**McGill University** December 2013
**Faculty of Science** Final Examination

# Abstract Algebra

## Math 235

Thursday, December 5, 2012

Time: 6:00 pm - 9:00 pm

Examiner: Prof. Henri Darmon Associate Examiner: Prof. Eyal Goren

### INSTRUCTIONS

1. Please answer questions in the exam booklet provided.
2. This exam consists of 10 questions. Each question is worth 10 points.
3. This a CLOSED BOOK exam. No notes, textbooks or crib sheets are permitted.
4. Calculators, cell phones, smart phones, tablets and laptops are not permitted.
5. Please write your name clearly on the examination booklet.

This exam comprises the cover page and two page of questions.

1. Let $\mathbf{F} = \mathbf{Z}/3\mathbf{Z}$ be the field with three elements.

a) Prove that the ring $\mathbf{F}[x]/(x^3 + x^2 + 2)$ is a field.

b) Prove that the ring $\mathbf{F}[x]/(x^3 + x + 2)$ is not a field, and write down a non-zero non-invertible element in this ring.

c) Show that the rings in a) and b) have the same cardinality (and say what that cardinality is), but that they are not isomorphic.

2. Define what it means for an ideal to be principal, and give an example of a ring $R$ and an ideal $I$ in it which is *not* principal. (You should justify your claims.)

3. State and prove Fermat's Little Theorem.

4. Write down *all* the distinct solutions of the equation

$$x^2 + 1 = 0$$

in the ring $\mathbf{Z}/n\mathbf{Z}$ with $n = 1717$.

5. a) Explain how to construct a field with 25 elements.

b) How many roots does the polynomial $x^{24} - 1$ have in this field?

6. a) List the conjugacy classes in the dihedral group

$$D_8 = \{1, r, r^2, r^3, D_1, D_2, V, H\}$$

consisting of the symmetries of the square. (The notations here are the ones used in class: $r$ is the rotation by an angle of $\pi/2$ about the center of the square, $D_1$ and $D_2$ are the reflections about its two diagonals, and $V$ and $H$ are the reflections about its vertical and horizontal axes of symmetry.)

b) Give a complete list of the normal subgroups of $D_8$.

7. a) Show that the group $G = S_4$ has a subgroup $H$ of cardinality 8.

b) For the group $H$ you found in part a), what is the cardinality of the set $G/H$ of cosets of $H$ in $G$?

8. a) Use the result of exercise 7 (especially, part b) to write down a non-trivial homomorphism

$$\varphi : S_4 \longrightarrow S_3.$$

b) What is the kernel of $\varphi$?

c) Show that $\varphi$ is surjective.

9. Let $G = S_n$ be the permutation group on $n$ elements. An element $x \in \{1, \ldots, n\}$ is called a *fixed point* of a permutation $g \in G$ if $g(x) = x$.

a) Show that if $g_1$ and $g_2$ are conjugate to each other in $G$, then they have the same number of fixed points.

b) Write down two elements in $S_4$ which have no fixed points but are not conjugate to each other.

10. Let $G$ be a non-abelian group. Show that there is an isomorphism $f : G \longrightarrow G$ from $G$ to itself which is not the identity function.

**Extra credit question**. Let $G = \mathbf{GL}_3(\mathbf{Z}/p\mathbf{Z})$ be the group of invertible $3 \times 3$ matrices with entries in $\mathbf{Z}/p\mathbf{Z}$. Write down the cardinality of $G$, and show that $G$ has at least $p^2(p-1)$ distinct conjugacy classes.