# 189-251B: Algebra 2
# Assignment 8
## Due: Wednesday, March 12

1. Let $V$ be the vector space over the reals consisting of polynomials of degree at most $n-1$, and let $x_1, \cdots, x_n$ be $n$ distinct real numbers.

a) Show that the functionals $\ell_1, \ldots, \ell_n$ on $V$ defined by

$$\ell_j(p(x)) = p(x_j) \qquad 1 \leq j \leq n$$

form a basis for the dual space $V^*$.

b) (*Polynomial interpolation*). Given real numbers $y_1, \ldots, y_n$, show that there is a unique polynomial of degree at most $n-1$ satisfying $p(x_j) = y_j$ for all $1 \leq j \leq n$, and that it is given by the formula

$$p(x) = p(x_1)p_1(x) + \cdots + p(x_n)p_n(x),$$

where $(p_1(x), \ldots, p_n(x))$ is the basis of $V$ which is dual to $(\ell_1, \ldots, \ell_n)$, i.e., satisfies

$$(p_1^*, \ldots, p_n^*) = (\ell_1, \ldots, \ell_n).$$

c) Write down a formula for the basis $(p_1, \ldots, p_n)$.

2. Proposition 7.2.1. of Eyal Goren's notes shows that there is a natural injective linear transformation from $V$ to $V^{**}$, the dual of the dual of $V$. It then shows furthermore that this map is an *isomorphism* when $V$ is finite-dimensional. Read the proof of Proposition 7.2.1. carefully and note the crucial role played by the finite-dimensionality assumption. (You do not need to write anything, yet...)

Now, give an example to show that the natural map $V \longrightarrow V^{**}$ need not be surjective in general, if $V$ is not assumed to be finite dimensional. (Warning: you might need some form of the axiom of choice...)

3. Let $\mathbf{H} = \mathbf{R}\langle i, j, k\rangle$ denote the usual ring of quaternions. (Recall that a typical quaternion is an element of the form

$$z = a + bi + cj + dk, \quad a, b, c, d \in \mathbf{R}.$$

Recall the trace and norm functions on $\mathbf{H}$ defined by

$$\operatorname{trace}(z) = 2a = z + \bar{z}, \quad \operatorname{norm}(z) = a^2 + b^2 + c^2 + d^2 = z\bar{z},$$

where $\bar{z} = a - bi - cj - dk$.

Show that $\mathbf{H}$, equipped with the rule

$$\langle x, y \rangle := \operatorname{trace}(x\bar{y})$$

is a real inner product space.

4. Note that $\mathbf{H}$ can also be viewed as a complex vector space by viewing the elements of the form $a + bi$ as complex numbers and using the multiplication in $\mathbf{H}$ to define the scalar multiplication. Is $\mathbf{H}$ equipped with the pairing above a Hermitian space? Explain.

5. Let $V$ be the real vector space of real-valued functions on the interval $[-\pi, \pi]$ equipped with the inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(t)g(t)dt.$$

Let $W$ be the subspace of functions spanned by $f_0 = 1/\sqrt{2\pi}$, $f_j(t) := 1/\sqrt{\pi}\cos(jt)$, with $1 \leq j \leq N$ and $g_j(t) := 1/\sqrt{\pi}\sin(jt)$ with $1 \leq j \leq N$. Show that $f_0, f_1, \ldots, f_N, g_1, \ldots, g_N$ is an orthonormal basis for $W$. Given $f \in V$, give a formula for the coefficients of the function

$$a_0 f_0 + a_1 f_1 + \cdots + a_N f_N + b_1 g_1 + \cdots + b_N g_N$$

in $W$ which best approximates $f$. (These coefficients are called the *Fourier coefficients* attached to $f$.) Compute these coefficients in the case of the function $f(x) = x$.

5. Let $V$ be the space of polynomials of degree $\leq 2$ equipped with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Find an orthonormal basis for $V$, by applying the Gramm-Schmidt procedure to the ordered basis $(1, x, x^2)$.

6. Let $S = (x_1, x_2, \ldots, x_N)$ and $(y_1, \ldots, y_N)$ be sequences of real numbers. Fix an integer $k$ less than $N$. In general, there need not be a polynomial $p$ of

degree $\leq k$ such that $p(x_j) = y_j$ for $j = 1, \ldots, N$. The next-best thing one might ask for is a polynomial $p$ of degree $\leq k$ for which the quantity

$$(p(x_1) - y_1)^2 + (p(x_2) - y_2)^2 + \cdots + (p(x_N) - y_N)^2$$

is minimised. Describe an approach that would produce such a $p$. In the case $k = 1$, give a formula for the coefficients $a, b$ of $p(x) = ax + b$ in terms of $x_1, \ldots, x_N, y_1, \ldots, y_N$.