix, which depends on a choice of basis and is only well-defined up to conjugation.)*

c) Two matrices with the same characteristic polynomial are necessarily conjugate.

False. For instance the identity matrix and the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ have the same characteristic polynomial $(x-1)^2$, but are not conjugate, since only the identity is conjugate to the identity.

d) Two matrices with the same minimal polynomial are necessarily conjugate.

False. It is less obvious to find a counterexample, since the smallest such is a 4×4 matrix (I think...). Indeed, if M_1 is a direct sum of two Jordan matrices $M(\lambda, 2)$ of size two, while M_2 is the direct sum of $M(\lambda, 2)$ with two copies of $M(\lambda, 1)$, then both 4×4 matrices have the minimal polynomial $(x - \lambda)^2$ but are not conjugate.

6. Choose one answer (no need to justify it). A matrix is diagonalisable if

and only if

a) Its minimal polynomial has no multiple roots.
b) Its characteristic polynomial has no multiple roots.
c) Its minimal polynomial factors into distinct linear factors.
d) Its characteristic polynomial factors into distinct linear factors.
e) Its minimal polynomial is equal to its characteristic polynomial.

The answer was c. A few of you chose a, which is not so *far off, but you had to remember that the underlying field of scalars is not necessarily algebraically closed....*

The next problem is a Bonus Question. Only attempt it if you are confident that you've answered the first 6 questions completely.

7. Let G be the group of invertible 3×3 matrices with entries in the field $\mathbf{Z}_p$ with p elements. What is its cardinality?

I won't spoil your fun by giving a solution to this one. If you didn't get it in the exam, take some time to think about it...